Lower Bounds for the Number of Bends in Three-Dimensional Orthogonal Graph Drawings

David R. Wood*

Basser Department of Computer Science
The University of Sydney
Sydney NSW 2006, Australia
davidw@cs.usyd.edu.au

Abstract. In this paper we present the first non-trivial lower bounds for the total number of bends in 3-D orthogonal drawings of maximum degree six graphs. In particular, we prove lower bounds for the number of bends in 3-D orthogonal drawings of complete simple graphs and multigraphs, which are tight in most cases. These results are used as the basis for the construction of infinite classes of c -connected simple graphs and multigraphs ($2 \leq c \leq 6$) of maximum degree Δ ($3 \leq \Delta \leq 6$) with lower bounds on the total number of bends for all members of the class. We also present lower bounds for the number of bends in general position 3-D orthogonal graph drawings. These results have significant ramifications for the ‘2-bends’ problem, which is one of the most important open problems in the field.

1 Introduction

The *3-D orthogonal grid* consists of *grid-points* in 3-space with integer coordinates, together with the axis-parallel *grid-lines* determined by these points. A *3-D orthogonal drawing* of a graph places the vertices at grid-points and routes the edges along sequences of contiguous segments of grid-lines. Edges are allowed to contain bends and can only intersect at a common vertex. 3-D orthogonal drawings have been studied in [2,3,4,5,6,7,8,12,13,14]. For brevity we say a 3-D orthogonal graph drawing is a *drawing*. A drawing with no more than b bends per edge is called a *b-bend drawing*. The graph-theoretic terms ‘vertex’ and ‘edge’ also refer to their representation in a drawing. The *ports* at a vertex v are the six directions, denoted by $\{X^+, X^-, Y^+, Y^-, Z^+, Z^-\}$, which the edges incident with v can use. For each dimension $I \in \{X, Y, Z\}$, the I^+ (respectively, I^-) port at a vertex v is said to be *extremal* if v has maximum (minimum) I -coordinate taken over all vertices. Clearly, orthogonal drawings can only exist for graphs with maximum degree six.

Drawings with many bends appear cluttered and are difficult to visualise. Therefore minimising the number of bends, along with minimising the bounding box volume, have been the most commonly proposed aesthetic criteria for

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measuring the quality of a drawing. Using straightforward extensions of the corresponding 2-D NP-hardness results, optimising each of these criteria is NP-hard [4]. Kolmogorov and Barzdin [6] established a lower bound of $\Omega(n^{3/2})$ for the bounding box volume of a drawings of n -vertex graphs. In this paper we establish the first non-trivial lower bounds for the number of bends in 3-D orthogonal drawings. Lower bounds for the number of bends in 2-D orthogonal graph drawings have been established by Tamassia *et al.* [9] and Biedl [1].

Lower bounds for the maximum number of bends per edge: Obviously every drawing of K_3 has at least one bend. It follows from results in multi-dimensional orthogonal graph drawing by Wood [11] that every drawing of K_5 has an edge with at least two bends. It is well known that every drawing of $6K_2$ has an edge with at least three bends, and easily seen that $2K_2$ and $3K_2$ have at least one edge with at least one and two bends, respectively. Here kK_2 is the 2-vertex multigraph with k edges.

A natural candidate for the existence of a simple graph with a 3-bend edge in every drawing is K_7 , as originally conjectured by Eades *et al.* [5]. A counterexample to this conjecture, namely a drawing of K_7 with at most two bends per edge, was first exhibited by Wood [11]. A more symmetric drawing of K_7 with at most two bends per edge is shown in Fig. 1(a). This drawing has the interesting feature of rotational symmetry about the line $X = Y = Z$. One may consider the other 6-regular complete multi-partite graphs $K_{6,6}$, $K_{3,3,3}$ and $K_{2,2,2,2}$ to be potential examples of simple graphs with a 3-bend edge in every drawing. 2-bend drawings of these graphs are presented in [14].

Fig. 1. (a) 2-bend drawing of K_7 (b) 4-bend drawing of K_7 with 24 bends.

Lower bounds for the total number of bends: In this paper we prove that drawings of the complete graphs K_4 , K_5 , K_6 and K_7 have at least 3, 7, 12 and 20 bends, respectively. For each of these graphs except K_7 there are well-

known drawings with the corresponding number of bends. Figure 1(b) shows a drawing of K_7 with a total of 24 bends (compared with the total of 42 bends for the 2-bend drawing). We conjecture that there is no drawing of K_7 with fewer than 24 bends.

We use these lower bounds for the number of bends in complete graphs as the basis for the construction of infinite families of c -connected graphs of maximum degree Δ with lower bounds on the number of bends for each member of the class. Table 1 summarises these lower bounds.

Table 1. Lower bounds for the number of bends in drawings of m -edge c -connected graphs with maximum degree Δ .

Connectivity c	Simple Graphs				Multigraphs			
	$\Delta = 6$	$\Delta = 5$	$\Delta = 4$	$\Delta = 3$	$\Delta = 6$	$\Delta = 5$	$\Delta = 4$	$\Delta = 3$
0	$\frac{20}{21}m$	$\frac{4}{5}m$	$\frac{7}{10}m$	$\frac{1}{2}m$	$2m$	$\frac{8}{5}m$	$\frac{3}{2}m$	$\frac{4}{3}m$
2	$\frac{3}{4}m$	$\frac{7}{11}m$	$\frac{3}{7}m$	$\frac{1}{4}m$	$\frac{4}{3}m$	$\frac{6}{5}m$	m	$\frac{2}{3}m$
3	$\frac{8}{11}m$	$\frac{14}{23}m$	$\frac{2}{5}m$	$\frac{2}{9}m$	m	$\frac{4}{5}m$	$\frac{1}{2}m$	-
4	$\frac{12}{17}m$	$\frac{7}{12}m$	$\frac{3}{8}m$	-	$\frac{2}{3}m$	$\frac{2}{5}m$	-	-
5	$\frac{24}{35}m$	$\frac{14}{25}m$	-	-	$\frac{1}{3}m$	-	-	-
6	$\frac{2}{3}m$	-	-	-	-	-	-	-

Upper bounds: A number of algorithms have been proposed for 3-D orthogonal graph drawing [2,3,5,6,8,12,14] which explore the apparent tradeoff between the maximum number of bends per edge and the bounding box volume (see [14] for an overview). We now summarise the known upper bounds on the number of bends in the drawings produced by these algorithms. The 3-BENDS algorithm of Eades *et al.* [5] and the INCREMENTAL algorithm of Papakostas and Tollis [7] both produce 3-bend drawings¹ of multigraphs² with maximum degree six. As discussed above there exists simple graphs with at least one edge having at least two bends in every drawing. The following problem is therefore of considerable interest:

2-Bends Problem: Does every simple graph with maximum degree six admit a 2-bend drawing? [5]

¹ The 3-BENDS algorithm [5] produces drawings with $27n^3$ volume. By deleting grid-planes not containing a vertex or a bend the volume is reduced to $8n^3$. The INCREMENTAL algorithm [7] produces drawings with $4.63n^3$ volume. A modification of the 3-BENDS algorithm by Wood [14] produces drawings with $n^3 + o(n^3)$ volume.

² The 3-BENDS algorithm [5] explicitly works for multigraphs. The INCREMENTAL algorithm, as stated in [7], only works for simple graphs; with a suitable modification it also works for multigraphs [A. Papakostas, private communication, 1998].

The DIAGONAL LAYOUT AND MOVEMENT algorithm of Wood [12] solves the 2-bends problem in the affirmative for simple graphs with maximum degree five. For maximum degree six simple graphs, the same algorithm uses a total of at most $7m/3$ bends, which is the best known upper bound for the total number of bends in 3-D orthogonal drawings.

In this paper we provide a negative result related to the 2-bends problem. A 3-D orthogonal graph drawing is said to be in *general position* if no two vertices lie in a common grid-plane. The general position model is used in the 3-BENDS and DIAGONAL LAYOUT AND MOVEMENT algorithms. In this paper we show that the general position model, and the natural variation of this model where pairs of vertices share a common plane, cannot be used to solve the 2-bends problem, at least for 2-connected graphs.

The remainder of this paper is organised as follows. In Sect. 2 we establish a number of introductory results concerning 0-bend drawings of cycles. These results are used to prove our lower bounds on the total number of bends in drawings of complete graphs, established in Sect. 3. In Sect. 4 we use these lower bounds as the basis for lower bounds on the number of bends in infinite families of graphs of varying connectivity and maximum degree. In Sect. 5 we present lower bounds for the number of bends in general position drawings and their implications for the 2-bends problem.

Throughout this paper we consider n -vertex m -edge loop-free graphs with maximum degree six. By C_m we denote the m -edge cycle. A *chord* of a cycle C is an edge not in C whose end-vertices are both in C . We say two cycles are *chord-disjoint* if they do not have a chord in common. In this paper most proofs are outlined, and the proofs of the results in Table 1 concerning multigraphs are omitted; see [13] for details of all the proofs.

2 Drawings of Cycles

In this section we characterise the 0-bend drawings of the cycles C_k ($k \leq 7$). We then show that if a drawing of a complete graph contains such a 0-bend drawing of a cycle then there must be many edges with at least three bends in the drawing of the complete graph. These results are used in Sect. 3 in our lower bounds for the total number of bends in drawings of complete graphs.

A straight-line path in a 0-bend drawing of a cycle is called a *side*. A side parallel to the I -axis for some $I \in \{X, Y, Z\}$ is called an I -side, and I is called the *dimension* of the side. Clearly the dimension of adjacent sides is different, thus in a 2-dimensional drawing the dimension of the sides alternate around the cycle. Hence, there is no 2-dimensional 0-bend drawing of a cycle with an odd number of sides. If there is an I -side in a drawing of a cycle for some $I \in \{X, Y, Z\}$ then clearly there is at least two I -sides. Therefore a drawing of a cycle with X -, Y - and Z -sides, which we call *truly 3-dimensional*, must have at least six sides. Hence there is no truly 3-dimensional 3-, 4- or 5-sided 0-bend drawing of a cycle. There is also no two-dimensional 3- or 5-sided 0-bend drawing of a cycle. We therefore have the following observation.

Observation 1 (a) *There is no 3- or 5-sided 0-bend drawing of a cycle,*
 (b) *there is no 0-bend drawing of C_3 , and*
 (c) *all 0-bend drawings of C_4 and C_5 have four sides.* $\square$

The next result forms an important component of the lower bounds to follow.

Lemma 1. *If a drawing of a complete graph contains a 0-bend 4-cycle (respectively, 5-cycle) then there are at least two (four) chords of the cycle each with at least three bends.*

Proof. By Obs. 1(c) all 0-bend drawings of C_4 and of C_5 have four sides. As illustrated in Fig. 2(a), the chord connecting each pair of diagonally opposite vertices in a 4-sided drawing of a cycle has at least three bends. Hence, if a drawing of a complete graph contains a 0-bend C_4 , then the two chords each have at least three bends. Also, in the case of C_5 , the two edges from the vertex not at the intersection of two sides to the diagonally opposite vertices both have at least three bends, as in Fig. 2(b). Hence, if a drawing of a complete graph contains a 0-bend C_5 , then the four chords each have at least three bends. $\square$

Fig. 2. 3-bend edges ‘across’ (a) C_4 and (b) C_5 . (c) The graph H_1

Consider the graph shown in Fig. 2(c), which we call H_1 .

Observation 2 H_1 *does not have a 0-bend drawing.*

Proof. H_1 contains C_4 . As in Lemma 1, an edge between the non-adjacent vertices of a 4-sided cycle needs at least three bends. Hence the 2-path in H_1 between the non-adjacent vertices of the 4-cycle has at least one bend. Therefore H_1 does not have a 0-bend drawing. $\square$

We now consider 6-sided 0-bend drawings of a cycle.

Lemma 2. *If a drawing of a complete graph contains a 0-bend 6-cycle then there are at least four chords of the cycle each with at least three bends.*

Proof. We can assume without loss of generality that there is a 0-bend drawing of C_6 contained in a drawing of K_6 . By Obs. 1(a), all 0-bend drawings of C_6 have four or six sides. In a 4-sided 0-bend drawing of C_6 the two vertices not at the intersection of adjacent sides can be (a) on the same side, (b) on adjacent sides, or (c) on opposite sides. In each case there are at least six chords from a

vertex at the intersection of two sides to a vertex on neither of these sides, each with at least three bends.

Case analysis shows that the only 6-sided 0-bend drawings of C_6 (up to symmetry) are those in Fig. 3. For each such drawing, the chords of C_6 shown in Fig. 3 each require at least three bends. In the case of the drawing in Fig. 3(c) there are at least six chords each requiring at least three bends. For the drawing in Fig. 3(a) (respectively, Fig. 3(b)) it can be shown, by considering certain sets of chords for which all edge routes with at most two bends pass through a single grid point, that a further two (four) chords have at least three bends. Hence there are at least six chords of the cycle each with at least three bends. $\square$

Fig. 3. Edges with at least 3 bends in a drawing of K_6 containing a 6-sided 0-bend C_6 .

We now consider 7-sided 0-bend drawings of a cycle.

Lemma 3. *If a drawing of K_7 contains a 0-bend 7-cycle then there are at least four chords of the cycle each with at least three bends.*

Proof. By Obs. 1(a), a 0-bend drawing of C_7 has four, six or seven sides. In a 4-sided 0-bend drawing of C_7 the three vertices not at the intersection of two adjacent sides can be (a) all on the same side, (b) two on one side and one on an adjacent side, (c) two on one side and one on the opposite side, or (d) all on different sides. In each case there are at least eight chords from a vertex at the intersection of two sides to a vertex on neither of these sides, each with at least three bends. The 6-sided 0-bend drawings of C_7 can be obtained from the 6-sided 0-bend drawings of C_6 by placing one new vertex at each of the essentially different edges of each drawing. By Lemma 2, at least six of the chords of C_6 , and therefore of C_7 , have at least three bends. Case analysis shows that the only 7-sided 0-bend drawings of C_7 are those shown in Fig. 4. For each drawing there are at least four chords which need at least three bends. $\square$

Fig. 4. Edges with at least three bends in a 7-sided 0-bend drawing of C_7 .

3 Drawings of Complete Graphs

In this section we establish lower bounds for the total number of bends in 3-D orthogonal drawings of K_4 , K_5 , K_6 and K_7 . We omit our proofs that the well-known drawings of K_4 , K_5 and K_6 shown in Fig. 5 are bend-minimum. They are similar to the proof of the lower bound for K_7 which follows.

Fig. 5. (a) K_4 with 3 bends, (b) K_5 with 7 bends and (c) K_6 with 12 bends.

Theorem 1. *Every drawing of K_4 has at least three bends. Every drawing of K_5 has at least seven bends. Every drawing of K_6 has at least twelve bends. $\square$*

Figure 1(b) shows a drawing of K_7 with a total of 24 bends.

Theorem 2. *Every drawing of K_7 has at least 20 bends.*

Proof. Suppose to the contrary, that there is a drawing of K_7 with at most 19 bends. The subgraph of K_7 consisting of the 0-bend edges is called the *0-bend subgraph*. Let k_i ($i \geq 0$) be the number of i -bend edges. Hence

$$\sum_{i \geq 0} k_i = 21, \text{ and } \sum_{i \geq 1} i k_i \leq 19. \quad (1)$$

Case analysis shows that every subgraph of K_7 with at least ten edges contains C_3 or H_1 . By Obs. 1(b) and Obs. 2, the graphs C_3 and H_1 do not have 0-bend drawings. Hence $k_0 \leq 9$. Suppose $k_0 = 8$ or $k_0 = 9$. By (1)

$$12 \leq \sum_{i \geq 1} k_i \leq 19 - \sum_{i \geq 1} (i-1)k_i$$

$$\sum_{i \geq 2} (i-1)k_i \leq 7. \quad (2)$$

Case analysis shows that every subgraph of K_7 with at least eight edges contains a cycle C_k ($k \neq 4$), two chord-disjoint cycles, or an H_1 subgraph. Therefore the 0-bend subgraph contains a cycle C_k ($k \geq 5$) or two chord-disjoint subgraphs (since C_3 and H_1 do not have 0-bend drawings by Obs. 1(b) and Obs. 2, respectively). If the 0-bend subgraph contains a cycle C_k ($k \geq 5$) then by Lemma 1, Lemma 2 and Lemma 3 there are at least four chords of the

cycle each with at least three bends. On the other hand if the 0-bend subgraph contains two chord-disjoint cycles then these cycles have length at least four; thus by Lemma 1, Lemma 2 and Lemma 3, two chords from each of these cycles each have at least three bends. In either case the drawing of K_7 has at least four edges each with at least three bends; that is,

$$\begin{aligned} 4 &\leq \sum_{i \geq 3} k_i \\ k_2 + 8 &\leq k_2 + \sum_{i \geq 3} 2k_i \leq \sum_{i \geq 2} (i-1)k_i \\ k_2 + 8 &\leq 7 \quad (\text{by (2)}) \end{aligned}$$

Hence $k_2 \leq -1$, which is a contradiction. Therefore $k_0 \leq 7$. By (1) with $k_0 \leq 7$

$$\begin{aligned} 14 &\leq \sum_{i \geq 1} k_i \leq 19 - \sum_{i \geq 1} (i-1)k_i \\ \sum_{i \geq 2} (i-1)k_i &\leq 5 \\ k_2 + 2 \sum_{i \geq 3} k_i &\leq 5 \end{aligned} \tag{3}$$

Let A be the set of edges of K_7 routed using an extremal port at exactly one end-vertex. Let B be the set of edges routed using extremal ports in the same direction at its end-vertices. Let C be the set of edges routed using extremal ports in differing directions at its end-vertices. K_7 is 6-regular, thus all ports are used. Since there is at least one extremal port in each direction, we have $|A| + |B| + 2|C| \geq 6$. It is easily seen that an edge in A or B has at least two bends, and that an edge in C has at least three bends. Hence

$$k_2 + 2 \sum_{i \geq 3} k_i \geq 6 .$$

This contradicts (3). The result follows. $\square$

4 Constructing Large Graphs

In this section we use the lower bounds for the total number of bends in drawings of the complete graphs established in Sect. 3 as building blocks to construct infinite families of graphs with lower bounds for the number of bends.

Given graphs G and H with $|V(G)| \geq \Delta(H)$, we define $H\langle G \rangle$ to be the graph obtained by replacing each vertex of H by a copy of G , and connecting the vertices adjacent to a vertex $v \in V(H)$ to different vertices in the copy of G corresponding to v . In most cases, H is regular and G is a complete graph, thus $H\langle G \rangle$ is well-defined. In other cases we shall specify the mapping between edges incident to v and the vertices in the copy of G corresponding to v . We

also employ the *cartesian product* $G \times H$ of graphs G and H to construct larger graphs. $G \times H$ has vertex set $V(H) \times V(G)$ with (v_1, w_1) and (v_2, w_2) adjacent in $G \times H$ if either $v_1 = v_2$ and $w_1 w_2 \in E(H)$, or $w_1 = w_2$ and $v_1 v_2 \in E(G)$.

By taking disjoint copies of K_7 , K_6 , K_5 and K_4 the next result follows immediately from Theorem 1 and Theorem 2.

Theorem 3. *There exists infinite families of simple (disconnected) m -edge graphs with maximum degree six (respectively, five, four and three) with at least $\frac{20}{21}m$ ($\frac{4}{5}m$, $\frac{7}{10}m$ and $\frac{1}{2}m$) bends in every drawing. $\square$*

To obtain our lower bounds for 2-, 3- and 4-connected graphs consider the graphs $C_r \langle K_p \rangle$ ($p \geq 3$), $(C_r \times K_2) \langle K_p \rangle$ ($p \geq 3$) and $(C_r \times C_3) \langle K_p \rangle$ ($p \geq 4$) for some $r \geq 3$, as illustrated in Fig. 6.

Fig. 6. (a) 2-connected $C_r \langle K_p \rangle$, (b) 3-connected $(C_r \times K_2) \langle K_p \rangle$, (c) 4-connected $(C_r \times C_3) \langle K_p \rangle$.

Theorem 4. *There exists infinite families of simple 2-connected m -edge graphs with maximum degree six (respectively, five, four and three) with at least $\frac{3}{4}m$ ($\frac{7}{11}m$, $\frac{3}{7}m$ and $\frac{1}{4}m$) bends in every drawing.*

Proof. The graphs $C_r \langle K_6 \rangle$ with $r \geq 2$ have maximum degree six and $m = 16r$ edges. By Theorem 1, K_6 has at least 12 bends in every drawing, thus $C_r \langle K_6 \rangle$ has at least $12r = \frac{3}{4}m$ bends in every drawing. The graphs $C_r \langle K_5 \rangle$ with $r \geq 2$ have maximum degree five and $m = 11r$ edges. By Theorem 1, K_5 has at least 7 bends in every drawing, thus $C_r \langle K_5 \rangle$ has at least $7r = \frac{7}{11}m$ bends in every drawing. The graphs $C_r \langle K_4 \rangle$ with $r \geq 2$ have maximum degree four and $m = 7r$ edges. By Theorem 1, K_4 has at least 3 bends in every drawing, thus $C_r \langle K_4 \rangle$ has at least $3r = \frac{3}{7}m$ bends in every drawing. The graphs $C_r \langle K_3 \rangle$ with $r \geq 2$ have maximum degree three and $m = 4r$ edges. By Obs. 1(b), K_3 has at least 1 bend in every drawing, thus $C_r \langle K_3 \rangle$ has at least $r = \frac{1}{4}m$ bends in every drawing. Clearly $C_r \langle K_p \rangle$ is 2-connected. $\square$

The proofs of the following results for 3- and 4-connected graphs are similar to the proof of Theorem 4.

Theorem 5. *There exists infinite families of simple 3-connected m -edge graphs with maximum degree six (respectively, five, four and three) with at least $\frac{8}{11}m$ ($\frac{14}{23}m$, $\frac{2}{5}m$ and $\frac{2}{9}m$) bends in every drawing. $\square$*

Theorem 6. *There exists an infinite family of simple 4-connected m -edge graphs with maximum degree six (respectively, five and four) with at least $\frac{12}{17}m$ ($\frac{7}{12}m$ and $\frac{3}{8}m$) bends in every drawing. $\square$*

To obtain our lower bounds for 5- and 6-connected graphs consider the graphs $(C_r \times C_3 \times K_2)\langle K_p \rangle$ ($p \geq 5$) and $(C_r \times C_3 \times C_3)\langle K_6 \rangle$ for some $r \geq 3$, as illustrated in Fig. 7. Again the proofs are very similar to that of Theorem 4.

Fig. 7. (a) 5-connected $(C_r \times C_3 \times K_2)\langle K_p \rangle$, (b) 6-connected $(C_r \times C_3 \times C_3)\langle K_6 \rangle$.

Theorem 7. *There exists infinite families of simple 5-connected m -edge graphs with maximum degree six (respectively, five) with at least $\frac{24}{35}m$ ($\frac{14}{25}m$) bends in every drawing. $\square$*

Theorem 8. *There exists an infinite family of simple 6-connected m -edge graphs with maximum degree six with at least $\frac{2}{3}m$ bends in every drawing. $\square$*

5 General Position Drawings and the 2-Bends Problem

Recall that a 3-D orthogonal graph drawing is in general position if no two vertices lie in a common grid-plane. In this section we establish lower bounds for the number of bends in general position drawings.

Lemma 4. *If the graph G has at least k bends in every general position drawing then for any edge e of G the graph $G \setminus e$ has at least $k - 4$ bends in every general position drawing.*

Proof. If $G \setminus e$ has a drawing with b bends then, the edge e can be inserted into the drawing with at most four bends and the edges rerouted so that there are no edge crossings [12,14]. Thus, there is a general position drawing of G with $b + 4$ bends. Every general position drawing of G has at least k bends, hence $b + 4 \geq k$ and $b \geq k - 4$. $\square$

Clearly every edge in a general position drawing has at least two bends. The following lower bounds for general position drawings are based on the observation that an edge routed using an extremal port in a general position drawing has at least three bends. For 6-regular m -edge graphs all ports must be used, thus such a graph has at least $2m + 6$ bends in every general position drawing. Hence the graphs consisting of disjoint copies of K_7 provide the following lower bound.

Lemma 5. *There exists an infinite family of n -vertex m -edge simple graphs, each with at least $2m + 6n/7$ bends in every general position drawing.* $\square$

Note that for 6-regular graphs the above lower bound is within $m/21$ of the upper bound of $7m/3$ for the total number of bends in general position drawings established by the DIAGONAL LAYOUT AND MOVEMENT algorithm [12].

To establish our lower bound for general position drawings of 2-connected graphs consider the graph $C_r \langle K_7 \setminus e \rangle$ for $r \geq 2$, where the non-adjacent vertices of each $K_7 \setminus e$ are incident to the edges of C_r , as illustrated in Fig. 8(a).

Fig. 8. (a) 2-connected $C_r \langle K_7 \setminus e \rangle$, (b) 4-connected $(C_r \times C_3) \langle K_7 \setminus \{e_1, e_2\} \rangle$.

Lemma 6. *There is an infinite family of n -vertex m -edge simple 2-connected graphs, each with at least $2m + 4n/7$ bends in every general position drawing.*

Proof. Clearly $C_r \langle K_7 \setminus e \rangle$ is 2-connected. K_7 has at least $2|E(K_7)| + 6$ bends in every general position drawing. Thus by Lemma 4, a general position drawing of $K_7 \setminus e$ has at least $2|E(K_7)| + 6 - 4 = 2|E(K_7 \setminus e)| + 4$ bends. The edges of C_r each have at least two bends, thus $C_r \langle K_7 \setminus e \rangle$ has at least $2m + 4n/7$ bends in every general position drawing. $\square$

To establish our lower bound for general position drawings of 4-connected graphs consider the graph $(C_r \times C_3) \langle K_7 \setminus \{e_1, e_2\} \rangle$ for $r \geq 2$, where for each copy of $K_7 \setminus \{e_1, e_2\}$, e_1 and e_2 are edges of K_7 with no common end-vertices, and these end-vertices are incident to different edges in $C_r \times C_3$, as illustrated in Fig. 8(b).

The proof of the next result is very similar to that of Lemma 6.

Lemma 7. *There is an infinite family of n -vertex m -edge simple 4-connected graphs, each with at least $2m + 2n/7$ bends in every general position drawing.* $\square$

We now look at the ramifications of the above general position lower bounds for the 2-bends problem. Edges with at most two bends can be classified as 0-bend, 1-bend, 2-bend planar or 2-bend non-planar. As illustrated in Fig. 9, a given 2-bend drawing can be transformed into a general position drawing whose number of bends depends on the number of 0-bend and 2-bend planar edge routes in the 2-bend drawing. We omit the proof.

Fig. 9. Removing a plane containing many vertices.

Lemma 8. *If in a 2-bend drawing of an m -edge graph G the number of 0-bend edges is k_0 and the number of 2-bend planar edges is k'_2 , then there exists a general position 3-D orthogonal drawing of G with $2m + k_0 + k'_2$ bends.* $\square$

Corollary 1. *There exists an infinite family of 6-regular n -vertex graphs, such that in a 2-bend drawing of any of the graphs, $k_0 + k'_2 \geq 6n/7$.*

Proof. By Lemma 5, there exists an infinite family of graphs, each with at least $2m + 6n/7$ bends in any general position drawing. If there is a 2-bend drawing of such a graph, then by Lemma 8 there is a general position drawing with $2m + k_0 + k'_2$ bends. Hence $2m + k_0 + k'_2 \geq 2m + 6n/7$ and $k_0 + k'_2 \geq 6n/7$. $\square$

The following two results are obtained using the same argument used in the proof of Corollary 1 applied with Lemma 6 and Lemma 7, respectively.

Corollary 2. *There exists an infinite family of 6-regular 2-connected n -vertex graphs, such that in a 2-bend drawing of any of the graphs, $k_0 + k'_2 \geq 4n/7$.* $\square$

Corollary 3. *There exists an infinite family of 6-regular 4-connected n -vertex graphs, such that in a 2-bend drawing of any of the graphs, $k_0 + k'_2 \geq 2n/7$.* $\square$

A natural variation of the general position model allows at most two vertices in any one grid-plane with each vertex being coplanar with at most one other vertex. In this model there is at most $n/2$ pairs of coplanar vertices and hence at most $n/2$ planar edge routes. Since $n/2 \leq 4n/7$, it follows from Corollary 2 that there exists graphs which do not have 2-bend drawings in this model.

Theorem 9. *There exists an infinite family of 2-connected graphs each of which does not have a 2-bend drawing with at most two vertices in any one grid-plane and with each vertex being coplanar with at most one other vertex.* $\square$

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