Digital n -Pseudomanifold and n -Weakmanifold in a Binary $(n + 1)$ -Digital Image

Mohammed Khachan¹, Patrick Chenin², and Hafsa Deddi³

¹ University of Poitiers, Department of Computer Science (IRCOM-SIC).
Bd 3, Teleport 2, Bp 179, 86960 Futuroscope cedex, France

khachan@sic.sp2mi.univ-poitiers.fr

² University Joseph Fourier, LMC-IMAG,
51 Rue des Mathématiques, BP 53, 38420 Grenoble cedex 9, France
pchenin@imag.fr

³ University of Quebec at Montreal, LACIM,
C.P 8888, Succ. Centre-ville. Montreal (Quebec) H3C 3P8 Canada
deddi@math.uqam.ca

Abstract. We introduce the notion of digital n -pseudomanifold and digital n -weakmanifold in $(n+1)$ -digital image, in the context of $(2n, 3^n - 1)$ -adjacency, and prove the digital version of the Jordan-Brouwer separation theorem for those classes. To accomplish this objective, we construct a polyhedral representation of the n -digital image, based on cubical complex decomposition. This enables us to translate some results from polyhedral topology into the digital space. Our main result extends the class of “thin” objects that are defined locally and verifies the Jordan-Brouwer separation theorem.

Keywords: digital topology, combinatorial topology, discrete spaces, combinatorial manifolds

1 Introduction

A binary $(n + 1)$ -digital image is an ordered uplet $\mathcal{P} = (\mathbb{Z}^{n+1}, \mathcal{R}, H)$, where H is a finite subset of $\mathbb{Z}^{n+1}$ and $\mathcal{R}$ represents the adjacency relation in the whole lattice.

Morgenthaler and Rosenfeld in [16] introduced for a binary 3-digital image the concept of a **simple surface point**, where $\mathcal{R} \in \{(6, 26), (26, 6)\}$, to characterize a class of objects H that verifies the digital version of the Jordan-Brouwer separation theorem. They define a simple surface point axiomatically by three local conditions, these conditions reflect basic properties of triangulated simple and closed surface’s vertices in $\mathbb{R}^3$. They show that any finite and connected subset of $\mathbb{Z}^3$ consisting entirely of simple surfaces points verifies the digital version of the Jordan-Brouwer separation theorem. Thus, simple surface points criterion determines a class of thin object called **digital simple surface** of $\mathbb{Z}^3$.

Kong and Roscoe in [11] reveal precisely the geometric meaning of the surface simple points and extend its properties to other classes of $\mathcal{R}$ -adjacency, i.e.,

$\mathcal{R} \in \{(18, 6), (6, 18), (26, 18), (18, 26)\}$. They introduce the notion of continuous analog for transferring statements from continuous topology to digital space, and show that a connected finite subset H of $\mathbb{Z}^3$ is a digital simple surface if and only if the continuous analogs of H is a simple and closed polyhedral surface of $\mathbb{R}^3$. Note that [14] extends the concept of continuous analogs to the class of strongly normal 3D digital image.

In [6] Bertrand and Mourgouyres defined strong surfaces and in [7] Couprie and Bertrand introduced simplicity surfaces. Those classes of surfaces are more richer than Morgenthaler-Rosenfeld surfaces because they contain more local configurations and discrete analogs for these classes of surfaces can be built (see [5] for strong surfaces).

In [4] Ayala, Dominguez, Frances and Quintero prove that any digital n -manifold in $\mathbb{Z}^{n+1}$ verifies the Jordan-Brouwer separation theorem (the digital n -manifold is the generalization of the notion of digital simple surface for $n \geq 2$).

Mourgouyres in [15] investigated the reverse problem in $\mathbb{Z}^3$, in the context of the (26,6)-adjacency relation. He shows that there is not a local characterization of subset of $\mathbb{Z}^3$ which separates $\mathbb{Z}^3$ in exactly two 6-connected components.

It is not difficult to realize that there is still a large margin between the local characterization of digital simple surface and the global characterization represented by the digital version of the Jordan-Brouwer separation theorem. Intuitively, the problem is that there exists polyhedra with singularity which verify the Jordan-Brouwer separation theorem. In this context, the notion of n -pseudomanifold is very interesting, it is a formulation of weak singularities corresponding to degenerate n -manifold (see property 4 in remark 1).

This work is a first attempt to bring closer local and global properties that define the class of object verifying the digital version of the Jordan-Brouwer separation theorem. Our approach is based on combinatorial topology and uses topological properties in addition to local neighborhood structures. Definitions and statements of n -manifold and n -pseudomanifold are given in section 2. Section 3 is devoted to review basic of digital topology in $\mathbb{Z}^{n+1}$. In section 4, we give a simple method to construct the continuous analogs of binary $(n + 1)$ -digital image based on cubical complex decomposition. This enables us to translate results from polyhedral topology into digital topology. Section 5 is aimed to state our main results.

2 n -Manifold and n -Pseudomanifold

In this section we review some definitions of algebraic topology.

Definition 1. (see [1], pp. 42)

A triangulation of a space X is a pair (K, ϕ) , where K is a simplicial complex and $\phi : |K| \rightarrow X$ is a homeomorphism. The complex K is said to triangulate X . A polyhedron is any space which admits a triangulation.

Definition 2. (see [8], pp.20)

Let K be an homogeneous n -dimensional simplicial complex. A combinatorial n -

sphere is a polyhedron $|K|$ such that certain subdivision of K is simplicially homeomorphic to a subdivision of the boundary of the $(n+1)$ -simplex.

Let K be a simplicial complex and $e \in K$, the link of e in K , denoted by $Lk(e; K)$, is a subcomplex of K such that $\sigma \in Lk(e; K)$ if and only if

1. e is not a face of σ , and
2. $\exists \sigma' \in K$ such that
  - 2.1. σ is a face of σ' , and
  - 2.2. e is a face of σ' .

Definition 3. (see [8], pp. 26)

A connected polyhedron X is said to be an n -manifold without boundary if it admits a triangulation (K, ϕ) satisfying:

1. K is homogeneously n -dimensional.
2. for all $\sigma \in K$, $Lk(\sigma; K)$ is a combinatorial $(n - \dim(\sigma) - 1)$ -sphere.

K is called a combinatorial n -manifold without boundary.

Definition 4. (see [1], pp. 195)

A connected polyhedron X is said to be an n -pseudomanifold without boundary if it admits a triangulation (K, ϕ) such that:

1. K is homogeneously n -dimensional.
2. Every $(n-1)$ -simplex of K is a face of precisely two n -simplices of K .
3. If σ and σ' are two distinct n -simplices of K , then there exists a sequence $\sigma_1, \dots, \sigma_u$ of n -simplices in K such that $\sigma_1 = \sigma$, $\sigma_u = \sigma'$ and σ_i meets σ_{i+1} in an $(n-1)$ -dimensional face for $1 \leq i < u$.

K is called a combinatorial n -pseudomanifold without boundary.

Remark 1. (see [1])

1. An n -sphere, denoted by $\mathbb{S}^n$, is an n -manifold without boundary.
2. Every n -manifold without boundary is an n -pseudomanifold without boundary.
3. Let X be an n -pseudomanifold and let (L, ψ) be any triangulation of X , then L is a combinatorial n -pseudomanifold without boundary.
4. An octahedron is a 2-manifold without boundary. If we identify two opposite vertices, we obtain a 2-dimensional curved polyhedron, whose triangulations are examples of 2-pseudomanifold without boundary which are not closed 2-manifolds without boundary.

Theorem 1. (see [3], pp. 94)

Every n -pseudomanifold without boundary in $\mathbb{S}^{n+1}$ divides $\mathbb{S}^{n+1}$ into two domains¹ and is the common boundary of both domains.

This theorem obviously remains true if $\mathbb{R}^{n+1}$ is substituted for $\mathbb{S}^{n+1}$: it is sufficient to map $\mathbb{S}^{n+1}$ onto $\mathbb{R}^{n+1}$ by means of a stereographic projection with center of projection lying outside the given pseudomanifold, see [3] page 94.

Theorem 2. Every n -pseudomanifold without boundary in $\mathbb{R}^{n+1}$ divides $\mathbb{R}^{n+1}$ into two domains and is the common boundary of both domains.

¹ A domain of $\mathbb{S}^{n+1}$ is an open and connected subset of $\mathbb{S}^{n+1}$

3 General Definitions

In digital image, the study of neighborhood is a very important and significant concept. For a given lattice point, a neighborhood of a point is defined typically using a metric distance. Let $p, q \in \mathbb{Z}^{n+1}$ with the coordinates $(x_i(p))_{i=1}^{n+1}$ and $(x_i(q))_{i=1}^{n+1}$ respectively. We will consider two types of distance between elements in $\mathbb{Z}^{n+1}$:

$$d_1(p, q) = \sum_{i=1}^{n+1} |x_i(p) - x_i(q)| \text{ and } d_\infty(p, q) = \max_{i=1 \dots n+1} |x_i(p) - x_i(q)|.$$

Let $\beta \in \{1, \infty\}$; two points $p, q \in \mathbb{Z}^{n+1}$ are said to be d_β -adjacent if $d_\beta(p, q) = 1$; we denote by $\mathbb{V}_\beta(p)$ the set of all d_β -neighbors of p . Consequently we have:

- i) $\mathbb{V}_1(p) = \{q \in \mathbb{Z}^{n+1} / d_1(p, q) = 1\}$ and $\text{Card}(\mathbb{V}_1(p)) = 2(n+1)$.
- ii) $\mathbb{V}_\infty(p) = \{q \in \mathbb{Z}^{n+1} / d_\infty(p, q) = 1\}$ and $\text{Card}(\mathbb{V}_\infty(p)) = 3^{n+1} - 1$.

In the literature ([13], [14] and [17]) the d_1 -neighbor is referred as $(2(n+1))$ -neighbor and the d_∞ -neighbor as $(3^{n+1} - 1)$ -neighbor.

Let $T \subset \mathbb{Z}^{n+1}$, we denote by T^c the **complement** of T in $\mathbb{Z}^{n+1}$; $T^c = \mathbb{Z}^{n+1} - T$.

A binary $(n+1)$ -digital image is an ordered uplet $\mathcal{P} = (\mathbb{Z}^{n+1}, \mathcal{R}, H)$, where H is a finite subset of $\mathbb{Z}^{n+1}$ and $\mathcal{R}$ represents the adjacency relation in the whole lattice. In this paper the adjacency relation $\mathcal{R}$ will be taken as follows :

two elements of H are said to be $\mathcal{R}$ -adjacent **if** they are d_1 -adjacent, two elements of H^c are said to be $\mathcal{R}$ -adjacent **if** they are d_∞ -adjacent, and an element in H is $\mathcal{R}$ -adjacent to an element in H^c **if** they are d_∞ -adjacent.

Let $T \subset \mathbb{Z}^{n+1}$ and $p \in \mathbb{Z}^{n+1}$, p is said to be $\mathcal{R}$ -adjacent to T if p is $\mathcal{R}$ -adjacent to some point in T . T is said to be **strongly thin** if and only if any element of T is $\mathcal{R}$ -adjacent to all $\mathcal{R}$ -components of T^c . It is said to be **separating** if and only if T^c has exactly two $\mathcal{R}$ -components.

If $T \subset \mathbb{Z}^{n+1}$ and $p, q \in T$ then an $\mathcal{R}$ -**path** from p to q in T is a sequence of distincts points $(p_1, \dots, p_m)$ in T such that $p_1 = p$ and $p_m = q$ and p_i is $\mathcal{R}$ -adjacent to p_{i+1} , $1 \leq i < m$. T is $\mathcal{R}$ -**connected** if given any two elements p and q in T there is an $\mathcal{R}$ -path in T from p to q ; an $\mathcal{R}$ -**component** of T is a maximal $\mathcal{R}$ -connected subset of T .

4 Polyhedral Representation of a Binary $(n+1)$ -Digital Image

Let I_i denote the open unit interval $]r_i, r_{i+1}[$ or the single point r_i for some integer r_i . A k -**cube** e_k is defined as $e_k = \prod_{i=1}^{n+1} I_i$ where k of the I_i 's are intervals and $n+1-k$ are single points. Thus e_k is an open k -cube embedded in $\mathbb{R}^n$. The closure of e_k , will be denoted by $\overline{e_k}$. Note that $\overline{e_k} = \prod_{i=1}^{n+1} \overline{I_i}$. We denote by $\text{Som}(e_k)$ the set of all vertices of $\overline{e_k}$. Obviously we have $\text{Som}(e_k) \subset \mathbb{Z}^{n+1}$. The

subscript of a k -cube will be omitted when irrelevant to the argument at hand. A face of a k -cube e is a k' -cube e' such that $Som(e') \subseteq Som(e)$. We will write $e' \preceq e$.

A **cubical complex** K , is a finite collection of k -cubes in some $\mathbb{R}^{n+1}$, $0 \leq k \leq n+1$, such that:

- ★ if $e \in K$, then all faces of e belong to K , and
- ★ if $e, e' \in K$, then $e \cap e' = \emptyset$.

Now, we will construct the continuous analog of a binary $(n+1)$ -digital image $\mathcal{P} = (\mathbb{Z}^{n+1}, \mathcal{R}, H)$. Intuitively, this construction consists of 'filling in the gaps' between black points of $\mathcal{P}$, and must be consistent with the $\mathcal{R}$ -adjacency relation of $\mathcal{P}$. Precisely, let $\mathbb{C}(H)$ be a collection of k -cubes in $\mathbb{R}^{n+1}$, $0 \leq k \leq n+1$, defined as follows:

$$\mathbb{C}(H) = \{e : k\text{-cube}, 0 \leq k \leq n+1 / Som(e) \subset H\}$$

$\mathbb{C}(H)$ is a cubical complex, the underlying polyhedron $|\mathbb{C}(H)|$ will be called the **polyhedral representation** of $\mathcal{P}$.

For each $x \in \mathbb{R}^{n+1}$, $e(x)$ will denote the k -cube, $0 \leq k \leq n+1$, that contains x . From the construction of $|\mathbb{C}(H)|$, we can deduce some natural properties:

Remark 2. Let $\mathcal{P} = (\mathbb{Z}^{n+1}, \mathcal{R}, H)$. We have :

1. if $x \in |\mathbb{C}(H)|$, then $e(x) \in \mathbb{C}(H)$.
2. if $x \in \mathbb{R}^{n+1} - |\mathbb{C}(H)|$, then at least one vertex of $e(x)$ belongs to H^c .
3. each component of $|\mathbb{C}(H)|$ or of $\mathbb{R}^{n+1} - |\mathbb{C}(H)|$ meets $\mathbb{Z}^{n+1}$.

The following theorem expresses the fundamental properties that permit us to relate digital topology to Euclidean space topology.

Theorem 3. Let $\mathcal{P} = (\mathbb{Z}^{n+1}, \mathcal{R}, H)$.

1. $|\mathbb{C}(H)| \cap \mathbb{Z}^{n+1} = H$ and $(\mathbb{R}^{n+1} - |\mathbb{C}(H)|) \cap \mathbb{Z}^{n+1} = H^c$.
2. Two points in H are in the same $\mathcal{R}$ -component of H iff they are in the same component of $|\mathbb{C}(H)|$.
3. Two points in H^c are in the same $\mathcal{R}$ -component of H^c iff they are in the same component of $\mathbb{R}^{n+1} - |\mathbb{C}(H)|$.

The boundary of a component A of $|\mathbb{C}(H)|$ meets the boundary of a component B of $\mathbb{R}^{n+1} - |\mathbb{C}(H)|$ if and only if there is a point in $A \cap \mathbb{Z}^{n+1}$ that is $\mathcal{R}$ -adjacent to a point in $B \cap \mathbb{Z}^{n+1}$.

The proof of the above theorem and other properties related to the concept of continuous analogs in a binary $(n+1)$ -digital image are given in [10].

5 Digital n -Weakmanifold and Digital n -Pseudomanifold

We denote by K a cubical complex such that $|K|$ is a connected polyhedron.

Definition 5. *n*-weakmanifold

K will be called a cubical *n*-weakmanifold without boundary if and only if

1. *K* is homogeneously *n*-dimensional.
  2. for each vertex $p \in K$, $Lk(p; K)$ is a combinatorial $(n - 1)$ -sphere.
- The underlying polyhedron $|K|$ is called an *n*-weakmanifold without boundary.

The definition of the *n*-weakmanifold is a weak formulation of *n*-manifold given in definition 3, it uses uniquely the property of link in vertices of *K*. For $1 \leq n \leq 2$ those two notions are equivalent.

Definition 6. *n*-pseudomanifold

K will be called a cubical *n*-pseudomanifold without boundary if and only if

1. *K* is homogeneously *n*-dimensional.
2. Every $(n - 1)$ -cube of *K* is a face of exactly two *n*-cubes of *K*.
3. If *e* and *e'* are two distinct *n*-cubes of *K*, then there exists a sequence $e_1, \dots, e_u$ of *n*-cubes in *K* such that $e_1 = e$, $e_u = e'$ and e_i meets e_{i+1} in an $(n - 1)$ -cube for $1 \leq i < u$.

The underlying polyhedron $|K|$ is called an *n*-pseudomanifold without boundary.

Remark 3. It is not difficult to see that

1. If *K* is a cubical *n*-pseudomanifold without boundary then the barycentric subdivision of *K* is a combinatorial *n*-pseudomanifold without boundary.
2. Let *p* be a vertex of *K*. If $Lk(p; K)$ is a combinatorial $(n - 1)$ -sphere, then $Lk(p; K)$ is a cubical $(n - 1)$ -pseudomanifold without boundary.

Let σ_1 be a *k*-cube of *K* and *p* be a vertex of *K* such that $p \notin \text{Som}(\sigma_1)$. We denote by $p.\sigma_1$ the $(k + 1)$ -cube, if it exists, such that $(\{p\} \cup \text{Som}(\sigma_1)) \subset \text{Som}(p.\sigma_1)$.

Let σ and σ_1 be respectively a $(k + 1)$ -cube and a *k*-cube of *K* such that $\sigma_1 \prec \sigma$. If $p \in \text{Som}(\sigma) - \text{Som}(\sigma_1)$ then $p.\sigma_1 = \sigma$.

Proposition 1. Any cubical *n*-weakmanifold without boundary *K* is a cubical *n*-pseudomanifold without boundary.

Proof. We have to prove properties 2 and 3 in the definition 6.

★ Let σ be an $(n - 1)$ -cube of *K*, we will show that σ is a face of exactly two *n*-cubes of *K*.

Let $p \in \text{Som}(\sigma)$, and σ' an $(n - 2)$ -cube of $Lk(p, K)$ such that $\sigma' \prec \sigma$. It is easy to see that $p.\sigma' = \sigma$. $Lk(p, K)$ is a combinatorial $(n - 1)$ -sphere, by using part 2 of remark 2, we deduce that σ' is a face of exactly two $(n - 1)$ -cubes (σ_1, σ_2) of $Lk(p, K)$. This implies that $p.\sigma' = \sigma$ is a face of exactly two *n*-cubes ($p.\sigma_1, p.\sigma_2$) of *K*, otherwise σ would be a face for more than two $(n - 1)$ -cubes of $Lk(p, K)$.

★ Let σ and σ' be two *n*-cubes of *K*, we will show that there exists a sequence $e_1, \dots, e_u$ of *n*-cubes in *K* such that $e_1 = \sigma$, $e_u = \sigma'$ and e_i meets e_{i+1} in an $(n - 1)$ -cube for $1 \leq i < u$.

Let $p \in \text{Som}(\sigma)$ and $p' \in \text{Som}(\sigma')$. Since $|K|$ is a connected polyhedron, there

is a sequence $(p_0, \dots, p_k)$ of vertices of K that joins $p = p_0$ to $p' = p_k$, i.e., $\bigcup_{i=0}^{k-1} [p_i, p_{i+1}] \subset |K|$. For i , $0 < i < k$, p_{i-1} and p_{i+1} belong to $\text{Lk}(p_i, K)$.

Moreover, any two $(n-1)$ -cubes e_i and e'_i of $\text{Lk}(p_i, K)$ can be joined by a sequence of $(n-1)$ -cubes $e_i^0, \dots, e_i^{k_i}$ in $\text{Lk}(p_i, K)$ such that consecutive $(n-1)$ -cubes of this sequence share a common $(n-2)$ -cube. So, any two n -cubes of K having p_i as a vertex can be joined by a sequence of n -cubes in K such that consecutive n -cubes of this sequence share a common $(n-1)$ -cube.

Let σ_{n,p_i} denotes an n -cube of K such that p_i and p_{i+1} belong to $\text{Som}(\sigma_{n,p_i})$. Thus, σ_{n,p_i} and $\sigma_{n,p_{i+1}}$ can be joined by a sequence of n -cubes in K such that consecutive n -cubes of this sequence share a common $(n-1)$ -cube, for $0 \leq i < k$. It is the same thing for the pair (σ, σ_{n,p_0}) and $(\sigma_{n,p_k}, \sigma')$. So there exists a sequence of n -cubes in K that joins σ to σ' such that any consecutive n -cubes of this sequence share a common $(n-1)$ -cube.

This completes the proof. $\square$

Definition 7. Let $\mathcal{P} = (\mathbb{Z}^{n+1}, \mathcal{R}, H)$, and H be $\mathcal{R}$ -connected.

1. H will be called **digital n -weakmanifold** if $\mathbb{C}(H)$ is a cubical n -weakmanifold without boundary.
2. H will be called **digital n -pseudomanifold** if $\mathbb{C}(H)$ is a cubical n -pseudomanifold without boundary.

Proposition 2. Let $\mathcal{P} = (\mathbb{Z}^{n+1}, \mathcal{R}, H)$.

1. If H is a digital n -weakmanifold, then H is a digital n -pseudomanifold.
2. If H is a digital n -pseudomanifold, then H is a separating set (see section 3 for the definition of separating set).
3. If H is a digital n -pseudomanifold, then H is strongly thin.

Proof.

1. Let H be a digital n -weakmanifold.

$\mathbb{C}(H)$ is a cubical n -weakmanifold without boundary. By using proposition 1, we deduce that $\mathbb{C}(H)$ is a cubical n -pseudomanifold without boundary. This implies that H is a digital n -pseudomanifold.

2. Let H be a digital n -pseudomanifold.

$\mathbb{C}(H)$ is a cubical n -pseudomanifold without boundary. By using the part 1 of the remark 3 and part 3 of the remark 1, we deduce that $|\mathbb{C}(H)|$ is an n -pseudomanifold without boundary. So, the theorem 2 enables us to assert that $|\mathbb{C}(H)|$ divides $\mathbb{R}^{n+1}$ in two domains (Int , Ext) and is the common boundary of both domains. By using Theorem 3, we deduce that H^c has exactly two $\mathcal{R}$ -components ($\text{Int} \cap \mathbb{Z}^{n+1}$ denoted by $\text{Int}(H)$, and $\text{Ext} \cap \mathbb{Z}^{n+1}$ denoted by $\text{Ext}(H)$). Furthermore $\partial \text{Int} = |\mathbb{C}(H)| = \partial \text{Ext}$.

3. Let $p \in H$. Since $\partial \text{Int} = |\mathbb{C}(H)|$ and $p \in |\mathbb{C}(H)|$, we can assert that :

$$\forall \epsilon > 0, \mathbb{B}(p, \epsilon) \neq \emptyset, \text{ with } \mathbb{B}(p, \epsilon) = \{x \in \mathbb{R}^{n+1} / d_\infty(p, x) \leq \epsilon\}.$$

Let $\epsilon = \frac{1}{3}$. $\exists x \in \text{Int} / d_\infty(p, x) \leq \frac{1}{3}$.

Let e be an $(n+1)$ -cube such that $p \in \text{Som}(e)$ and $x \in \bar{e}$.

$e(x) \notin \mathbb{C}(H)$, otherwise $x \in |\mathbb{C}(H)|$ (for the definition of $e(x)$ see section 4). So, $\exists p_1 \in \text{Som}(e(x)) / p_1 \notin H$. Note that $e(x) \preceq e$.

$[x, p_1] \subset e(x)$. Since $p_1 \in H^c$ and $x \in \text{Int}$, then $p_1 \in \text{Int}$ (Int is a connected component of $\mathbb{R}^{n+1} - |\mathbb{C}(H)|$). $\text{Int} \cap \mathbb{Z}^{n+1} = \text{Int}(H)$, so $p_1 \in \text{Int}(H)$.

Since $e(x) \preceq e$, then p and p' are vertices of e . This implies that p is d_∞ -adjacent to p' , i.e, p is $\mathcal{R}$ -adjacent to $\text{Int}(H)$.

In the same way, we prove that p is $\mathcal{R}$ -adjacent to $\text{Ext}(H)$.

This completes the proof. $\square$

Theorem 4. Let $\mathcal{P} = (\mathbb{Z}^{n+1}, \mathcal{R}, H)$.

1. If H is a digital n -pseudomanifold, then H verifies the digital version of the Jordan-Brouwer separation theorem.
2. If H is a digital n -weakmanifold, then H verifies the digital version of the Jordan-Brouwer separation theorem.

Proof. It is easy to deduce this theorem from proposition 2.

6 Conclusion

We have defined in a binary $(n + 1)$ -digital image $\mathcal{P} = (\mathbb{Z}^{n+1}, \mathcal{R}, H)$, where $\mathcal{R} = (2(n + 1), 3^{n+1} - 1)$, a new class of objects H (digital n -weakmanifold and digital n -pseudomanifold) that verifies the digital version of the Jordan-Brouwer separation theorem, for $n \geq 2$.

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