Discrete-Log With Compressible Exponents

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Abstract

Many key distribution systems are based on the assumption that the Discrete-Log (DL) problem is hard. The implementations could be more efficient if a significantly smaller exponent could be used, without lowering the complexity of the DL problem. When the exponent is known to reside in interval of size w , the DL problem can be computed in time $O(\sqrt{w})$, using Pollard's "Lambda method for catching Kangaroos".

Suppose we want a level of security of 300 years on a 1 MIP machine, with 1K bit operations per instruction. Then $w = 2^{127}$ currently seems sufficient (with 512 bit modulus). It is not clear, however, whether methods other than "Kangaroo" exist, with lower complexity.

Let s and m denote the number of squarings and multiplications, respectively, required to exponentiate. It is well known that s roughly equals the size in bits of the exponent (L), and m is roughly $1.5 \cdot L / \lg_2(L)$, for the most efficient methods, in the practical range.

We show that by using an exponent which is known to be compressible by a factor η , using the Ziv-Lempel method, we reduce m exponentially in η , on the average (integer multiplications may be more than twice as expensive as squarings, hence this is not negligible).

This can be used to speed up cryptographic key distribution systems of the Diffie-Hellman family. However, it is not clear how safe compressible exponents are.

1 Introduction

Many key distribution systems are based on the assumption that the Discrete-Log (DL) problem is hard. The implementations could be more efficient if a significantly smaller exponent could be used, without lowering the complexity of the DL problem. When the exponent is known to reside in interval of size w , the DL problem can be computed in time $O(\sqrt{w})$, using Pollard's "Lambda method for catching Kangaroos" [P]. This is a randomized algorithm, with controllable error probability $\epsilon > 0$, which

tree, where the path e_i from the root to node i is a segment of the exponent, and node i contains the partial result x^{e_i} . The structure of the tree makes it easy to add a new leaf, given its parent. We describe first a subroutine that creates the binary tree, and then the main program, which uses the tree to compute x^n .

The exponent is called here *seq*, and is treated sometimes as an integer, and sometimes as a sequence of bits. Each segment e_i of the exponent is an odd number of length $L_i \geq 1$, and is preceded (to the right) by $Z_i \geq 0$ zeroes.

subroutine **build-tree**
begin

- **Init:** Store $e_0 = 1$; $Z_0 \leftarrow L_{-1} \leftarrow 0$ in node 0, and set $L_0 \leftarrow 0$. Put parse 0 right of *seq*; $i \leftarrow 1$;
- **While** there are more symbols in *seq* **do**
- **begin**
 - $L_i \leftarrow 0$; $Z_i \leftarrow 0$.
 - Scan *seq* from the $i - 1$ th parse to the left.
 - While the new symbol (*ns*) = 0 $Z_i \leftarrow Z_i + 1$.
 - (*ns*=1) $L_i \leftarrow L_i + 1$; start following the path of the tree defined by *seq*, incrementing L_i , until a leaf is reached.
 - Scan one more symbol (*ns*), add a new arc from the last visited leaf, label it *ns*, add a new leaf and a new parse, and label them i . e_i is the integer represented by the segment which starts at parse $i - 1$ and ends at parse i , not including leading zeroes to the right.
 - Compute x^{e_i} , and store it together with $L_{i-1} = |e_{i-1}|$, e_{i-1} , and Z_i in leaf i .
 - $i \leftarrow i + 1$;
- **end;**
- end.**

Let $k = \max(i)$. The exponent looks as follows. $n = e_k 0^{z_k} e_{k-1} 0^{z_{k-1}} \dots e_1 0^{z_1}$

The following theorem and its proof appear in [Y]. As before, let $L = \log(n)$ the length of the exponent.

Theorem: The average time complexity of Algorithm 1 is
 $l(n) = L - (\log(L) - \log\log(L))/2 + 1.5 \cdot (L/\log(L) + o(L/\log(L)))$.

3 Compressible exponents

Roughly speaking, if a sequence is ZL-compressible then its compression tree is not balanced, and on the average, node-names are shorter than the corresponding exponent segments, i.e. the tree has relatively few nodes. Let k be the number of nodes of a noncompressible sequence, and k' the number of nodes of another sequence of the same length, which is compressible by a factor $\eta < 1$. Let N denote the expected length of a node-name (similarly N' for the compressible case). Then roughly $k = 2^N$, and since $N' = \eta \cdot N$, we get $k' = 2^{N'} = k^\eta$. The number of multiplications in the compressible case is $m' = c \cdot k'$, $1 < c < 2$ compared with $m = 1.5 \cdot k$, i.e. the saving is asymptotically exponential in η .

If the tree is skewed to the right (fewer ones), then $c < 1.5$, while if the tree is skewed to the left, $c > 1.5$.

Some implementors use exponents of low Hamming weight; thus gaining in the linear factor ($c < 1.5$), this may be combined with the new algorithm to gain twice: use low weight exponents ($c < 1.5$), which are also ZL-compressible ($\eta < 1$).

4 References

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