Orthogonal 3D Shapes of Theta Graphs[*]

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Abstract. The recent interest in three dimensional graph drawing has been motivating studies on how to extend two dimensional techniques to higher dimensions. A common approach for computing a 2D orthogonal drawing of a graph separates the task of defining the shape of the drawing from the task of computing its coordinates. First results towards finding a three-dimensional counterpart of this approach are presented in [8,9], where characterizations of orthogonal representations of paths and cycles are studied. In this note we show that the known characterization for cycles does not immediately extend to even seemingly simple graphs such as theta graphs. A sufficient condition for recognizing three-dimensional orthogonal representations of theta graphs is also presented.

1 Introduction

The recent interest in three-dimensional graph drawing has been motivating studies on how to extend two dimensional techniques to 3D space. Work in this direction includes extensions of simulated annealing techniques, spring embedder techniques, and incremental techniques (see e.g., [3,5,11,15]). However, while a rich body of literature is devoted to three-dimensional orthogonal drawings, little is known on the challenging task of extending to 3D the well-known topology-shape-metrics approach [18].

In order to reach such an ambitious research target, a crucial step is to find a three-dimensional counterpart of the well-known two-dimensional characterization of orthogonal representations (see, e.g. [18,19]). More precisely, a solution to the following problem has to be found.

Let G be a graph such that each edge is directed and labeled with a label in the set $\{Up, Down, East, West, North, South\}$. Does a three-dimensional orthogonal drawing of G exist such that no two edges intersect and each edge has a direction consistent with its associated label?

This question has been addressed by Di Battista et al. [7,8,9] for simple classes of graphs, namely paths and cycles. The goal of this note is to shed some more light on the above basic question. Namely, we show that the known

characterization for cycles does not immediately extend to even seemingly simple graphs such as theta graphs, that is, graphs composed by two nodes of degree three and three disjoint paths, of length at least two, joining them. Also, we give a sufficient condition for recognizing three-dimensional orthogonal representations of theta graphs.

We remark that theta graphs have been studied extensively in the literature. For example, they arise in problems concerning graph planarity (see, e.g., [1,4,17]), graph bandwidth (see, e.g., [12,14,16]), and chromatic polynomials (see, e.g., [2]).

2   Preliminaries

We assume familiarity with basic graph drawing terminology (see, e.g. [6]). A *direction label* is a label in the set $\{U, D, E, W, N, S\}$ specifying the directions *Up*, *Down*, *East*, *West*, *North*, *South*, respectively. Three distinct labels are also used to identify an octant of the reference system, which is assumed to be open.

Let e be an undirected edge of a graph whose end-vertices are u and v. We use the term *darts* for the two possible orientations (u, v) and (v, u) of edge e. A *3D shape graph* γ is an undirected labeled graph such that (i) each dart is associated with one direction label; (ii) the two darts of the same edge have opposite labels; and (iii) each label in the set $\{U, D, E, W, N, S\}$ appears at least once in γ.

A *three dimensional orthogonal drawing* of a graph is such that vertices are mapped to grid points of an integer three dimensional grid and edges are segments along the integer grid lines connecting the end points. An *intersection* in a three dimensional orthogonal drawing is a pair of edges that overlap in at least a point that does not correspond to a common end-vertex.

A 3D shape graph γ is *simple* if there exists an orthogonal drawing Γ of γ such that no two edges of Γ intersect and each oriented edge satisfies the direction constraint defined by the direction labels associated with its darts. The *simplicity testing* problem for a 3D shape graph γ is to decide whether γ is simple.

Obviously, if γ is simple, no vertex of γ has two entering (or leaving) darts with the same label. For example, this implies that no vertex of γ can have degree greater than six. Also, we restrict our attention to the class of shape graphs that do not have vertices of degree two whose entering darts are given opposite direction labels, since it can be shown (we omit the details for the sake of brevity) that each instance for which this property does not hold has an equivalent instance in this class.

Let γ be a 3D shape graph. A *flat* $F \subset \gamma$ is a connected subgraph of γ that is maximal with respect to the property that in every orthogonal drawing with the shape of γ all edges in F are co-planar. Observe that the labels of γ come from the union of two oppositely directed pairs of directions. Also, the definition above extends the analogous definitions given in [7,8,9].

Let π be a 3D shape path with vertices $p_1, p_2, \ldots, p_k$ and suppose to walk along π from p_1 to p_k. The *label path* of π, denoted by σ_π, is the sequence of labels associated with darts $(p_1, p_2), \ldots, (p_i, p_{i+1}), \ldots, (p_{k-1}, p_k)$. Similarly, let C be a 3D shape cycle with vertices $p_1, p_2, \ldots, p_k$ and suppose to walk along C from p_1 to p_k. The *label cycle* of C, denoted by σ_C, is the sequence of label associated with darts $(p_1, p_2), \ldots, (p_i, p_{i+1}), \ldots, (p_{k-1}, p_k), (p_k, p_1)$.

Given a label path or cycle σ, a not necessarily consecutive subsequence $\tau \subseteq \sigma$, where τ consists of k elements, is a *canonical sequence* [9,8] provided that: (i) $1 \leq k \leq 6$; (ii) the labels of τ are distinct; (iii) no flat of σ contains more than three labels of τ; and (iv) if a flat F of σ contains one or more labels of τ, then $\tau \cap F$ form a consecutive subsequence of σ.

The following basic result for paths is given in [9]:

Theorem 1. [9] *Let π be a 3D shape path with vertices $p_1, p_2, \ldots, p_k$ and let σ_π be the label path associated with the sequence of darts $(p_1, p_2), \ldots, (p_i, p_{i+1})$, $\ldots, (p_{k-1}, p_k)$. Shape path π admits an intersection free orthogonal drawing with p_1 at the origin and p_k in the UNE octant if and only if σ_π contains a canonical sequence of three labels in the set $\{U, N, E\}$.*

A characterization of simple shape cycles is given in [8]:

Theorem 2. [8] *A 3D shape cycle C described by a sequence of direction labels σ_C is simple if and only if it contains a canonical sequence of length six.*

A *theta graph* is a graph with two non adjacent nodes of degree three and all other nodes of degree two [13]. Thus, a theta graph consists of two nodes of degree three and three disjoint paths, of length at least two, joining them. We will call *3D theta shape* a 3D shape for a theta graph.

In the following we will denote by p and q the two degree-three nodes of a theta shape. Also, given a shape path π_x from p to q, and a shape path π_y from q to p, we denote $C_{x,y}$ the shape cycle obtained by joining π_x and π_y. Observe that given a theta shape, three paths π_x, π_y, and π_z, and three cycles $C_{x,y}$, $C_{x,z}$, and $C_{y,z}$, are defined.

3   A Forbidden Theta Shape

In this section we show that the simplicity of the cycles composing a theta shape does not imply the simplicity of the theta shape itself. We use the following notation: given a shape path π (say $DWUNE$) we will denote by $\overline{\pi}$ ($\overline{DWUNE}$ in the example) the shape path obtained by orienting each edge in the opposite direction and changing its label with the opposite one ($\overline{DWUNE} = WSDEU$). Also, we will use a dot to denote a series composition of paths (for example $\pi_1 = \pi_2 \cdot \overline{\pi}_3$). Given two distinct vertices v and w of a shape path (shape cycle, theta shape, respectively) γ, we say that v *is Y with respect to* w, where $Y \in \{U, D, E, W, N, S\}$, if in any drawing of γ, denoted Π_v and Π_w the two planes orthogonal to Y containing v and w respectively, the two planes may be joined by a segment oriented from Π_w to Π_v which has direction Y. Observe that, if v is Y with respect to w, w is $\overline{Y}$ with respect to v.
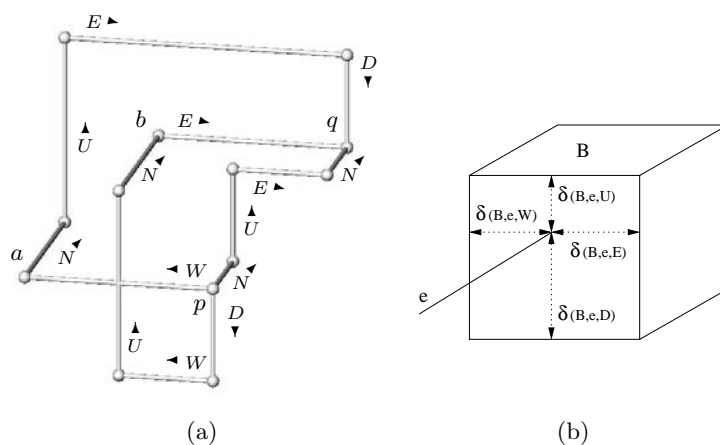
Fig. 1. (a) A not simple theta shape and (b) the definition of $\delta(B, e, X)$.

Theorem 3. *There exists a 3D shape graph that is not simple even if all its induced cycles are simple.*

Proof. Let ϑ be the theta shape composed by the three paths: $\pi_1 = WNUED$, $\pi_2 = DWUNE$, and $\pi_3 = NUEN$ (see Fig. 1.a). The three cycles $C_{1,2}$, $C_{1,3}$, and $C_{2,3}$ defined by ϑ are simple. Namely, we have:

$$C_{1,2} = WNUED \cdot \overline{DWUNE} = WNUEDWSDEU$$
$$C_{1,3} = WNUED \cdot \overline{NUEN} = WNUEDSWDS$$
$$C_{2,3} = DWUNE \cdot \overline{NUEN} = DWUNESWDS$$

and each of them satisfies the hypotheses of Theorem 2. In fact, $C_{1,2}$ contains, for example, the canonical sequence identified by the checked labels in the sequence $W\check{N}UE\check{D}\check{W}\check{S}D\check{E}\check{U}$; $C_{1,3}$ contains, for example, the canonical sequence identified by the checked labels in the sequence $W\check{N}\check{U}\check{E}D\check{S}\check{W}\check{D}S$; and $C_{2,3}$ contains, for example, the canonical sequence identified by the checked labels in the sequence $D\check{W}\check{U}\check{N}\check{E}SW\check{D}\check{S}$.

Suppose for a contradiction that ϑ is simple. Consider a non intersecting orthogonal drawing Γ of ϑ. Let σ be the label path $ENUENW$ from a to b. By Theorem 1, since the only canonical sequences of length three contained in σ are of type NUW or NUE, then b is NUW or NUE with respect to a in each non intersecting drawing of σ. Let σ' be the label path $NUEDW$ from a to b. By Theorem 1, since the only canonical sequences of length three contained in σ' are of type NUE, NDE, or NDW, then b is NUE, NDE, or NDW with respect to a in each non intersecting drawing of σ'. Let σ'' be the label path $EDWUN$ from a to b. By Theorem 1, since the only canonical sequences of length three contained in σ'' are of type NUW, NDE, or NDW, then b is NUW, NDE, or NDW with respect to a in each non intersecting drawing of σ''. We have a contradiction, the three conditions above can not be simultaneously verified and thus one intersection necessarily occurs.

4   Triply Expanding Drawings

Let $\pi_1, \pi_2, \ldots, \pi_n$ be n shape paths starting from a common point p. Denote by m_i the number of edges of π_i, and by $e_{i,h}$, with $h = 1, \ldots, m_i$, the h-th edge of π_i starting from p. An *expanding drawing* of $\pi_1, \pi_2, \ldots, \pi_n$ is a simple 3D orthogonal drawing for which edges $e_{1,m_1}, e_{2,m_2}, \ldots, e_{n,m_n}$ can be replaced by arbitrarily long segments without creating any intersection with the drawing. The *bounding box* of an expanding drawing is the bounding box of the drawing when edges $e_{1,m_1}, e_{2,m_2}, \ldots, e_{n,m_n}$ are removed. In [8] Di Battista et al. showed the following sufficient condition for the existence of an expanding drawing of two paths (*doubly expanding drawing*).

Theorem 4. [8] *A shape path π with n edges admits a doubly expanding drawing if it either consists of exactly two edges or contains at least two flats.*

A *triply expanding drawing* is an expanding drawing of three paths. The next theorem extends Theorem 4 to triply expanding drawings.

Theorem 5. *Let π_x, π_y, and π_z be three shape paths starting from a common point p. If the paths $\overline{\pi_x} \cdot \pi_y$, $\overline{\pi_x} \cdot \pi_z$, and $\overline{\pi_y} \cdot \pi_z$ consist of exactly two edges or contain at least two flats, then π_x, π_y, and π_z admit a triply expanding drawing.*

Due to space limitations we omit the proof of Theorem 5 and refer to [10] the interested reader. Roughly, the theorem can be proved by construction, after a suitable case analysis, and by using the following lemma.

Lemma 1. *A shape graph γ that admits a three dimensional orthogonal drawing such that no intersection occurs between two edges of the same flat is simple.*

Proof. The statement can be proved by construction. Namely, given a three dimensional orthogonal drawing Γ such that no intersection occurs between two edges of the same flat, an iterative process can be used to produce an orthogonal drawing Γ' without intersections. Consider an intersection between two edges e_1 and e_2 that do not belong to the same flat. The following technique eliminates the intersection, without introducing a new one. Consider a plane p common to e_1 and e_2 and a direction d orthogonal to p. Move one unit in the d direction all the nodes in the open half-space determined by p and d, the end-points of e_1, and the end-points of all the edges of the flat F of e_1 perpendicular to d, if any. It can be shown that the obtained drawing is a 3D orthogonal drawing of γ, that the intersection between e_1 and e_2 has been removed, and that no other intersection has been introduced (see [10] for details).

By virtue of the preceding lemma, intersections occurring between edges (and nodes) that do not share a flat can be neglected, assuming that they could be easily eliminated in a post processing step.

5   A Sufficient Condition for Theta Shapes Simplicity

Let Θ be a theta shape composed by three shape paths π_1, π_2, and π_3 from point p to point q. Let e_{ij}, with $j = 1, \ldots, m_i$, the j-th edge of π_i. In the following we denote by l_{ij} the label associated with e_{ij} when directed according to π_i. Recall that when π_i is reversed, as in $C_{k,i} = \pi_k \cdot \overline{\pi_i}$, the label associated with e_{ij} is the opposite of l_{ij}, that is $\overline{l_{ij}}$.

Theorem 6. *Let Θ be a theta shape composed by three shape paths π_1, π_2, and π_3 from point p to point q. If for each π_i there exist three edges e_{i1}, e_{i2}, and e_{i3}, such that for each pair of paths π_i and π_j, $i, j = 1, 2, 3$, $i \neq j$, the six labels l_{i1}, l_{i2}, l_{i3}, $\overline{l_{j1}}$, $\overline{l_{j2}}$, and $\overline{l_{j3}}$ form a canonical sequence $\tau_{i,j}$ for the shape cycle $C_{i,j} = \pi_i \cdot \overline{\pi_j}$, then Θ is simple .*

Proof. By using the canonicity of $\tau_{1,2}$, $\tau_{1,3}$, and $\tau_{2,3}$, the following properties for labels l_{ij} can be verified:

1. The labels l_{ij} of the same path are different, i.e., $l_{ij} \neq l_{ik}$, $i, j, k = 1, 2, 3$, $j \neq k$.
2. No two labels l_{ij} of the same path are opposite, i.e., $l_{ij} \neq \overline{l_{ik}}$, $i, j, k = 1, 2, 3$, $j \neq k$.
3. No two labels l_{ij} of different paths are opposite, i.e., $l_{ij} \neq \overline{l_{hk}}$, $i, j, h, k = 1, 2, 3$, $i \neq h$.

It follows that, for each $i = 1, 2, 3$, labels $l_{i1}\ l_{i2}\ l_{i3}$ are a permutation of the same three labels, one for each oppositely directed pair. We assume, without loss of generality, that $l_{ij} \in \{U, N, E\}$ with $i, j = 1, 2, 3$.

The nine edges e_{ij} $(i, j = 1, 2, 3)$ bound eight connected subgraphs of the theta graph. We call $G_{i,j}$ the subpath of π_i from e_{ij} to $e_{i(j+1)}$, $(i = 1, 2, 3, j = 1, 2)$. Further, we call G_p the subgraph composed by the three paths from p to e_{i1} with $i = 1, 2, 3$, and G_q the subgraph composed by the three paths from e_{i3} with $i = 1, 2, 3$ to q.

Observe that $G_{i,j}$, with $i = 1, 2, 3$, and $j = 1, 2$ admits a doubly expanding drawing. In fact, since the labels associated with e_{ij} and $e_{i(j+1)}$ are canonical, it follows that, if they are not consecutive, they do not share a flat, and Theorem 4 applies. Also, observe that G_p admits a triply expanding drawing. In fact, since, for each pair i, j, e_{i1} and e_{j1} are canonical in $C_{i,j}$, it follows that, if they are not consecutive, they do not share a flat, and Theorem 5 applies. Analogously, G_q admits a triply expanding drawing.

We draw a doubly expanding drawing $\Gamma_{i,j}$ for each $G_{i,j}$, with $i = 1, 2, 3$, and $j = 1, 2$.Theorem 4. Let L be the maximum length of a side of a bounding box of the drawings obtained above. We draw a triply expanding drawing Γ_p of G_p and a triply expanding drawing Γ_q of G_q.

Now we show how to set the lengths λ_{ij}, $i, j = 1, 2, 3$, of the part of each edge e_{ij} that is not contained in the bounding box of any expanding drawing, since the actual lengths of the edges can be easily computed from it.

Denote by B_p, B_q, and $B_{i,j}$, with $i = 1, 2, 3$ and $j = 1, 2$, the bounding boxes of Γ_p, Γ_q, and $\Gamma_{i,j}$, respectively. Since edges e_{i1}, with $i = 1, 2, 3$, have labels in

$\{U, N, E\}$, there are three pairwise orthogonal sides of B_p from which edges e_{i1} may come out. Denote by v_{B_p} the only point common to these sides. Denote by v_{B_q} the point common to the analogous sides of B_q. Place the drawings Γ_p and Γ_q in such a way that v_{B_p} has coordinates $(-2L-1, -2L-1, -2L-1)$ and v_{B_q} has coordinates $(2L+1, 2L+1, 2L+1)$.

Now we show how to add the drawing of $\Gamma_{i,1}$ and $\Gamma_{i,2}$, for $i = 1, 2, 3$. Given a bounding box B, an edge e protruding from it, and a direction label X orthogonal to e, we denote by $\delta(B, e, X)$ the distance of e from the side of B in the X direction (see Figure 1.b). We determine λ_{ij}, with $i, j = 1, 2, 3$, by considering the following equalities.

$$\lambda_{i,1} = 4L + 2 + \delta(B_q, e_{i3}, \overline{l_{i1}}) - \delta(B_{i,2}, e_{i3}, \overline{l_{i1}}) + \\ + \delta(B_{i,2}, e_{i2}, \overline{l_{i1}}) - \delta(B_{i,1}, e_{i2}, \overline{l_{i1}}) \tag{1}$$

$$\lambda_{i,2} = 4L + 2 + \delta(B_p, e_{i1}, l_{i2}) + \delta(B_q, e_{i3}, \overline{l_{i2}}) - \\ - \delta(B_{i,1}, e_{i1}, l_{i2}) - \delta(B_{i,2}, e_{i3}, \overline{l_{i2}}) \tag{2}$$

$$\lambda_{i,3} = 4L + 2 + \delta(B_p, e_{i1}, l_{i3}) - \delta(B_{i,1}, e_{i1}, l_{i3}) + \\ + \delta(B_{i,1}, e_{i2}, l_{i3}) - \delta(B_{i,2}, e_{i2}, l_{i3}) \tag{3}$$

Now we show that no intersection between edges sharing a flat has been introduced in the drawing. By Theorems 4 and 5 no intersection occurs in each (doubly or triply) expanding drawing.

There is no intersection between two edges of the same π_i. In fact, since $0 \le \delta(B_{i,j}, e, X) \le L$, and by equations 1, 2, and 3, each λ_{ij}, with $i, j = 1, 2, 3$ is at least $2L + 2$ and each one of B_p, B_q, $B_{i,1}$, and $B_{i,2}$ is in a different octant.

Intersections between edges belonging to different paths involve edges that do not share a flat, and thus, by Lemma 1, can be removed. Indeed, suppose for a contradiction that e_x and e_y are two intersecting edges belonging to two different paths π_x and π_y and that e_x and e_y are on the same flat F. Edges e_x and e_y do not belong to the same expanding drawing, since otherwise they could not intersect.

There must exist a path π^* joining e_x and e_y, entirely contained in F, and containing p or q. In π^* there can be at most three of the edges e_{ij}, with $i = x, y$, and $j = 1, 2, 3$, since otherwise there would be more than three canonical labels on the same flat for the cycle $C_{x,y}$, contradicting the hypothesis that $\tau_{x,y}$ is a canonical sequence. Since edges e_{ij} in π^* are on the same flat F and are canonical, they are necessarily adjacent, and thus if they are consecutive they are orthogonal with each other. If e_x and e_y coincide with two edges e_{ij}, then they can not intersect since they are parallel or adjacent and orthogonal. Otherwise, since edges e_{ij} are the only ones to transition between octants, then e_x and e_y belong to two different octants and they can not intersect.

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