COMBINATORIAL HYPERMAP REWRITING

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Abstract: Combinatorial hypermaps may be viewed as topological representations of hypergraphs. In this paper, we introduce a hypermap rewriting model, based on a purely combinatorial formulation of the rewriting mechanism. We illustrate this definition by providing a hypermap grammar which generates the set of all connected planar maps. We also investigate a special kind of hypermap grammars, the H-grammars, for which we give a Pumping Theorem enlightening the combinatorial structure of the generated hypermap languages.

0. INTRODUCTION

The appearance of graph grammars, in the early 70's, was essentially motivated by practical concerns in picture processing (recognition of chromosomes or manuscript letters). They were influenced by many application areas such as databases, compiling techniques, abstract data types, distributed systems or the development of biological cells [2,6]. The diversity of applicability domains of such grammars gives rise to the definition of many graph rewriting systems, which deal with rather different kinds of problems. Theoretical research essentially attempts to retrieve the main results from formal language theory. Some models now possess a well-developed mathematical theory, such as the algebraic approach of Ehrig [5] or the NLC (Node-Label-Controlled) grammars of Janssens and Rozenberg [8].

Most models of graph grammars are based on a graph, or subgraph, rewriting operation and the iteration of such a process. Due to the very structure of graphs, the rewriting mechanism, and especially the "embedding" operation, is presented in a more complex, and sometimes less rigorous way, than in formal language theory. The recent results of Bauderon and Courcelle [1] provide a simplification of the graph rewriting mechanism, based on a representation of graphs by algebraic expressions.

One must point out that, in addition to problems issued from formal language theory (classification of graph languages, properties of derivation sequences, decidability problems), the graph grammar theoretician deals with many problems related to the structure of generated graphs (planar graphs, Hamiltonian graphs, connected graphs).

Representation of graphs on surfaces takes a prominent part in graph theory. Meanwhile, we notice the lack of rewriting systems dealing with topological maps, even anterior to the concept of graph (see for

example the four colour problem). In this paper, we attempt to fill this gap by defining a map (or more generally a hypermap) generating system based on a combinatorial representation of them, introduced by Edmonds [4]. Combinatorial hypermaps may be viewed as topological representations of hypergraphs, and are defined by a pair (σ, α) of permutations acting on a finite set of darts [3] (darts represent pairs of incident edges and vertices ; permutation σ (resp. α) gives an order to the edges (resp. vertices) incident to a given vertex (resp. edge)). The interest in such a model essentially lies in the simplicity of the rewriting mechanism, expressed in a strictly combinatorial way, as a product of permutations.

In the first section, we introduce the main objects of our proposal: hypermaps and hypergraphs, and point out the links between them by notions of representation of hypergraphs and the underlying hypergraph of a given hypermap.

The second section is devoted to the definition of our hypermap rewriting model. A production p is made of a pair (L, R) of labelled hypermaps. A labelled hypermap H can then be rewritten by application of production p in three steps: localise an occurrence μL of left-hand-side L in H , isolate each dart of occurrence μL and then glue them together using the "scheme" given by right-hand-side R (this step may introduce new darts). The resulting hypermap is expressed as a combinatorial product of the form: $\xi D \otimes \mu L^{-1} \otimes H$.

In the third section we illustrate these definitions by providing a hypermap grammar which generates the set of all connected planar maps. The last section deals with a special kind of hypermap grammars: the H-grammars. Each production realize the rewriting of a full hyperedge. We show that these grammars may be considered as Context-Free hypermap grammars and provide a theorem on the combinatorial structure of generated languages, generalizing a well-known result (Pumping Lemma) from formal language theory.

Due to the limitation of space, we will omit the proofs which can be found in [9].

1. BASIC DEFINITIONS

In this work, we will denote by $[n]$ the set $\{1, 2, \dots, n\}$, and by $[n_1, n_2]$ the set $\{n_1, n_1+1, \dots, n_2\}$.

Let B be a finite set ; a permutation β acting on B is a bijection from B to itself. A cycle of β is a sequence $(b_1, \dots, b_k)$ of elements of B such that $\beta b_k = b_1$ and $\forall i \in [k-1], \beta b_i = b_{i+1}$; the set $\{b_1, \dots, b_k\}$ is called an orbit of β .

Let A be a subset of B ; the restriction of β to A , denoted by $\beta|_A$, is the permutation acting on A defined by: $\forall a \in A, \beta|_A a = \beta^k a$ where k is the smallest positive integer such that $\beta^k a \in A$.

Let C be a set containing B ; the extension of β to C , denoted by $\text{ext}_C(\beta)$ is the permutation acting on C defined by: $\forall c \in C$, if $c \in B$ then

$\text{ext}_c(\beta)c = \beta c$ else $\text{ext}_c(\beta)c = c$. Whenever the set C is given by context, we simply write β to denote $\text{ext}_c(\beta)$.

Example: Let β be the permutation defined on $B = [6]$ by:

$$\beta = (1, 3)(2, 4, 6)(5),$$

thus we have:

$$\beta_{\{1,4\}} = (1, 3)(2, 4) \quad \text{and} \quad \text{ext}_{\{8\}}(\beta) = (1, 3)(2, 4, 6)(5)(7)(8).$$

A *hypermap* is a triple $H = (D, \sigma, \alpha)$ where D is a finite set of *darts*, σ and α are two permutations over D . The orbits of σ (resp. α) are called the *vertices* (resp. *hyperedges*) of the hypermap. Let $\alpha\sigma$ be the permutation defined by $\alpha\sigma d = \alpha(\sigma(d))$ for each d in D . The orbits of $\alpha\sigma$ are called the *faces* of the hypermap.

The cardinality $\#D$ of set D is the *order* of the hypermap. We call the *degree* of a vertex (resp. hyperedge, face) the number of darts belonging to this vertex (resp. hyperedge, face). A vertex and a hyperedge are said to be *incident* if they have one dart at least in common.

If the group $G_{\sigma, \alpha}$ generated by $\langle \sigma, \alpha \rangle$ is transitive, we say the hypermap H is *connected* (i.e. $\forall d_1, d_2 \in D, \exists \tau \in G_{\sigma, \alpha} / d_1 = \tau d_2$).

A hypermap $H = (D, \sigma, \alpha)$ is called a *map* iff the permutation α is a fixpoint free involution (i.e. $\forall d \in D, \alpha d \neq d$ and $\alpha^2(d) = d$).

Let $H = (D, \sigma, \alpha)$ be a hypermap. d_0 and d_p two darts of D ; a *chain* from d_0 to d_p is a sequence $d_0, d_1, \dots, d_p$ of darts such that: $\forall i \in [p], d_i = \sigma d_{i-1}$ or $d_i = \alpha d_{i-1}$. Thus, a hypermap is connected iff for every pair (d, d') of darts there exists a chain from d to d' . Moreover, the relation $\mathcal{R}$ defined by: $d \mathcal{R} d'$ iff there exists a chain from d to d' is an equivalence relation. Let C be an equivalence class of $\mathcal{R}$; the hypermap $(C, \sigma|_C, \alpha|_C)$ is a *connected component* of H . We will denote by $CC(H)$ the set of all connected components of H .

The *genus* of a connected hypermap H is the quantity:

$$g(H) = 1 + 1/2 [\#D - z(\sigma) - z(\alpha) - z(\alpha\sigma)],$$

where for any permutation β , $z(\beta)$ denotes the number of cycles of β .

The genus of a general hypermap is then: $g(H) = \max \{g(C) / C \in CC(H)\}$.

A hypermap with genus zero is said to be *planar*. Thus, a hypermap is planar iff each of its connected components is planar.

In order to ensure homogeneity, every hypermap of order n will be defined over $D = [n]$.

A *hypergraph* is a pair $K = (V, E)$ where V is a finite set of *vertices* and E a finite set of *edges*. An edge is a family $e = \{v_1, \dots, v_p\}$ of elements of V (note that the same vertex may appear twice or more).

Let $K = (V, E)$ be a hypergraph; without loss of generality, one can suppose that each vertex v of V belongs to one edge e of E at least. From K , we define a set of darts as follows: for any vertex v and any edge e incident to v , let k be the number of occurrences of v in e ; then we define k darts denoted by $\langle v, e \rangle_1, \dots, \langle v, e \rangle_k$. Thus, V and E determine two partitions over D whose classes are respectively associated with the vertices and edges of K . By giving an arbitrary

order to each of these classes, we define two permutations over D . Let σ and α be these two permutations ; we will say that the hypermap $H=(D,\sigma,\alpha)$ is a *representation* of hypergraph K .

Example: Figure 1 shows a hypergraph K (1.a) and a hypermap H (1.b) which is a representation of K (vertices of H are drawn as circles with darts on their circumferences ; for vertices and hyperedges, positive orientation has been chosen as clockwise).

- figure 1 -

Conversely, let $H=(D,\sigma,\alpha)$ be a hypermap ; the underlying hypergraph K , denoted by $Und(H)$, is built in the following way: orbits of σ are the vertices of K ; the edges are obtained from the orbits of α by replacing every dart by the vertex it belongs to.

If H is a connected hypermap, the underlying hypergraph $Und(H)$ is clearly connected. The underlying hypergraph of a map is a *graph*. The underlying hypergraph of a planar hypermap is planar, but a hypermap which is a representation of a planar hypergraph is not necessarily planar (figure 1.c shows a non planar representation of hypergraph K from figure 1.a).

Let Σ be a non-empty finite set of *labels*. A *labelled hypermap* over Σ is a 4-tuple $H=(D,\sigma,\alpha,\lambda)$ such that: $H=(D,\sigma,\alpha)$ is a hypermap and λ a mapping from D to Σ called the *labelling function*. The set of all labelled hypermaps over Σ will be denoted by $\mathbb{H}_\Sigma$. A *hypermap language* is then a subset of $\mathbb{H}_\Sigma$.

Let $H_1=(D_1,\sigma_1,\alpha_1,\lambda_1)$ and $H_2=(D_2,\sigma_2,\alpha_2,\lambda_2)$ be two hypermaps of $\mathbb{H}_\Sigma$. We say that H_1 and H_2 are *isomorphic*, denoted by $H_1 \sim H_2$, if there exists a bijection μ from D_1 to D_2 such that: $\alpha_2 = \mu \alpha_1 \mu^{-1}$, $\sigma_2 = \mu \sigma_1 \mu^{-1}$, $\lambda_2 = \lambda_1 \mu^{-1}$. For such hypermaps, we will write $H_2 = \mu H_1$.

Let $H=(D,\sigma,\alpha,\lambda)$ be an element of $\mathbb{H}_\Sigma$, and A a subset of D ; the *restriction* of H to A , denoted by $H|_A$, is the hypermap $H'=(D',\sigma',\alpha',\lambda')$ given by: $D'=A$, $\sigma'=\sigma|_A$, $\alpha'=\alpha|_A$ and $\lambda'=\lambda|_A$.

Let H_1 and H_2 be two hypermaps of $\mathbb{H}_\Sigma$. We will say that H_2 is a *sub-hypermap* of H_1 iff there exists a subset A_1 of D_1 such that $H_2 = H_1|_{A_1}$.

The combinatorial product of two hypermaps, defined below, will enable us to give a very simple expression of hypermap rewriting mechanism.

Let $H_1 = (D_1, \sigma_1, \alpha_1, \lambda_1)$ and $H_2 = (D_2, \sigma_2, \alpha_2, \lambda_2)$ be two elements of $\mathbb{H}_x$ (D_1 and D_2 not necessarily disjoint). We define the hypermap $H' = H_1 \otimes H_2$ as follows: $D' = D_1 \cup D_2$, $\sigma' = \sigma_1 \circ \sigma_2$, $\alpha' = \alpha_1 \circ \alpha_2$, if $d \in D_1$ then $\lambda' d = \lambda_1 d$ else $\lambda' d = \lambda_2 d$.

Let H be a hypermap ; we will denote by H^{-1} the hypermap defined by $H^{-1} = (D, \sigma^{-1}, \alpha^{-1}, \lambda)$. Note that H^{-1} is the unique hypermap verifying $H \otimes H^{-1} = H^{-1} \otimes H = I$, with $I = (D, Id, Id, \lambda)$ where Id denotes the identity over D (hypermap H^{-1} may be viewed like hypermap H but with counterclockwise orientation).

2. HYPERMAP GRAMMARS

In this section, we introduce the concept of hypermap grammars, based on a strictly combinatorial formulation of the rewriting mechanism.

2.1 Definition: A hypermap grammar is a 4-tuple $G = \langle \Sigma, \Delta, P, Z \rangle$ where Σ is a finite non-empty set (the total alphabet), Δ a non-empty subset of Σ (the terminal alphabet), $P \subset \mathbb{H}_x \times \mathbb{H}_x$ the set of productions and $Z \in \mathbb{H}_x$ the axiom.

Let $p = (L, R)$ be a production ; the hypermap L (resp. R) is called *left-hand-side* (resp. *right-hand-side*) of production p and is denoted by $LHS(p)$ (resp. $RHS(p)$). In order to simplify the derivation concept, we will assume that $D_L \subset D_R$ (i.e. in each derivation step, all darts are preserved).

2.2 Definition: Let $G = \langle \Sigma, \Delta, P, Z \rangle$ be a hypermap grammar ; a *derivation step* is a 5-tuple $\langle H, p, \mu, \xi, H' \rangle$ verifying:

- (2.2.1) $H = (D, \sigma, \alpha, \lambda)$ and $H' = (D', \sigma', \alpha', \lambda')$ are elements of $\mathbb{H}_x$,
- (2.2.2) $p = (L, R)$ is a production of P ,
- (2.2.3) μ is an injective mapping from D_L to D such that μL is a sub-hypermap of H ,
- (2.2.4) ξ is a bijection from D_R to $D' \setminus D \cup \mu D_L$ such that $\xi|_{\mu D_L} = \mu$,
- (2.2.5) $H' = \xi D \otimes \mu L^{-1} \otimes H$.

We will say that H *directly derives* H' and denote it by $H \mapsto_{p, \mu} H'$. Whenever the values of p, μ are of no importance we simply note $H \mapsto H'$.

Interpretation: The rewriting of H onto H' by applying production $p = (L, R)$ may be decomposed in the following way:

- Step 1: find an occurrence μL of $LHS(p)$ in H (condition 2.2.3),
- Step 2: define a bijection ξ which associates dart 1 (resp. 2, 3, ...) of R with dart $\mu(1)$ (resp. $\mu(2), \mu(3), \dots$) of occurrence μL (condition 2.2.4),
- Step 3: H' is then obtained by the combinatorial product $\xi D \otimes \mu L^{-1} \otimes H$.

This product can be interpreted as:

- i) $\mu L^{-1} \otimes H$: "isolate" in H each dart of μL ,
- ii) $\xi D \otimes (\mu L^{-1} \otimes H)$: "glue" different portions together using the scheme given by $RHS(p)$.

Note that according to the choice of ξ , we obtain different hypermaps which are isomorphic.

2.3 Example: Figure 2 shows a production p and two hypermaps H and H' such that $H \mapsto H'$ by applying production p (hypermap $\mu L^{-1} \otimes H$ is given as an intermediate step of the derivation).

- figure 2 -

2.4 Definition: Let $G = \langle \Sigma, \Delta, P, Z \rangle$ be a grammar, $H = (D, \sigma, \alpha, \lambda)$ and $H' = (D', \sigma', \alpha', \lambda')$ two elements of $\mathbb{H}_\Sigma$; we will say that H derives H' , denoted by $H \mapsto^* H'$, if either $H = H'$ or there exists a sequence $H_0, H_1, \dots, H_t$ of hypermaps such that: $H_0 = H$, $H_t = H'$, and $\forall i \in [t]$, $H_{i-1} \mapsto H_i$. The language generated by G is then the set:

$$L(G) = \{ H \in \mathbb{H}_\Delta / \exists H_1 \in \mathbb{H}_\Delta, Z \mapsto^* H_1 \text{ and } H \sim H_1 \}.$$

We call H_Δ -language all subset L of $\mathbb{H}_\Delta$ such that there exists a hypermap grammar G verifying $L(G) = L$. We will denote by $\mathbf{LH}_\Delta$ the set of all H_Δ -languages.

3. EXAMPLES

In this section, we illustrate the previous concepts by providing two hypermap grammars which generate respectively the set of all planar maps with only one vertex, and the set of all connected planar maps.

3.1 Example (planar maps with only one vertex):

Let us consider the grammar $G = \langle \Sigma, \Delta, P, Z \rangle$ defined by:

1. $\Sigma = \Delta = \{a\}$,
2. $P = \{p\}$ with $p = (L, R)$ given by:

3. axiom Z is

Let M be a planar map with only one vertex; applying production p consists in choosing a dart d in M and adding a loop made of the two darts created by p (this loop "follows", in the sense of permutation σ , dart d in the resulting hypermap M'). Applying p clearly preserves planarity and the number of vertices. Moreover we have:

3.2 Proposition: $L(G)$ is exactly the set of all planar maps with only one vertex.

Figure 3 shows an example of derivation sequence (darts marked $\bullet$ denote darts of occurrence μL , dotted edges denote created edges).

- figure 3 -

3.3 Example (connected planar maps):

Let us consider the grammar $G = \langle \Sigma, \Delta, P, Z \rangle$ defined by:

1. $\Sigma = \{a, b\}$, $\Delta = \{a\}$,
2. $P = \{p_1, p_2, p_3, p_4, p_5\}$ where p_1, p_2, p_3, p_4, p_5 are given by:

3. axiom $\mathcal{Z}$ is

Productions p_1 and p_2 generate the two only connected planar maps of order 2 (label b ensures applicability of these productions in the first step only). Production p_3 (from the preceding example) allows us to add loops on a vertex. Productions p_4 and p_5 allow us to cut off a vertex in two parts joined by a new edge (production p_5 (resp. p_4) is used when the two darts indicating the scission axis are joined (resp. not joined) by an edge).

3.4 Proposition: $L(G)$ is exactly the set of all connected planar maps.

Figure 4 shows an example of derivation sequence applied to the map issued from the previous example (dotted lines denote axis of scission).

- figure 4 -

4. HYPERMAP H-GRAMMARS

In this section we deal with a particular subclass of hypermap grammars, called H-grammars, based upon hyperedge rewriting (note that a system based on vertex rewriting can be obtained as a dual form of H-grammars, acting on "dual hypermaps" $H^d = (D, \alpha, \sigma, \lambda)$). We show that these grammars may be viewed as Context-Free hypermap grammars, and exhibit a Pumping Theorem which generalizes a classical result from formal language theory.

We use in this section a ranked alphabet Σ and denote by $\tau: \Sigma \rightarrow \mathbb{N} - \{0\}$ the rank function. This alphabet allows us to introduce the concept of H-hypermaps which is essential in our definition of H-grammars.

4.1 Definition: A *H-hypermap* is a hypermap $H=(D,\sigma,\alpha,\lambda)$ of $\mathbb{H}_2$ verifying: for any hyperedge C in H , for any dart d in C , $\tau\lambda d=k$ where k is the degree of C . Thus, the degree of any hyperedge in a *H-hypermap* is given by the labels of its darts.

4.2 Definition: A hypermap grammar $G=\langle \Sigma, \Delta, P, Z \rangle$ is a *H-grammar* iff:

(4.2.1) Z is a *H-hypermap*,

(4.2.2) for every production $p=\langle L, R \rangle$ we have:

- L and R are *H-hypermaps*,
- α_L is a circular permutation (L contains only one hyperedge),
- $\sigma_{R|DL} = \sigma_L$.

Condition $\sigma_{R|DL} = \sigma_L$ prohibits us from cutting off the vertices of a rewritten hypermap. The valuation of Σ (and the concept of *H-hypermaps*) allows us to ensure the rewriting of full hyperedges.

As in the general case, we define notions of derivation step, derivation sequence and generated language of a *H-grammar* (note that all generated hypermaps are *H-hypermaps*). We call *HH_A-language* any subset L of $\mathbb{H}_A$ such that there exists a *H-grammar* G verifying $L(G) = L$. The set of all *HH_A-languages* is denoted by LHH_A .

The following lemma shows that the rewriting mechanism of *H-grammars* is strictly local and does not affect the context of rewritten hypermaps.

4.3 Lemma: Let $G=\langle \Sigma, \Delta, P, Z \rangle$ be a *H-grammar*, $\langle H, p, \mu, \xi, H' \rangle$ a derivation step and C_0 the hyperedge of H rewritten by production p . Then, every hyperedge C of H , distinct from C_0 , remains unchanged in H' and we have $H'_{|C} = H_{|C}$.

This lemma induces the following corollary:

4.4 Corollary: Let $G=\langle \Sigma, \Delta, P, Z \rangle$ be a *H-grammar*; then we have:

$$\{ H_{|C} / \exists H \in \mathbb{H}_2, Z \mapsto H, C \text{ hyperedge of } H \} \\ \sim \{ Z_{|C} / C \text{ hyperedge of } Z \} \cup \{ R_{|C} / \exists \langle L, R \rangle \in P, C \text{ hyperedge of } R \}$$

Note: Let L_1 and L_2 be two sets of hypermaps; by $L_1 \sim L_2$ we mean:

- $\forall H_1 \in L_1, \exists H_2 \in L_2, H_2 \sim H_1$,
- $\forall H_2 \in L_2, \exists H_1 \in L_1, H_1 \sim H_2$.

Thus, every hyperedge in a hypermap derived from Z is of bounded degree (the maximal degree is an integer k_0 depending on grammar G only).

4.5 Remark: To avoid ambiguity when rewriting a given hypermap we will use the alphabet $\Sigma \times [k_0]$, where k_0 is the integer introduced in preceding corollary. Moreover, by using techniques from formal language theory, we can give any *H-grammar* G in a normal form verifying: every hyperedge C of axiom Z or right-hand-side R of a production p may be given as $C=(d_1, \dots, d_k)$ such that:

- i) $\pi_1 \lambda d_1 = \dots = \pi_1 \lambda d_k = a \in \Sigma, \tau a = k$,
- ii) if $C'=(d'_1, \dots, d'_k)$ is another such hyperedge of same degree than C

$H_n = \text{Iter}(H, M, E_M, n)$ as follows:

i) $H_0 = H$,

ii) $H_1 = (D_1, \sigma_1, \alpha_1, \lambda_1, A_1)$,

with $(D_1, \sigma_1, \alpha_1, \lambda_1) = F_\alpha(H_0, M, E_M) = \xi_1 M \otimes H_0^{-1} \mid_{A_0} \otimes H_0$

and $A_1 = \xi_1 A_0$,

iii) for every $t \geq 1$, $H_t = \text{Iter}(H_{t-1}, M, E_M, 1)$.

Hypermap H (resp. M) is called *base* (resp. *pattern*) of the iteration.

4.10 Example: Figure 5 shows two hypermaps H and M and the two first corresponding iterations $H_1 = \text{Iter}(H, M, E_M, 1)$ and $H_2 = \text{Iter}(H, M, E_M, 2)$.

- figure 5 -

4.11 Theorem (Pumping Theorem): Let $G = \langle \Sigma, \Delta, P, Z \rangle$ be a H -grammar such that $L(G)$ is infinite. Then, there exists two positive integers $K1$ and $K2$ verifying: for every hypermap H in $L(G)$ such that $\#D_H > K1$, there exists a positive integer k , a base X and a pattern M , an H -hypermap Y and a sequence E_Y of k distinct darts of Y such that:

(4.11.1) $\#D_M + \#D_Y \leq K2$,

(4.11.2) $\#D_M \geq k$,

(4.11.3) $H = F_\alpha(\text{Iter}(X, M, E_M, 1), Y, E_Y)$,

(4.11.4) $\forall n \geq 0, F_\alpha(\text{Iter}(X, M, E_M, n), Y, E_Y) \in L(G)$.

This situation is depicted below:

but of different "shape" (i.e. $\sigma_{1C} \neq \sigma_{1C'}$) then $\pi_1 \lambda d_1 \neq \pi_1 \lambda d'_1$,
 iii) $\pi_2 \lambda d_1 = i$ for any i in $[k]$.
 where π_1 and π_2 denote respectively projections on first and second component.

We assume now that alphabet Σ is such that every H-grammar G can be given in normal form.

For every hyperedge in a hypermap derived from Z , the labels of its darts determine its degree but also its shape ($\# \Sigma$ represents the number of different hyperedge shapes (including their degree) that may appear in a HH-language).

The following lemma indicates that H-grammars verify a property of Church-Rosser type.

4.6 Lemma: Let $H = (D, \sigma, \alpha, \lambda)$ be an H-hypermap, p_1 and p_2 two productions which may be applied to H over two distinct hyperedges C_1 and C_2 ; we know (lemma 4.3) that if H directly derives H_1 (resp. H_2) by application of p_1 (resp. p_2), then p_2 (resp. p_1) may be applied in H_1 (resp. H_2) to the same hyperedge C_2 (resp. C_1). Moreover, we have:

$$\text{if } H \xrightarrow{p_1} H_1 \xrightarrow{p_2} H_{12} \text{ and } H \xrightarrow{p_2} H_2 \xrightarrow{p_1} H_{21}, \text{ then } H_{12} \sim H_{21}.$$

In our pumping theorem, we will use the following construction, based on the concept of α_k -pointed hypermaps, to describe the structure of HH-languages.

4.7 Definition: A α_k -pointed hypermap is a 5-tuple $H = (D, \sigma, \alpha, \lambda, A)$ where $H = (D, \sigma, \alpha, \lambda)$ is a H-hypermap and A a sequence of k distinct darts $a_1, \dots, a_k$ such that $(a_1, \dots, a_k)$ is a hyperedge and $\pi_2 \lambda a_i = i, \forall i \in [k]$.

4.8 Definition: Let $H = (D, \sigma, \alpha, \lambda, A)$ be a α_k -pointed hypermap, $H' = (D', \sigma', \alpha', \lambda')$ a hypermap such that $D' \cap D = \emptyset$, and $E' = e'_1, \dots, e'_k$ a sequence of k distinct darts of H' . We call α -fusion along E' of hypermaps H and H' , denoted $F_\alpha(H, H', E')$ the hypermap defined over $(D \cup D') \setminus E'$ by:

$$F_\alpha(H, H', E') = \xi H' \otimes H_A^{-1} \otimes H,$$

where ξ is the mapping from D' to $(D' \cup A) \setminus E'$ defined by: $\forall d \in D',$ if $d = e'_i$ then $\xi d = a_i$ else $\xi d = d$.

This operation is only a slightly different formulation of the hyperedge rewriting mechanism we have defined (if $G = \langle \Sigma, \Delta, P, Z \rangle$ is a H-grammar and $\langle H, p, \mu, \xi, H' \rangle$ a derivation step, then we have $H' = F_\alpha(H, R, \{1, \dots, \#D_L\})$).

The notion we introduce now allows us to generalize to hypermaps the concept of concatenation from formal language theory.

4.9 Definition: Let $H = (D_H, \sigma_H, \alpha_H, \lambda_H, A_H)$ and $M = (D_M, \sigma_M, \alpha_M, \lambda_M, A_M)$ be two α_k -pointed hypermaps and $E_M = e_1, \dots, e_k$ a sequence of k distinct darts of D_M . For any integer $n \geq 0$, we define the α_k -pointed hypermap

Theorem 4.11 induces the following corollary:

4.12 Corollary: Let $G = \langle \Sigma, \Delta, P, Z \rangle$ be a H-grammar such that $L(G)$ is infinite ; let $NB(G) = \{ n \in \mathbb{N} / \exists H \in L(G), \#D_H = n \}$. Then, there exists integers n_0 and k such that: $\forall i > n_0, [i, i+k] \cap NB(G) \neq \emptyset$.

This corollary enable us to state that there exists hypermap families that are not HH-languages. It is namely the case for the following:

4.13 Example: The set of all hypermaps of the following form cannot be generated by an H-grammar:

The following theorem shows that H-grammars allow us to retrieve hypergraph languages generated by hyperedge replacement, introduced by Habel and Kreowski [7]. We will denote by *HK-grammar* such a system.

4.14 Theorem: For any HK-grammar HKG , there exists a H-grammar HG such that: $L(HKG) = Und(L(HG)) = \{ Und(H) / H \in L(HG) \}$.

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