Parameter Preserving Data Type Specifications

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Abstract

Term rewriting methods are used for solving the persistency problem of parameterized data type specifications. Such a specification is called persistent if the parameter part of its algebraic semantics agrees with the semantics of the parameter specification. Since persistency mostly cannot be guaranteed for the whole equational variety of the parameter specification, the persistency criteria developed here mainly concern classes of parameter algebras with "built-in" logic.

1. Persistency, extensions and inductive theories

Starting from a many-sorted signature <S,OP> with sorts S and operation symbols OP an algebraic specification in this sense of ADJ /1/ is given by a triple SPEC = <S,OP,E> where E is a set of equations between OP-terms. Algebras with signature <S,OP> which satisfy E are called SPEC-algebras. For reasons discussed extensively in the literature (e.g. in /1/) the isomorphism class of initial SPEC-algebras plays a dominant role.
A parameterized specification PAR is a pair of two specifications PSPEC and SPEC where the **parameter** PSPEC is part of the **target** SPEC. The role of initial algebras is taken over by a class of target algebras each of which is "freely generated" over some algebra in a given class K of parameter algebras (cf. /2/).
Such a class of target algebras is called a parameterized data type. /10/ deals with the proof-theoretical characterization of the equational variety of parameterized data types. This variety turned out to be a certain "inductive" theory of the target specification.
In many cases this characterization works only if PAR is persistent, i.e. if each algebra in the corresponding data type "preserves" the parameter algebra where it is "freely generated" upon. Persistency is also a sufficient criterion for the "passing compatibility" of PAR with actual parameter specifications (cf. ADJ /3/). So this paper is devoted to decidable and powerful criteria for persistency. The first step towards such conditions is the decomposition of PAR into a "base" specification BPAR and the remaining operations & equations of PAR. BPAR is supposed to contain those operations & equations of PAR that are necessary for the "construction" of data. Following this strategy it

is mostly simple to show that BPAR is persistent. Then PAR is persistent, too, if BPAR is a conservative extension of PAR which means that the "base" part of the data type specified by PAR agrees with the data type specified by BPAR.
The extension property is separated into two parts: completeness and consistency. So the tools for solving the persistency problem are the criteria for persistency of BPAR given by Theorem 2.12 and the completeness and consistency conditions of Thms. 3.4/4.7 and 3.5/5.14, respectively. They involve normalization and confluence properties of term reductions and are tailor-made for parameter algebras with "built-in" logic where the proof-theoretical characterization of parameterized data types given in /10/, section 3, is based upon, too.
Besides well-known notions in term rewriting theory like "confluence" and "critical pair" we use some recently introduced ones like "coherence" (cf. /8/), "contextual reductions" (cf. /12/) and "recursive critical pairs" (cf. /11/). They should support the reader's intuition, although their definitions sometimes deviate from their meaning in the cited papers. Moreover, the corresponding results presented here are different from those given there.
The paper is organized as follows: Section 2 contains basic definitions and proof-theoretical characterizations of conservative extensions (2.9) and persistency (2.12). In section 3 general completeness and consistency theorems (3.4/5) are given that refer to term reductions. Sections 4 and 5 focus on parameter algebras with "built-in" logic and adapt the notions of section 3 to this case. Decidable criteria for the crucial confluence criteria of Thm 3.5 are developed in section 5 that culminates in the Critical Pair Thm. 5.12. The main results of sections 4 and 5 are summarized by Completeness Thm. 4.8, Consistency Thm. 5.14 and Persistency Thm. 5.16.
Former versions of these results are part of the author's Ph.D.thesis /9/.
id, inc and nat denote identity, inclusion and natural mappings, respectively.
The first occurrences of notions used throughout the whole paper are printed in boldface.

2. The syntax and semantics of parameterized specifications

Let SIG = <S,OP> be a many-sorted signature with a set S of sorts and an $(S^* \times S)$-sorted set OP of operation symbols. If $\sigma \in OP_{w,s}$, then arity(σ) = w, sort(σ) = s, and we often write σ:w $\rightarrow$ s. If w = ε (empty word), σ is called a constant. **T(SIG)** denotes the free S-sorted algebra of OP-terms over a fixed infinite S-sorted set X of variables. If t,t' $\in$ T(SIG) and x $\in$ X, then t[t'/x] is t with x replaced by t'.
For every S-sorted set A and all s1,...,sn $\in$ S, $A_{s1...sn} := A_{s1} \times \ldots \times A_{sn}$. Let w $\in S^*$, s $\in$ S, $\sigma \in OP_{w,s}$ and t $\in T(SIG)_w$. Then root(σt) = σ, arg(σt) = t, sort(σt) = s, and op(σt) resp. var(σt) denote the set of operation symbols resp. variables of σt. Size(t) is the number of operation symbol occurrences in t. A SIG-equation l=r is a pair of SIG-terms l and r with sort(l) = sort(r). Let A be a SIG-algebra. **Z(A)** denotes the S-sorted set of functions from X to A. The unique homomorphic extension of f $\in$ Z(A) to T(SIG) is also written f. If f $\in$ Z(T(SIG)), t $\in$ T(SIG) and x $\in$ X, then f[t/x] $\in$ Z(T(SIG)) is defined by f[t/x](x) = t and f[t/x](y) = fy for all y $\in$ X-{x}.
A satisfies a SIG-equation l=r if for all f $\in$ Z(A) fl=fr. (This definition extends to classes of algebras and sets of equations as usual.)

2.1. Definitions (specification & semantics)

An (equational) **specification** SPEC = <S,OP,E> consists of a many-sorted signature SIG = <S,OP> and a set E of SIG-equations. **Alg(SPEC)** denotes the class of SIG-algebras that satify E. The **free SPEC-congruence** $=_{SPEC}$ is the smallest SIG-congruence on T(SIG) that contains all pairs <fl,fr> with l=r in E and f $\in$ Z(T(SIG)). $=_{SPEC}$ is also called the **free theory of SPEC.**
G(SIG) denotes the free S-sorted algebra of OP-terms over the empty set. **Gen(SPEC)** is the class of "finitely generated" SIG-algebras that satisfy E, i.e. every a $\in$ A is the interpretation of some t $\in$ G(SIG). The **inductive SPEC-congruence** $\equiv_{SPEC}$ is given by all pairs <t,t'> $\in$ $T(SIG)^2$ such that for all f $\in$ Z(G(SIG)) ft $=_{SPEC}$ ft'. $\equiv_{SPEC}$ is also called the **inductive theory of SPEC.** Note that the restriction of $\equiv_{SPEC}$ to $G(SIG)^2$ coincides with $=_{SPEC}$.

Two facts are well-known (cf. /4/ resp. /1/):

2.2. Theorem

1. Alg(SPEC) satisfies t=t' iff $t =_{SPEC} t'$.
2. Gen(SPEC) satisfies t=t' iff $t \equiv_{SPEC} t'$. ☐

2.3. Definitions (parameterized data types)

A **parameterized specification PAR** is a pair of two specifications PSPEC and SPEC. The forgetful functor from Alg(SPEC) to Alg(PSPEC) is denoted by U_{PAR}, while F_{PAR} stands for its left adjoint. For every class K of PSPEC-algebras the **parameterized data type specified** by **<PAR,K>** is given by

PDT(PAR,K) = $\{F_{PAR}(A) \mid A \in K\}$.

Let **PSIG** = <PS,POP>, **PSPEC** = <PS,POP,PE>, **SPEC** = <S,OP,E> and **PX** = $\{x \in X \mid sort(x) \in PS\}$. Regarding PX as constants we obtain the signature **SIGX** = <S,OP $\cup$ PX> and the specification **SPECX** = <S,OP $\cup$ PX,E>. $\equiv_{SPECX}$ is called the **inductive theory of PAR.**

Analogously to Thm. 2.2, there is the following proof-theoretical characterization of the data type specified by <PAR,Alg(PSPEC)>:

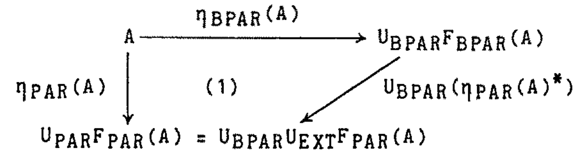
2.4. Theorem (/10/,1.7)

PDT(PAR,Alg(PSPEC)) satisfies t = t' iff $t \equiv_{SPECX} t'$. ☐

2.5. Definitions (persistency & extension)

Let ID be the identity functor on Alg(PSPEC). We recall from category theory that there is a functor transformation η_{PAR}: ID $\to$ $U_{PAR}F_{PAR}$ such that for all B $\in$ Alg(SPEC) each homomorphism h: A $\to$ $U_{PAR}(B)$ uniquely extends to a homomorphism h^*: $F_{PAR}(A) \to B$ such that $U_{PAR}(h^*) \circ \eta_{PAR}(A) = h$.
Let K be a class of PSPEC-algebras. <PAR,K> is **persistent** if for all A $\in$ K $\eta_{PAR}(A)$ is bijective.
Let **BPAR** = <PSPEC,BSPEC> be a parameterized subspecification of PAR, i.e. **BSPEC** = **<BS,BOP,BE>** is componentwise included in SPEC. Let **BSIG** = <BS,BOP> and **EXT** = <BSPEC,SPEC>. Since $U_{PAR} = U_{BPAR} \circ U_{EXT}$, $\eta_{PAR}(A)$: A $\to$ $U_{PAR}F_{PAR}(A)$ uniquely

extends to $\eta_{PAR(A)}^*$: $F_{BPAR}(A) \rightarrow U_{EXT}F_{PAR}(A)$ such that $U_{BPAR}(\eta_{PAR}(A)^*) \circ \eta_{BPAR}(A) = \eta_{PAR}(A)$:



PAR is **complete** (**consistent**) w.r.t. <BPAR,K> if for all $A \in K$ $\eta_{PAR}(A)^*$ is surjective (injective). PAR is a **conservative extension** of **<BPAR,K>** if PAR is complete and consistent w.r.t. <BPAR,K>.

An immediate consequence of these definitions is the following

2.6. Decomposition Lemma for Persistency

Let <BPAR,K> be persistent. PAR is a conservative extension of <BPAR,K> iff <PAR,K> is persistent. □

2.7. Example

Let BOOL be a specification of Boolean algebras, i.e. BOOL consists of a sort bool, constants true and false, operation symbols ¬, ∧, ∨, ⇒, ⇔ and the Boolean algebras axioms. Moreover,

```
DATA = BOOL +
   sorts: entry
   opns:  eq: entry entry --> bool
   eqns:  eq(x,x) = true                                    (e1)
          eq(x,y) = eq(y,x)                                 (e2)
          (eq(x,y) ∧ eq(y,z)) ⇒ eq(x,z) = true              (e3)


BSET = DATA +
   sorts: set
   opns : Ø: --> set
   ins: set entry --> set
   eqns : ins(ins(s,x),x) = ins(s,x)                        (e4)
          ins(ins(s,x),y) = ins(ins(s,y),x)                 (e5)

SET = BSET +
   opns: has: set entry --> bool
         del: set entry --> set
         if-bool: bool bool bool --> bool
         if-set: bool set set --> set
   eqns: has(Ø,x) = false                                   (e6)
         has(ins(s,x),y) = if-bool(eq(x,y),true,has(s,y))   (e7)
         del(Ø,x) = Ø                                       (e8)
         del(ins(s,x),y) = if-set(eq(x,y),del(s,y),ins(del(s,y),x))  (e9)
         if-bool(true,b,b') = b                             (e10)
         if-bool(false,b,b') = b'                           (e11)
         if-set(true,s,s') = s                              (e12)
         if-set(false,s,s') = s'                            (e13)
```

Following the strategy developed in this paper we will show that for a certain class Log of DATA-algebras, which will be given in section 4, <SET,Log(DATA)> is persistent. □

We proceed with the representation of parameterized data types by classes of initial algebras which is essential for the proof-theoretical characterization of conservative extensions (2.9).

2.8. Definition and Theorem (/10/,1.5)

Let $A \in$ Alg(PSPEC). The specification

$$\mathbf{SPEC(A)} = \langle S, OP \cup A, E \cup \triangle(A)\rangle$$

has all operation symbols of SPEC together with all elements of A as constants, while the set of equations of SPEC is extended by the **equational diagram** of A, $\triangle(A)$, that consists of all equations $\sigma(a) = \sigma_A(a)$ with $\sigma \in$ POP and $a \in A_{arity(\sigma)}$.
$F_{PAR}(A)$ gets an (OP $\cup$ A)-algebra by interpreting each constant $a \in A$ by $\eta_{PAR}(A)(a)$. Moreover, $F_{PAR}(A)$ is an initial object in Alg(SPEC(A)). □

Using the well-known quotient term algebra representation of initial algebras (cf. ADJ/1/) we can formulate completeness and consistency as free theory properties:

2.9. Theorem

Let **BSIG**(A) = <BS,BOP $\cup$ A> and **SIG**(A) = <S,OP $\cup$ A>.

1. PAR is complete w.r.t. <BPAR,K> iff for all $A \in K$, $s \in BS$ and $t \in G(SIG(A))_s$ some $t' \in G(BSIG(A))$ satisfies $t =_{SPEC(A)} t'$.
2. PAR is consistent w.r.t. <BPAR,K> iff for all $A \in K$ and $t,t' \in G(BSIG(A))$ $t =_{SPEC(A)} t'$ implies $t =_{BSPEC(A)} t'$. □

Besides persistency Thms. 2.8 & 2.9 provide a useful criterion for the validity of equations in parameterized data types: PDT(PAR,K) satisfies a set E' of SIG-equations if PAR is complete w.r.t. <BPAR,K> and <PSPEC,<S,OP,E $\cup$ E'>> is consistent w.r.t. <BPAR,K>.

The proof-theoretical conditions "maximal completeness" and "maximal consistency" defined below deal with variables instead of elements of a particular parameter algebra. Hence they characterize persistency of PAR with respect to **all** parameter algebras (Thm. 2.12).

2.10. Definitions

PAR is **maximally complete** if for all $s \in PS$ and $t \in G(SIGX)_s$ $t =_{SPECX} t'$ for some $t' \in T(PSIG)$ (cf.2.3.). PAR is **maximally consistent** if for all $t,t' \in T(PSIG)$ $t =_{SPECX} t'$ implies $t =_{PSPEC} t'$.

2.11. Definition

The **simple reduction relation generated** by E, $\overrightarrow{E}$, is the smallest relation on T(SIG) resp. Z(T(SIG)) such that

(i) for all l=r in E and $f \in Z(T(SIG))$ $fl \overrightarrow{E} fr$,

(ii) for all $\sigma \in OP$ $\sigma(t_1,\dots,t_i,\dots,t_n) \overrightarrow{E} \sigma(t_1',\dots,t_i',..t_n)$ if $t_i \overrightarrow{E} t_i'$,

(iii) for all $f,g \in Z(T(SIG))$ $f \xrightarrow[E]{} g$ if for all $x \in X$ $fx \xrightarrow[E]{} gx$.

$\xrightarrow[E]{=}$, $\xleftrightarrow[E]{}$ and $\xrightarrow[E]{*}$ denote the reflexive, symmetric and reflexive-transitive closures of $\xrightarrow[E]{}$, respectively.

2.12. Persistency Theorem I

<PAR,Alg(PSPEC)> is persistent iff PAR is maximally complete and maximally consistent.

Proof:
"only if": By assumption, $\eta_{PAR(T_{PSIG})/=_{PSPEC}}$ is an isomorphism. Since

$F_{PAR}(T_{PSIG})/=_{PSPEC} = G(SIGX)/=_{SPECX}$

(/10/,1.6), we conclude

$T_{PSIG}/=_{PSPEC} = U_{EXT}(G(SIGX)/=_{SPECX})$.

The surjective resp. injective part of this isomorphism is maximal completeness resp. consistency of PAR.
The "if"-part is more tedious and given in the extended version of this paper. Its main idea is due to H. Ganzinger (cf. /6/, Thm. 5). □

2.13. Corollary

<PAR,Alg(PSPEC)> is persistent if for all $\sigma \in OP$, sort(σ) $\in$ PS implies $\sigma \in POP$ and if for all l=r in E sort(l) $\in$ PS implies that l=r is in PE. □

2.14. Example (cf. 2.7)

Using Corollary 2.13 we immediately observe that <<DATA,BSET>,Alg(DATA)> is persistent. □

3. Extension Proofs by term rewriting

Assuming that the "base" <BPAR,K> is persistent we now turn to tools for extension proofs which are based on the proof-theoretical characterization of conservative extensions given in the last section (2.9).
From now on we suppose that S=BS and for all $A \in$ Alg(PSPEC) and $s \in S$ $G(SIG(A))_s$ is nonempty.

A first step towards extension criteria is the decomposition of the free SPEC(A)-congruence into simple reductions and the free BSPEC(A)-congruence:

3.1. Definition

Let $A \in$ Alg(PSPEC). A set R of SIG(A)-equations is **Church-Rosser w.r.t. A** if for all $s \in BS$ and $t,t' \in G(SIG(A))_s$

$$t =_{SPEC(A)} t' \text{ implies } \begin{array}{ccc} t & & t' \\ R \Big\downarrow * & & * \Big\downarrow R \\ u & =_{BSPEC(A)} & u' \end{array}$$

for some $u,u' \in G(BSIG(A))$.

3.2. Lemma

Suppose that for each $l=r$ in E-BE op(l) contains at least one operation symbol of OP-BOP. For all $A \in K$ let $E(A)$ be a subset of $=_{BSPEC(A)}$.
If $(E\text{-}BE) \cup E(A)$ is Church-Rosser w.r.t. A, then PAR is consistent w.r.t. <BPAR,K>. □

Localizing the Church-Rosser property by "confluence" and "coherence" conditions goes along with restricting equations to "normalizing" ones:

3.3. Definitions

Let $A \in Alg(PSPEC)$ and R be a set of SIG(A)-equations. $t' \in G(BSIG(A))$ is an **R-normal form of** $t \in G(SIG(A))$ if $t \xrightarrow[R]{*} t'$. R is **normalizing w.r.t. A** if for all $t \in G(SIG(A))$ t has an R-normal form. R is **confluent w.r.t. A** if for all $t \in G(SIG(A))$ all R-normal forms t_1,t_2 of t satisfy $t_1 =_{BSPEC(A)} t_2$. $\langle t_1,t_2\rangle \in G(SIG(A))$ is **uniformly R-convergent w.r.t. A** if some R-normal forms t_1', t_2' of t_1 resp. t_2 satisfy $t_1' =_{BSPEC(A)} t_2'$, written: $t_1 \downarrow_{R,A} t_2$.
R is **coherent w.r.t. A** if for all $t,t_1,t_2 \in G(SIG(A))$

$$\begin{array}{l} t \xrightarrow{R} t_1 \\ t \xrightarrow{BE} t_2 \end{array} \quad \text{implies} \quad t_1 \downarrow_{R,A} t_2.$$

R **commutes with** another set R' of SIG(A)-equations if for all $t,t_1,t_2 \in G(SIG(A))$

$$\begin{array}{l} t \xrightarrow[R]{*} t_1 \\ t \xrightarrow[R']{*} t_2 \end{array} \quad \text{implies} \quad \begin{array}{l} t_1 \xrightarrow[R']{*} t' \\ t_2 \xrightarrow[R]{*} t' \end{array}$$

for some $t' \in G(SIG(A))$.

3.4. Completeness Theorem I

For all $A \in K$ let $E(A)$ be a subset of $=_{SPEC(A)}$. If for all $A \in K$ $E \cup E(A)$ is normalizing w.r.t. <BPAR,K>, then PAR is complete w.r.t. <BPAR,K>.
Proof:
The statement immediately follows from Thm. 2.9.1. □

3.5. Consistency Theorem I

Suppose that for each $l = r$ in BE $var(r) \subseteq var(l)$ and for each $l = r$ in E-BE l contains at least one operation symbol of OP-BOP. For all $A \in K$ let $E(A)$ be a subset of $=_{BSPEC(A)}$. If $(E\text{-}BE) \cup E(A)$ is normalizing, confluent and coherent w.r.t. A and commutes with $\triangle(A) \cup \triangle(A)^{-1}$ (cf. 2.8), then PAR is consistent w.r.t. <BPAR,K>.

Proof:

By Lemma 3.2, it is sufficient to show that R = (E-BE) ∪ E(A) is Church-Rosser w.r.t. A. So let s ∈ BS and t,t' ∈ $G(SIG(A))_s$ such that t $=_{SPEC(A)}$ t'. There are a least number n and $t_1,\ldots,t_n$, $u_1,\ldots,u_n$ ∈ G(SIG(A)) with t_1=t, u_n=t' and for all $1 \le i<n$ $u_i \xrightarrow[E-BE]{*} t_i$ and

(i) $u_i \xrightarrow[E-BE]{} t_{i+1}$

or (ii) $u_i \xrightarrow[BE]{} t_{i+1}$

or (iii) $t_{i+1} \xrightarrow[BE]{} u_i$

or (iv) $u_i \xleftrightarrow[\triangle(A)]{} t_{i+1}$.

We prove t $\downarrow_R$ t' by induction on n. n = 1 implies $t' \xrightarrow[R]{*} t$, and t $\downarrow_R$ t' follows from normalization and confluence of R w.r.t. A. Since E(A) ⊆ $=_{BSPEC(A)}$ and for each l=r in E-BE op(l) ∩ (OP-BOP) ≠ ∅,

(*) for all u ∈ G(BSIG(A)) $u \xrightarrow[R]{} u'$ implies u $=_{BSPEC(A)}$ u'.

Let n>1. By induction hypothesis, $t_2 \downarrow_{R,A}$ t'. Hence by confluence of R, it remains to show $t_1 \downarrow_{R,A} t_2$.

The proof proceeds by deriving $t_1 \downarrow_{R,A} t_2$ in each of the cases (i)-(iv) for i=1.
□

4. Parameters with "built-in" logic

From now on we deal with parameters including Boolean operators and restrict parameter algebras to those where the Boolean operators are interpreted as in propositional logic. In addition, we use if-then-else operators to simulate conditional axioms by equations.

General Assumption

Suppose that **BOOL** (cf. 2.7) is a subspecification of PSPEC. Moreover, let **ifS** be a subset of S such that for all s ∈ ifS SIG contains an operation symbol if-s: bool s s → s and E includes the equations

if-s(true,x,y) = x and if-s(false,x,y) = y.

Vice versa, for each l = r in E

(i) sort(l) = bool implies l ∉ {true, false},

(ii) sort(l) ≠ bool implies t ∈ {true, false} for all bool-sorted subterms t of l.

4.1. Definitions

Let **PEXT** = <BOOL,PSPEC>. The class **Log(PSPEC)** is given by all PSPEC-algebras A such that $U_{PEXT}(A)$ is the Boolean algebra {true, false}. Hence we drop the equations true=true and false=false from the equational diagram of A (cf. 2.8). For all A ∈ Log(PSPEC) **LE(A)** denotes the set of BSIG(A)-equations l=r with l ∈ $G(BSIG(A))_{bool}$-{true,false}, r ∈ {true, false} and l $=_{BSPEC(A)}$ r.

4.2. Lemma

Let <BPAR,Log(PSPEC)> be persistent. Then for all $A \in Log(PSPEC)$ and $t \in G(BSIG(A))_{bool}$ either t=true or t=false is in LE(A).

Proof:

Let $A \in Log(PSPEC)$. By assumption, $A = U_{BPAR}F_{BPAR}(A)$. By Thm. 2.8, $F_{BPAR}(A) \approx G(BSIG(A))/{=}_{BSPEC(A)}$. Hence the statement follows from $U_{PEXT}(A) = \{true, false\}$.
□

Next we define a reduction relation with conditions (contexts) in order to simulate reductions via LE(A).

4.3. Definition

Let **LT** = $T(SIG)_{bool}$. The **contextual reduction relation generated by** E, $\{\xrightarrow[E;p]{*}\}_{p \in LT}$, is the family of smallest relations on T(SIG) such that

(i) for all $t \in T(SIG)$ and $p \in LT$ $t \xrightarrow[E;p]{*} t$,

(ii) for all l=r in E, $f \in Z(T(SIG))$ and $p \in LT$ $fl \xrightarrow[E;p]{*} fr$,

(iii) for all $\sigma \in OP$
$\sigma(t_1,\ldots,t_i,\ldots,t_n) \xrightarrow[E;p]{*} \sigma(t_1,\ldots,t_i',\ldots,t_n)$
if $t_i \xrightarrow[E;p]{*} t_i'$,

(iv) for all $s \in ifS$ and $t_1,t_2 \in T(SIG)_s$
$\text{if-s}(p,t_1,t_2) \xrightarrow[E;p]{*} t_1$ and $\text{if-s}(p,t_1,t_2) \xrightarrow[E;\neg p]{*} t_2$,

(v) $t \xrightarrow[E;p \wedge q]{*} t''$ if $t \xrightarrow[E;p]{*} t'$ and $t' \xrightarrow[E;q]{*} t''$,

(vi) $t \xrightarrow[E;p \vee q]{*} t'$ if $t \xrightarrow[E;p]{*} t'$ and $t \xrightarrow[E;q]{*} t'$,

(vii) for all $t,t' \in T(SIG)$ $t \xrightarrow[E;false]{*} t'$.

The following lemma draws the connection between contextual and LE(A)-reductions. Contexts are now restricted to "base" terms so that contextual reductions can be regarded as "hierarchical" ones.

4.4. Lemma

Let <BPAR,Log(PSPEC)> be persistent. Then all $A \in Log(PSPEC)$, $f \in Z(G(SIG(A)))$ and $f' \in Z(G(BSIG(A)))$ with $f \xrightarrow[E \cup LE(A)]{*} f'$ satisfy

(*) $ft \xrightarrow[E \cup LE(A)]{*} ft'$ if $t \xrightarrow[E;p]{*} t'$ and f'p=true is in LE(A). □

Contextual reduction properties that correspond to 3.3 are defined by 4.5 and 4.11 below.

4.5. Definition

Let $t \in G(SIGX)$ (cf. 2.3). t **has contextual E-normal forms** $t_1,\ldots,t_n \in G(BSIGX)$ if there are $n \in \mathbb{N}$ and $p_1,\ldots,p_n \in LT$ such that $p_1 \vee \ldots \vee p_n =_{SPEC} true$ and for all $1 \leq i \leq n$ $t \xrightarrow[E;p_i]{*} t_i$. E is **contextually normalizing** if all $t \in G(SIGX)$ have contextual E-normal forms.

To reduce normalization of E ∪ LE(A) to contextual normalization of E we have to guarantee that SPEC(A) does not identify true and false:

4.6. Definition

PAR is **logically consistent** if for all A ∈ Log(PSPEC) some B ∈ Alg(SPEC(A)) has different interpretations of true and false.

4.7. Lemma

Suppose that <BPAR,Log(PSPEC)> is persistent and PAR is logically consistent. Let A ∈ Log(PSPEC). E ∪ LE(A) is normalizing w.r.t. A if E is contextually normalizing. □

4.8. Completeness Theorem II

Suppose that <BPAR,Log(PSPEC)> is persistent and PAR is logically consistent. If E is contextually normalizing, then PAR is complete w.r.t. <BPAR,Log(PSPEC)>.

Proof:
The statement immediately follows from Lemma 4.7 and Completeness Theorem 3.4. □

4.9. Example (cf. 2.7)

Let E={e6,...,e13}. One easily observes that E is contextually normalizing if
(*) for all t,t' ∈ $G(BSIGX)_{set}$ has(t,x), del(t,x) and if-set(x,t,t') have contextual E-normal forms.
(*) follows by induction on size(t)+size(t') because we obtain

$$has(\emptyset,x) \xrightarrow[e8]{} false,$$

$$has(ins(t,x),y) \xrightarrow[e7;eq(x,y)]{*} true,$$

$$has(ins(t,x),y) \xrightarrow[e7;\neg eq(x,y)]{*} has(t,y),$$

$$del(\emptyset,x) \xrightarrow[e8]{} \emptyset,$$

$$del(ins(t,x),y) \xrightarrow[e9;eq(x,y)]{*} del(t,y),$$

$$del(ins(t,x),y) \xrightarrow[e9;\neg eq(x,y)]{*} ins(del(t,y),x),$$

$$if\text{-}set(x,t,t') \xrightarrow[E;x]{} t,$$

$$if\text{-}set(x,t,t') \xrightarrow[E;\neg x]{} t'$$

for all t,t' ∈ $G(BSIGX)_{set}$. Since <<DATA,BSET>,Log(DATA)> is persistent (cf. Ex. 2.14), we conclude from Thm. 4.8 that <DATA,SET> is complete w.r.t. <<DATA,BSET>,Log(DATA)>. □

Local criteria for confluence and coherence require the "new" equations E-BE to be normalizing (cf. Thm. 3.5). "Base" equations (BE) are often not normalizing. Hence we can use Noetherian induction - to lift local criteria - only with respect to E-BE. But BE must be considered, too. The lack of normalization of BE is circumvented by working with parallel BE-reductions which combine independent simple reductions in one step.

4.10. Definition

The **parallel reduction relation generated** by E, $\underset{E}{\Rightarrow}$, and its reflexive closure $\underset{E}{\overset{=}{\Rightarrow}}$ are the smallest relations on T(SIG) resp. Z(T(SIG)) such that

(i) for all $t \in$ T(SIG) $t \underset{E}{\overset{=}{\Rightarrow}} t$,

(ii) for all $f,g \in$ Z(T(SIG)) $f \underset{E}{\overset{=}{\Rightarrow}} g$ if for all $x \in X$ $fx \underset{E}{\overset{=}{\Rightarrow}} gx$,

(iii) for all l=r in E $fl \underset{E}{\Rightarrow} gr$ if $f \underset{E}{\overset{=}{\Rightarrow}} g$,

(iv) $t \underset{E}{\overset{=}{\Rightarrow}} t'$ if $t \underset{E}{\Rightarrow} t'$,

(v) for all $\sigma \in$ OP $\sigma(t_1,..,t_n) \underset{E}{\Rightarrow} \sigma(t_1',..,t_n')$

if $\exists\ 1 \le i \le n : t_i \underset{E}{\Rightarrow} t_i'$ and $\forall\ 1 \le i \le n : t_i \underset{E}{\overset{=}{\Rightarrow}} t_i'$.

4.11. Definition

$\langle t_1,t_2 \rangle \in T(SIG)^2$ is **contextually E-convergent** if there are $n \in \mathbb{N}$, $p_1,\ldots,p_n,q_1,\ldots,q_n \in$ LT, $t_1^1,..,t_n^1,t_1^2,..,t_n^2 \in$ T(SIG) such that
(i) $(p_1 \wedge q_1) \vee \ldots \vee (p_n \wedge q_n) =_{SPEC}$ true ,
(ii) for all $1 \le i \le n$

$$\begin{array}{ccc} t_1 & \overset{E;p_i}{\underset{*}{\longrightarrow}} & t_i^1 \\ & & \Big\Downarrow{=}\,BE \\ t_2 & \overset{*}{\underset{E;q_i}{\longrightarrow}} & t_i^2. \end{array}$$

4.12. Lemma

Let $\langle$BPAR,Log(PSPEC)$\rangle$ be persistent, let PAR be logically consistent and E be contextually normalizing. Let $A \in$ Log(PSPEC). If $\langle t_1,t_2 \rangle \in$ T(SIG) is contextually E-convergent, then for all $f,g \in$ Z(G(SIG(A))) $f \underset{BE}{\overset{=}{\Rightarrow}} g$ implies

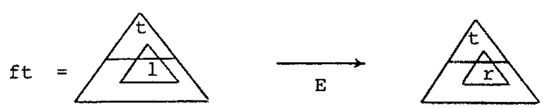
$$\begin{array}{ccc} ft_1 & \overset{E \cup LE(A)}{\underset{*}{\longrightarrow}} & u_1 \\ & & = \Big\Downarrow BE \\ gt_2 & \overset{*}{\underset{E \cup LE(A)}{\longrightarrow}} & u_2 \end{array}$$

for some $u_1,u_2 \in G(SIG(A))$. □

5. Critical pair conditions for consistency

This section is the most technical one. We show that contextual convergence of certain critical pairs is sufficient for confluence, coherence and commutativity of (E-BE) $\cup$ LE(A) (cf. 3.3/5). The assumptions of section 4 are still valid.

To prepare the critical pair conditions we introduce superposition relations (5.1 and 5.8) as those reductions where the lefthand side of the applied equation l=r overlaps a given prefix t of the term to reduce, pictorially:



5.1. Definition

The **simple superposition relation generated by E,**
$\{\xrightarrow[E;f;t]{}\}_{f \in Z(T(SIG)), t \in T(SIG)-X}$, is the family of smallest relations on T(SIG) such that
(i) for all l=r in E $ft \xrightarrow[E;f;t]{} fr$ if ft=fl,
(ii) for all $\sigma \in OP$
$$f\sigma(t_1,..,t_i,..,t_n) \xrightarrow[E;f;\sigma(t_1,..,t_i,..,t_n)]{} f\sigma(t_1,..,t_i',..,t_n)$$
if $ft_i \xrightarrow[E;f;t_i]{} ft_i'$.
An instance of 5.1(i) is **minimal** if f is a most general unifier of t and l.

Let $n(ft \xrightarrow[E;f;t]{} t')$ resp. $n(t \xrightarrow[E]{} t')$ denote the least number of derivation steps 5.1(i)&(ii) resp. 2.11(i)&(ii) that lead to $ft \xrightarrow[E;f;t]{} t'$ resp. $t \xrightarrow[E]{} t'$.

5.2. Proposition

If $ft \xrightarrow[E;f;t]{} t'$, then there are l=r in E, $t_0 \in T(SIG)$, $t_1 \in T(SIG)-X$ and $x \in X$ such that $t=t_0[t_1/x]$, $ft_1=fl$ and $t'=f[fr/x](t_0)$, i.e. l "overlaps" t in ft.
Proof: Straightforward induction on $n(ft \xrightarrow[E;f;t]{} t')$. □

5.3. Proposition

Let t, t' $\in$ T(SIG) and f $\in$ Z(T(SIG)) such that $ft \xrightarrow[E]{} t'$, but not $ft \xrightarrow[E;f;t]{} t'$. Then there are x $\in$ var(t) and $t_x \in$ T(SIG) such that $fx \xrightarrow[E]{} t_x$, $n(fx \xrightarrow[E]{} t_x) \leq n(ft \xrightarrow[E]{} t')$ and

(i) $t' = f[t_x/x](t)$ if t has unique variable occurrences,

(ii) $t' \xrightarrow[E]{*} f[t_x/x](t)$ otherwise.

Proof: Straightforward induction on $n(ft \xrightarrow[E]{} t')$. □

5.4. Definition

E is **linear** if for each l=r in E each variable occurs at most once in l.

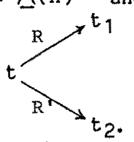
The next lemma provides a syntactical criterion for the commutativity property in Consistency Theorem 3.5.:

5.5. Lemma

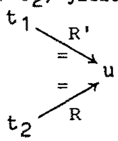
Suppose that <BPAR, Log(PSPEC)> is persistent, E is linear and for each l=r in E l does not contain operation symbols of POP-{true,false}. Then for all A $\in$ Log(PSPEC) E $\cup$ LE(A) commutes with $\triangle(A) \cup \triangle(A)^{-1}$.

Proof:

Let R = E $\cup$ LE(A), R' = $\triangle(A) \cup \triangle(A)^{-1}$ and



Induction on $n(t \xrightarrow[R]{} t_1) + n(t \xrightarrow[R']{} t_2)$ yields

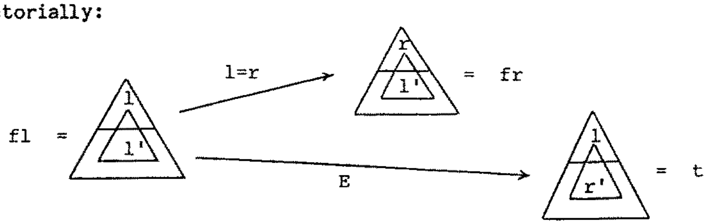


for some u. □

Using the superposition relation we can easily define a **critical pair** of E into l=r as a pair of

(i) the substituted righthand side fr and

(ii) the result t of reducing fl by some equation l'=r' where l' overlaps l, pictorially:



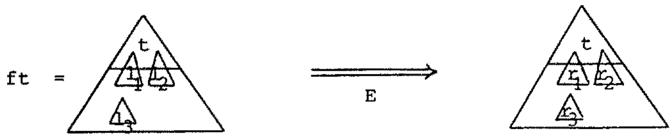
5.6. Definition

Let $fl \xrightarrow[E;f;l]{} t$ and l=r be a SIG-equation. <fr,t> is called a **critical pair of E into l=r.**

5.7. Lemma

Let $A \in Log(PSPEC)$. Suppose that for each $l=r$ in $E-BE$ $op(l)$ contains at least one operation symbol of $OP-BOP$. Then there are no critical pairs of $E-BE$ into $LE(A)$ or of $LE(A)$ into $E-BE$. □

In parallel reductions we may have several equations $l_i=r_i$ applied to the same term u. If all outermost l_i overlap a given prefix t of u, we get a "superposing" parallel reduction, pictorially:



5.8. Definition

The **parallel superposition relation generated by E**,

$\{\underset{E;f;t}{\Rightarrow}\}_{f \in Z(T(SIG)), t \in T(SIG)-X}$, is the family of smallest relations on $T(SIG)$ resp. $Z(T(SIG))$ such that

(i) for all $l=r$ in E $ft \underset{E;f;t}{\Rightarrow} gr$ if $ft=fl$ and $f \overset{=}{\underset{E}{\Rightarrow}} g$,

(ii) for all $\sigma \in OP$ $f\sigma(t_1,\dots,t_n) \underset{E;f;\sigma(t_1,\dots,t_n)}{\Rightarrow} \sigma(t_1',\dots,t_n')$

if $\exists\ 1 \le i \le n$: $ft_i \underset{E;f;t_i}{\Rightarrow} t_i'$ and $\forall\ 1 \le i \le n$: $ft_i \overset{=}{\underset{E;f;t_i}{\Rightarrow}} t_i'$.

Let $n(ft \underset{E;f;t}{\Rightarrow} t')$ resp. $n(t \underset{E}{\Rightarrow} t')$ denote the least number of derivation steps 5.8.(i)&(ii) resp. 4.9.(i)-(v) that lead to $ft \underset{E;f;t}{\Rightarrow} t'$ resp. $t \underset{E}{\Rightarrow} t'$.

5.9. Proposition

If $ft \underset{E;f;t}{\Rightarrow} t'$, then there are $t_0 \in T(SIG)$, $g \in Z(T(SIG))$, $n > 0$ and for all $1 \le i \le n$ $l_i=r_i$ in E, $t_i \in T(SIG)-X$ and $x_i \in X$ such that $t=t_0[t_i/x_i \mid 1 \le i \le n]$, $ft_i=fl_i$, $f \overset{=}{\underset{E}{\Rightarrow}} g$ and $t'=f[gr_i/x_i \mid 1 \le i \le n](t_0)$, i.e. $l_1,\dots,l_n$ "overlap" t in ft.

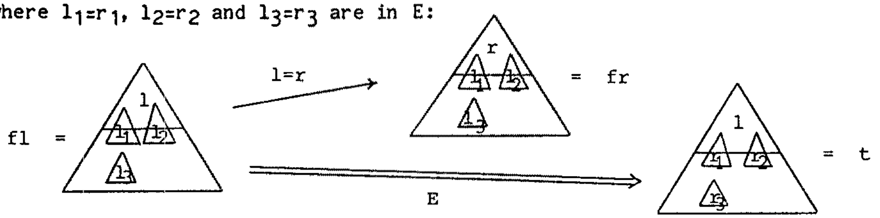
Proof: Straightforward induction on $n(ft \underset{E;f;t}{\Rightarrow} t')$. □

5.10. Proposition

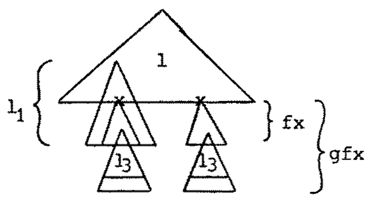
Let $t,t' \in T(SIG)$ and $f \in Z(T(SIG))$ such that t has unique variable occurrences. $ft \underset{E}{\Rightarrow} t'$, but not $ft \underset{E;f;t}{\Rightarrow} t'$. Then there are $n > 0$, $x_1,\ldots,x_n \in var(t)$ and $t_1,\ldots,t_n \in T(SIG)$ such that $fx_i \underset{E}{\Rightarrow} t_i$ and $t' = f[t_i/x_i \mid 1 \leq i \leq n](t)$.

Proof: Straightforward induction on $n(ft \underset{E}{\Rightarrow} t')$. □

Parallel critical pairs of E into l=r arise in situations like the following one where $l_1=r_1$, $l_2=r_2$ and $l_3=r_3$ are in E:



A more complicated case of a parallel overlapping can occur if l_1 shares a subterm of l and a prefix of l_3, e.g.



Applying $(l_1=r_1) \in E$ on one hand and $(l=r),(l_3=r_3) \in R$ on the other hand leads to a **recursive critical pair** of E into R.

5.11. Definitions

Let $fl \underset{E;f;l}{\Rightarrow} t$ and l=r be a SIG-equation. $\langle fr,t\rangle$ is called a **parallel critical pair of E into l=r.**

Let R be a set of SIG-equations and l=r in R. Suppose that $fl \xrightarrow[E;f;l]{} ft$ is derived from a minimal instance of 5.1(i). If for all $x \in X$ $gfx = hx$ or $gfx \underset{R;g;fx}{\Rightarrow} hx$, then $\langle gft,hr\rangle$ is a **recursive critical pair of E into R.**

E is **terminating** if there are no infinite sequences $t_1 \xrightarrow[E]{} t_2 \xrightarrow[E]{} t_3 \xrightarrow[E]{} \ldots$

t is **E-reducible** if $t \xrightarrow[E]{} t'$ for some t'.

5.12. Critical Pair Theorem

Suppose that <BPAR,Log(PSPEC)> is persistent and PAR is logically consistent.
Let E-BE be linear, terminating and contextually normalizing (cf. 4.5), for each l=r in BE var(r) $\subseteq$ var(l) and for each l=r in E-BE l contains at least one operation symbol of OP-BOP.
Let A $\in$ Log(PSPEC). (E-BE) $\cup$ LE(A) is confluent and coherent w.r.t. A (cf. 3.3) if

(i) all critical pairs of E-BE into E-BE,
(ii) all parallel critical pairs of BE into E-BE,
(iii) all recursive critical pairs of E-BE into BE

are contextually (E-BE)-convergent (cf. 4.11).

Proof: Let R = (E-BE) $\cup$ LE(A). A simple proof by Noetherian induction w.r.t. $\xrightarrow[R]{}$ shows that R is confluent w.r.t. A if for all $t, t_1, t_2 \in G(SIG(A))$

$$t \xrightarrow[R]{} t_1, \quad t \xrightarrow[R]{} t_2 \quad \text{implies} \quad t_1 \downarrow_{R,A} t_2 \text{ (cf. 3.3).} \qquad (1)$$

Suppose that for all $t, t_1, t_2 \in G(SIG(A))$

$$\left(t \xrightarrow[R]{} t_1,\ t \xrightarrow[R]{} t_2 \right) \text{ or } \left(t \xrightarrow[R]{} t_1,\ t \overset{BE}{\Longrightarrow} t_2 \right) \quad \text{implies} \quad (*) \quad \begin{array}{ccc} t_1 & \xrightarrow[*]{R} & t_3 \\ & & \Big\Downarrow BE \\ t_2 & \xrightarrow[R]{*} & t_4 \end{array} \qquad (2)$$

for some $t_3, t_4 \in G(SIG(A))$. We prove by Noetherian induction w.r.t. $\xrightarrow[R]{}$ that (2) implies (1) and coherence of R w.r.t. A.
By Lemma 4.6, R is normalizing w.r.t. A. Hence if t_3 is not R-reducible, we have $t_3 \in G(BSIG(A))$ and thus $t_4 \in G(BSIG(A))$ so that $t_1 \downarrow_{R,A} t_2$. If t_3 is R-reducible, then $t_3 \xrightarrow[R]{} t_5$ for some $t_5 \in G(SIG(A))$. We obtain $t_5 \downarrow_{R,A} t_4$ by induction hypothesis and thus $t_1 \downarrow_{R,A} t_2$.
Hence it remains to show (2).

a) Let $t \xrightarrow[R]{} t_1$, $t \xrightarrow[R]{} t_2$.

Induction on $n(t \xrightarrow[R]{} t_1) + n(t \xrightarrow[R]{} t_2)$ and a tedious case analysis leads to (*).

b) Let $t \xrightarrow[R]{} t_1$, $t \overset{BE}{\Longrightarrow} t_2$.

Induction on $n(t \xrightarrow[R]{} t_1) + n(t \underset{BE}{\Rightarrow} t_2)$ and a tedious case analysis leads to (*). □

5.13. Example (cf. 2.7)

Let PAR = <DATA,SET> and BPAR = <DATA,BSET>. One immediately verifies all assumptions of Thm. 5.12 (cf. Exs. 2.14 and 4.9) except for termination of E-BE and the critical pair conditions. For termination we refer to the recursive path ordering method (cf. /5/, /7/), which applied to E-BE = {(e6),...,(e13)} provides a straightforward termination proof.
Assume that there is a critical pair of E-BE into E-BE or a recursive critical pair of E-BE into BE. In both cases we would have l=r in E, $f \in Z(T(SIG))$ and $t \in T(SIG)$ such that $fl \xrightarrow[E\text{-}BE;f;l]{} t$. By Prop. 5.2, there would be l'=r' in E-BE, $t_0 \in T(SIG)$, $t_1 \in T(SIG)-X$ and $x \in X$ such that $l = t_0[t_1/x]$, $ft_1 = fl'$ and $t = f[fr'/x](t_0)$. Since for all l=r in E-BE $op(l) \cap (OP-BOP) = \{root(l)\}$, we conclude $t_0=x$, l=l' and r=r'. Thus we have no recursive critical pair of E-BE into BE, and if <fr,t> is a critical pair of E-BE into E-BE, then fr=fr'=t.
Let $<t_1,t_2>$ be a parallel critical pair of BE into E-BE. Then t_1=fr and

$fl \underset{BE;f;l}{\Rightarrow} t_2$ for some l=r in E-BE and $f \in Z(T(SIG))$. By Prop. 5.9, there are $t_0 \in T(SIG)$, $g \in Z(T(SIG))$, n>0 and for all $1 \le i \le n$ $l_i=r_i$ in BE, $u_i \in T(SIG)-X$ and $x_i \in X$ such that $l = t_0[u_i/x_i \mid 1 \le i \le n]$, $fu_i=fl_i$, $f \overset{=}{\underset{BE}{\Rightarrow}} g$ and $t_2 = f[gr_i/x_i \mid 1 \le i \le n](t_0)$.
Case 1: $t_0 = has(x_1,y)$, $u_1 = ins(s,x)$ and l=r is e7.
Case 1.1: fs = ins(fs',fx) and $l_1=r_1$ is e4. Then
t_1 = ifb(eq(fx,fy),true,has(fs,fy)),
t_2 = has(ins(gs',gx),fy)
so that

$$t_1 \xrightarrow{E\text{-}BE,\,eq(fx,fy)} true \xleftarrow{E\text{-}BE,\,eq(gx,fy)} t_2 \quad\text{and}\quad t_1 \xrightarrow{E\text{-}BE,\,\neg eq(fx,fy)} has(fs,fy) \overset{=}{\underset{BE}{\Downarrow}} t_2.$$

Since
(eq(fx,fy) ∧ eq(gx,fy)) ∨ ¬eq(fx,fy)
$=_{BSPEC}$ eq(fx,fy) ∨ ¬eq(fx,fy) $=_{BSPEC}$ true,
$<t_1,t_2>$ is contextually (E-BE)-convergent.
Case 1.2: fs = ins(fs',fx') and $l_1=r_1$ is e5. Then
t_1 = ifb(eq(fx,fy),true,has(fs,fy)),
t_2 = has(ins(ins(gs',gx),gx'),fy)
so that

$$t_1 \xrightarrow{E\text{-}BE,\ eq(fx,fy) \vee (\neg eq(fx,fy) \wedge eq(fx',fy))} true \xleftarrow{E\text{-}BE,\ eq(gx',fy) \vee (\neg eq(gx',fy) \wedge eq(gx,fy))} t_2$$

and

$$t_1 \xrightarrow{E\text{-}BE,\ \neg eq(fx,fy) \wedge \neg eq(fx',fy)} has(fs',fy) \overset{=}{\underset{BE}{\Downarrow}} has(gs',fy) \xleftarrow{E\text{-}BE,\ \neg eq(gx',fy) \wedge \neg eq(gx,fy)} t_2.$$

Since

$$((eq(fx,fy) \lor (\neg eq(fx,fy) \land eq(fx',fy))) \land (eq(gx',fy) \lor (\neg eq(gx',fy) \land eq(gx,fy)))) \lor (\neg eq(fx,fy) \land \neg eq(fx',fy) \land \neg eq(gx',fy) \land \neg eq(gx,fy))$$

$$=_{BSPEC} ((eq(fx,fy) \lor eq(fx',fy)) \land (eq(gx',fy) \lor eq(gx,fy))) \lor (\neg eq(fx,fy) \land \neg eq(fx',fy))$$

$$=_{BSPEC} eq(fx,fy) \lor eq(fx',fy) \lor (\neg eq(fx,fy) \land \neg eq(fx',fy))$$

$$=_{BSPEC} true,$$

$\langle t_1,t_2\rangle$ is contextually (E-BE)-convergent.

Case 2: $t_0 = del(x_1,y)$, $u_1 = ins(s,x)$ and l=r is e9.
Analogously to case 1 we can deduce that $\langle t_1,t_2\rangle$ is contextually (E-BE)-convergent.

Hence all parallel critical pairs of BE into E-BE are contextually (E-BE)-convergent, and we conclude from Thm. 5.12 that for all A $\in$ Log(PSPEC) (E-BE) $\cup$ LE(A) is confluent and coherent w.r.t. A. □

Thms. 3.5 & 5.12 and Lemmata 4.7 & 5.5 imply the

5.14. Consistency Theorem II

Suppose that <BPAR,Log(PSPEC)> is persistent and PAR is logically consistent.
Let E-BE be linear, terminating and contextually normalizing, for each l=r in BE var(r) $\subseteq$ var(l) and for each l=r in E-BE l contains at least one operation symbol of OP-BOP, but no operation symbols of POP-{true,false}.
If all critical pairs of E-BE into E-BE, all parallel critical pairs of BE into E-BE and all recursive critical pairs of E-BE into BE are contextually (E-BE)-convergent, then PAR is consistent w.r.t. <BPAR,Log(PSPEC)>. □

5.15. Example (cf. 2.7)

Let PAR = <DATA,SET> and BPAR = <DATA,BSET>. Using Thm. 5.14 we conclude from Ex. 5.13 that PAR is consistent w.r.t. <BPAR,Log(PSPEC)>. Hence by Ex. 2.14, PAR is a conservative extension of <BPAR,Log(PSPEC)>. Thus the Decomposition Lemma for Persistency (2.6) implies that <PAR,Log(PSPEC)> is persistent. □

Putting together all "syntactical" criteria developed in this paper we obtain the

5.16. Persistency Theorem II

<PAR,Log(PSPEC)> is persistent if PAR is logically consistent and contains a "base" specification BPAR such that

(i) for all $\sigma \in$ BOP sort(σ) $\in$ PS implies $\sigma \in$ POP,
(ii) for all l=r in BE var(r) $\subseteq$ var(l), and sort(l) $\in$ PS implies that l=r is in PE,
(iii) for all l=r in E-BE l contains at least one operation symbol of OP-BOP, but no operation symbols of POP-{true,false},
(iv) E-BE is linear, terminating and contextually normalizing,
(v) all critical pairs of E-BE into E-BE, all parallel critical pairs of BE into E-BE and all recursive critical pairs of E-BE into BE are contextually (E-BE)-convergent.

(Note also the "Boolean assumptions" at the beginning of section 4.) □

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