ON THE DENSITY OF COLOR-FAMILIES

E. Welzl

Institut für Informationsverarbeitung
Technische Universität Graz
Steyrergasse 17
A-8010 Graz/Austria

Introduction

Recent research of Maurer, Salomaa, and Wood, [2], in the area of finite grammar forms has established a connection between form theory and the theory of graphs.

More specially, the mechanism of interpretation - applied to languages containing only words of length two - comes out to be a generalization of the notion of n -coloring if the languages at issue are interpreted as graphs in the natural way (letters become vertices, a word 'ab' represents an edge from a to b). Thus, considerations concerning graphs yield results which can be applied to finite grammar forms, afterwards. In this paper we present an outline of the proof that color-families are dense (a color-family consists of the interpretations of an undirected graph). By virtue of the above mentioned connection of graph theory and form theory this result can be used to establish the density of certain families of finite grammar forms (see [4]).

For unexplained notions in graph theory I refer to [1]. [6] is a general introduction to the theory of forms.

Preliminaries

We consider finite, undirected, unlabeled graphs, possibly with loops, but without parallel edges.

An *elementary homomorphism* in such a graph G consists of identifying two vertices x and y and inserting edges between the identified vertex $x=y$ and all vertices z adjacent to either x or y in G (when x and y are adjacent, $x=y$ has a loop - this means an edge from the vertex to itself).

A graph G' is a *morphic image* of a graph G if it is obtained from G by finitely many elementary homomorphisms.

Note that this elementary homomorphism, finitely often applied, is an inverse dfl-substitution (μ is a disjoint-finite-letter substitution on

an alphabet V , if $\mu(v)$ is a finite set of letters for all v in V and $v \neq w$ implies $\mu(v) \cap \mu(w) = \emptyset$. Therefore the following definition is natural:

A graph G is an *interpretation* of a graph H if a morphic image G' of G is a subgraph of H .

A *color-family* $\mathcal{L}(G)$, defined by a graph G , is the set of all graphs H which are interpretations of G .

Obviously, the relation $\mathcal{L}(G) \subseteq \mathcal{L}(H)$ holds if and only if G is an interpretation of H , since 'interpretation' is transitive. Moreover, the relation 'interpretation' is decidable. Let K_n be the complete graph with n vertices and let C_m be the cycle with m vertices. Then we have a basic hierarchy among color-families established in [2] :

$$(I) \quad \mathcal{L}(K_2) \subsetneq \dots \mathcal{L}(C_{2m+1}) \subsetneq \mathcal{L}(C_{2m-1}) \subsetneq \dots \mathcal{L}(C_3) = \mathcal{L}(K_3) \subsetneq \\ \subsetneq \mathcal{L}(K_4) \subsetneq \dots \mathcal{L}(K_n) \subsetneq \mathcal{L}(K_{n+1}) \subsetneq \dots$$

Concerning this hierarchy the question arises (especially from the point of view of grammar forms) whether there are color-families strictly between these C_{2m+1} 's and K_n 's, or more general, whether for every pair of families

$$\mathcal{L}(G) \subsetneq \mathcal{L}(H)$$

there is a family $\mathcal{L}(F)$ for which

$$\mathcal{L}(G) \subsetneq \mathcal{L}(F) \subsetneq \mathcal{L}(H)$$

holds, i.e. whether color-families are *dense*. If such a family $\mathcal{L}(F)$ does not exist, we say G is a *predecessor* of H .

Outline of the Proof of the Density of Color-Families

MAIN THEOREM: Color-families are dense (excluding the pair $\mathcal{L}(K_1), \mathcal{L}(K_2)$).

Lemma 1:

(i) If P and Q are predecessors of a connected graph G , then

$$\mathcal{L}(P) = \mathcal{L}(Q) \quad \text{holds.}$$

(ii) Let H be a graph with $\mathcal{L}(H) \subsetneq \mathcal{L}(G)$, and let P be a predecessor of G . Then H is an interpretation of P .

Proof(outline): For two predecessors P and Q the relations $\mathcal{L}(P) \subsetneq \mathcal{L}(Q)$ or $\mathcal{L}(P) \supsetneq \mathcal{L}(Q)$ would be a contradiction to the definition. But if $\mathcal{L}(P)$ and $\mathcal{L}(Q)$ are incomparable, we have a contradiction with the relation $\mathcal{L}(P) \subsetneq \mathcal{L}(P \cup Q) \subsetneq \mathcal{L}(G)$, too ($P \cup Q$ is the disjoint union of the graphs P and Q). Consequently, $\mathcal{L}(P)$ equals $\mathcal{L}(Q)$. Proof of (ii) is similar. $\square$

If $\mathcal{L}(W)$ is a subset of $\mathcal{L}(G)$ and the relation $\mathcal{L}(W') \not\subseteq \mathcal{L}(G)$ holds for no morphic image W' ($\neq W$) of W , then W is called a *weak predecessor* of G . The following theorem shows the interrelation between predecessors and weak predecessors.

Theorem 2:

Let G be a connected graph. G has no predecessor if and only if G has infinitely many weak predecessors (Two weak predecessors V and W are termed different if $\mathcal{L}(V) \neq \mathcal{L}(W)$).

Proof: ($\Leftarrow$) Let W be a weak predecessor of G and let P be a predecessor of G . Then we know (see Lemma 1) that W is an interpretation of P . Thus there is a morphic image W' (of W) which is a subgraph of P . Then

$$\mathcal{L}(W') \subseteq \mathcal{L}(P) \subsetneq \mathcal{L}(G)$$

contradicts the definition of 'weak predecessor', except W' equals W . Therefore all weak predecessors of G are subgraphs of the predecessor P . Obviously, no finite graph P can have infinitely many weak predecessors as subgraphs.

($\Rightarrow$) Let $W_1, W_2, \dots, W_n$ be all the weak predecessors of G . It is easily seen that a graph H with $\mathcal{L}(H) \not\subseteq \mathcal{L}(G)$ has a morphic image H' which is a weak predecessor of G . Consequently such a graph H is an interpretation of $P = W_1 \cup W_2 \cup \dots \cup W_n$. Therefore P is a predecessor of G , which completes the proof. $\square$

Now the proof of the Main Theorem consists of constructing infinitely many weak predecessors for all graphs. We show this explicitly for C_3 .

Lemma 3:

C_3 has infinitely many weak predecessors.

Proof(outline): Construction of a set of graphs $\{S_n\}_{n=3,5,7,9,\dots}$ which are weak predecessors of C_3 :

(i) take a cycle C_{3n} (n an odd integer) and a vertex ' n ' and insert edges between ' n ' and every third vertex in C_{3n} .

(ii) color the graph with the numbers 1,2,and 3 as follows:

- 'n' is 2-colored,
- all vertices which are adjacent to 'n' are 1-colored,
- all vertices which follow a 1-colored vertex in C_{3n} clockwise are 2-colored,
- the remaining vertices are 3-colored.

(iii) insert edges between every 3-colored and every 2-colored vertex (not equal to 'n'), except those pairs which would form a C_3 with a 1-colored vertex. This completes the construction of S_n .

The graph S_5 with C_3 -coloring:

The identification of all vertices in S_n which are colored with the same number results in a C_3 . Thus S_n is an interpretation of C_3 . C_3 is not a subgraph of S_n and every morphic image of C_3 contains a loop. Consequently, C_3 is not an interpretation of S_n and we have the relation

$$\mathcal{L}(S_n) \subsetneq \mathcal{L}(C_3).$$

Moreover, every elementary homomorphism on S_n produces a morphic image S'_n , which has a C_3 or a loop as a subgraph. Thus $\mathcal{L}(C_3)$ is a subset of $\mathcal{L}(S'_n)$ and $\mathcal{L}(S'_n) \subsetneq \mathcal{L}(C_3)$ holds for no morphic image S'_n of S_n . $\square$

Similar constructions and Theorem 2 lead to

Lemma 4:

For $m \in \mathbb{N}$, C_{2m+1} has no predecessor.

Consider an arbitrary graph G with a minimal odd cycle of length $2m+1$ (if a graph has no odd cycle, G is bicolable and $\mathcal{L}(G) = \mathcal{L}(K_2)$). We construct weak predecessors of G by replacing an arbitrary C_{2m+1} by weak predecessors as described above. In this fashion the density of color-families can be shown.

Conclusion

When we come back to the notion of grammar forms the result presented here can be shown to be equivalent to

"($\mathcal{L}(G_1), \mathcal{L}(G_2)$) is a dense pair"

where $G_1 = (\{S\}, \{a, b\}, \{S \rightarrow ab | ba\}, S)$ and

$G_2 = (\{S\}, \{a\}, \{S \rightarrow aa\}, S)$ are two finite context-free grammar forms. The existence of such a dense pair of finite grammar forms was an open problem in the theory of forms, stated e.g. in [3] and [5].

Using this approach, Maurer, Salomaa and Wood, [4], have shown that $(\mathcal{L}(\text{REG}), \mathcal{L}(\text{CF}))$ is not maximally dense.

Perhaps, some other open problems for grammar forms (like (i) characterisation of subregular grammar forms, for which $(\mathcal{L}(G), \mathcal{L}(\text{REG}))$ is dense, (ii) properties of minimal grammar forms which induce a grammatical family or (iii) the complexity of membership in grammatical families) can be settled by a detour to "general coloring of graphs".

References

- [1] Harary F., Graph Theory. Addison-Wesley (1972).
- [2] Maurer H.A., Salomaa A., Wood D., Colorings and Interpretations: A Connection Between Graphs and Grammar Forms. Report 49, Institut für Informationsverarbeitung, TU Graz (1980).
- [3] Maurer H.A., Salomaa A., Wood D., On Finite Grammar Forms. Report 51, Institut für Informationsverarbeitung, TU Graz (1980).
- [4] Maurer H.A., Salomaa A., Wood D., On Predecessors of Finite Languages. Report 57, Institut für Informationsverarbeitung, TU Graz (1980).
- [5] Salomaa A., Morphisms on Free Monoids and Language Theory, in: Formal Language Theory - Perspectives and Open Problems, edited by R.V. Book. Academic Press (1980).
- [6] Wood D., Grammar and L Forms: An Introduction. Springer Lecture Notes in Computer Science 91 (1980).