Risch's Theorem

Introduction

Integration is probably the area of computer algebra with the greatest disparity between users' expectations and current systems' performance. Although heuristic techniques for integration are well developed (Moses,1967), there are still many advantages in having a decision procedure which can, for example, prove that an expression is unintegrable (Moses,1971).

This chapter describes an underlying body of theory to the area of finding (or proving non-existent) the elementary integrals of algebraic functions, where a function is *algebraic* if it can be generated from the variable of integration and constants by the arithmetic operations and the taking of roots of equations*, possibly with nesting (see Chapter 2 for a detailed discussion of such expressions and how we choose to represent them). By *elementary* we mean generated from the variable of integration and constants by the arithmetic operations and the taking of roots, exponentials and logarithms, possibly with nesting (this notion is made precise below). As a particular case of this problem, we have the question of deciding when an integral which at first sight appears to be elliptic can in fact be expressed in terms of elementary expressions. This problem was long thought to be insoluble, and Hardy summarised the classical position when he said

> "No method has been devised whereby we can always determine in a finite number of steps whether a given elliptic integral is pseudo-elliptic, and integrate it if it is, and there is reason to suppose that no such method can be given". (Hardy,1916 pp. 47-8).

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* The theory does not require that these roots should be expressible in terms of radicals. As mentioned in chapter 2, the implementation is currently restricted to algebraic quantities expressible in terms of square roots.

We shall explain a new algorithm for this problem, based on techniques from algebraic geometry (see the previous chapters), and discuss the implementation of this and the many open problems that remain in this area. We shall use the terminology and ideas of the previous chapters without comment. Note that this approach is different from Ng(1974), where the concern was with finding canonical forms for non-elementary elliptic and hyper-elliptic integrals. It is not certain quite what the relationships between these two techniques for attacking different but related problems are.

The problem "can we integrate this function?" might appear simple to pose, but this is not the case. In particular, suppose that we have an algorithm for deciding if algebraic functions over K are insoluble, where K is the field generated from the integers, log 2 and π by means of the operations of addition, subtraction, multiplication and division, and the functions $x \Rightarrow \log x$, $x \Rightarrow \exp x$, $x \Rightarrow |x|$. Then let $f(x)$ be any unintegrable function (e.g. $\sqrt{(1-x^2)(1-kx^2)}$ with k neither 0 nor 1) and A any expression which cannot be determined to be identically 0 or not (such expressions exist by the work of Richardson (1968)). Then consider whether $Af(x)$ is integrable: clearly an insoluble question. This is similar to Risch's exposition (Risch, 1969, Proposition 2.2), and the reader is referred to his work for further explanations of computability. Risch's work shows that we need a firm theoretical base for our discussion of integration, and this is provided by the subject of Differential Algebra, which we now describe.

Differential Algebra

Before we can discuss the problems of integration, we require a definition of what we mean by integration and integral. The obvious answer, and the definition we adopt, says that $F(x)$ is the integral of $G(x)\mathrm{d}x$ iff $G(x)$ is the derivative of $F(x)$ with respect to x. This merely converts the problem from the definition of integration into the problem of differentiation. The study of differentiation as an algebraic operation (rather than an analytic one) is the province of *Differential Algebra,* a field of Mathematics founded by J.F. Ritt, although elements of it go back to Liouville and Laplace. We give here a very brief summary of elementary differential algebra in order to introduce the terminology we will need for our studies of differentiation and integration. For further details see the introduction to the subject by Kaplansky (1957), the survey by Ritt (1950) or the unified exposition of differential algebra by Kolchin (1973).

By a *differential field* we shall mean a field* together with a family D_i of unary operators (written as $a \rightarrow Da$) on the field (the *differentiations* of the field) satisfying:

$$D(a + b) = Da + Db,$$

$$D(ab) = aDb + bDa$$

for all a and b in the field and for each differentiation D. We define K to be a *differential extension* of L iff K is a field extension of L and every differentiation of K can be regarded as a restriction of a differentiation of L. Where there is only one differentiation D, we sometimes write a' for Da.

Any element which has image 0 under all differentiations is said to be a *constant* of the field. We obtain immediately that $D0=0$, and then that $D1=0$, so that any integer is a constant in the sense just defined. Except in chapter 6 we will only be dealing with one differentiation, namely the one that is the opposite to the integration we wish to perform. If X is the variable we wish to integrate with respect to (called *the variable of integration*), then we can convert $\{K(X,Y) \mid F(X,Y) = 0\}$ into a differential field by defining a' to be 0 for all a in K, $X'=1$ and $Y' = -(F_X/F_Y)$, where by F_X we mean the partial derivative of F with respect to X done purely formally (i.e. the derivative of cX^n is cnX^{n-1} and this partial differentiation is an additive operation). More generally, e.g. in the case of multivariate representations, if $F(X,Y,Z,.,W) = 0$, then $0 = F' = X'F_X + Y'F_Y + Z'F_Z + \ldots + W'F_W$, and this determines W' in terms of $X,Y,Z,\ldots,W$. This clearly makes the field into a differential field, and integration is defined to be the inverse operation to differentiation where it is defined.

* Of characteristic 0 unless otherwise stated. "Ritt himself had no use for fields of non-zero characteristic and referred to them as 'monkey fields' " (Kolchin, 1973, p. xiii). I hope to be able to show later (see Chapter 8 in particular) that these fields are of some use in the theory of integration.

If K is a differential field and x,y belong to K with y non-zero, and $Dx=(Dy)/y$ for each differentiation D of K, we say that x is a *logarithm* of y, or that y is an *exponential* of x. Note that this agrees with the standard (analytic) definitions, in that the equation $Dx=(Dy/y)$ is normally taken as the definition of the exponential function of classical calculus, and that logarithms are defined as the inverse of exponentials. A differential extension field L of K is termed an *elementary extension* of K iff it is of the form $K(t_1,\ldots,t_n)$, where for $1 \leq i \leq n$ we have one of:

t_i is a logarithm of an element in $K(t_1,\ldots,t_{i-1})$

t_i is an exponential of an element in $K(t_1,\ldots,t_{i-1})$

t_i is algebraic over $K(t_1,\ldots,t_{i-1})$.

This definition makes precise the concept of an elementary function, in that it is one that can be expressed as a member of an elementary extension of a constant field.

Risch's Theorem

Just as the integral of a rational function is a rational function plus the sum of logarithms with constant coefficients, so the integral of an algebraic function, if it is elementary, is the sum of an algebraic function and logarithms of algebraic functions, these logarithms having constant coefficients. We shall say that such an integral consists of an *algebraic part* and a *logarithmic part.* An integral with no logarithmic part is said to be *purely algebraic* and one with no algebraic part is said to be *purely logarithmic.*

The first important result about the form of these integrals is Laplace's Principle (Hardy, 1916 pp. 9-10), which says that the integral of an algebraic function can be expressed so as to contain only those algebraic quantities which occurred in the integrand. This result can be proved by "elementary" means and Hardy gives a sketch of how to do so (pp. 36-42 and 46). However this is an easy consequence of our general work, see Corollary 2 below.

When one integrates a rational function, the logarithmic part comes precisely from the integration of terms such as $1/(X-k)$ in a partial fraction decomposition of the integrand, and the algebraic part comes precisely from the integration of all the other terms. This remark was generalised in a precise manner to algebraic functions by Risch, who stated the following Theorem (1970):

"Let w be a differential in $\{K(X,Y) \mid F(X,Y) = 0\}$. Let $r_1,\ldots,r_k$ be a basis for the $\mathbf{Z}$-module generated by the residues of w. Thus, at each K-place P, w has residue $\sum_{i=1}^{k} a_{iP} r_i$ with a_{iP} in $\mathbf{Z}$. Let the divisor d_i be given by multiplicities a_{iP} at the places P. Then, iff $\int w$ is elementary, there are $v_0,\ldots,v_k$ in $K(X,Y)$ and integers $j_1,\ldots,j_k$ such that d_i to the power* j_i is the divisor of the function v_i and

$$w = \mathrm{d}v_0 + \sum_{i=1}^{i=k} \frac{r_i}{j_i} \frac{\mathrm{d}v_i}{v_i}$$

i.e.

$$\int w = v_0 + \sum_{i=1}^{i=k} \frac{r_i}{j_i} \log v_i".$$

In this theorem, v_0 is the algebraic part of the answer, and the remainder is the logarithmic part, which depends only on the residues of the integrand. This is a straightforward generalisation of standard integration methods for rational functions, except that there is the possibility that the divisors may be rationally equivalent to 0, instead of linearly equivalent. If they are, then j_i is that integer multiplier which makes d_i into the divisor of a function. This theorem immediately gives rise to a possible algorithm for integrating algebraic functions:

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* in the sense of multiplication of divisors introduced in Chapter 3; i.e. with all the multiplicities multiplied by j_i and the places unchanged.

RISCH_ALGEBRAIC

Input:

F(X,Y): the equation of an algebraic curve

f(X,Y): a function of *X* and *Y*

(*w* will then be *f*(*X*,*Y*)d*X*)

Output:

I : The integral of f(X,Y)dX,

(or NOT_ELEMENTARY if there is no elementary integral)

[1] POTENTIAL_POLES:= "INFINITY"

combined with all factors of the denominator of f(X,Y)

(which is a polynomial in *X* by the remarks on representations of algebraics contained in Chapter 2).

[2] RESIDUES:=for each U in POTENTIAL_POLES

for each place V lying over U

Collect (V,Residue of f(X,Y) at V)

[3] Z_BASIS:=basis for the ***Z***-module of the residues formed in step [2]

[4] For each Ri in Z_BASIS do

[4.1] Di:= for each U in RESIDUES collect:

(Place.coefficient of Ri in residue).

This corresponds to the divisor d_i in Risch's Theorem.

[4.2] Ji:=FIND_ORDER Di.

By Risch's theorem, the integral can only exist if the divisor Di is of finite order. This tells us the order of the divisor Di, i.e. which is the least power of it which is linearly equivalent to 0. If there is none such, then this returns INFINITE.

[4.3] If Ji = INFINITE, then return NOT__ELEMENTARY.

[4.4] INTEGRAL:=INTEGRAL+
(Ri/Ji)*log(DIVISOR__TO__FUNCTION(F(X,Y),Di**Ji))

We have now found all the logarithmic parts of the integral, which we can now remove from the integrand in order to find the algebraic part.

[5] Now find the algebraic part of the answer.

[5.1] f(X,Y):=f(X,Y)−derivative of INTEGRAL.

[5.2] ALG__PART:=FIND__ALGEBRAIC__PART f(X,Y)

[5.3] If ALG__PART = NOT__ALGEBRAIC
then return NOT__ELEMENTARY.
In this case, we could find all the logarithmic parts of the integral, but not the algebraic part.

[6] return INTEGRAL + ALG__PART.

This is, in fact, a perfectly workable algorithm, subject to the definition of the subsidiary algorithms. Many of these, although their precise description may appear strange, are perfectly familiar algorithms of computer algebra. The factorisation in step [1] is merely in $K[X]$ because of the remark in Chapter 2 which says that we can represent an element of $\{K(X,Y) \mid F(X,Y) = 0\}$ as a polynomial in X and Y divided by a polynomial in X. The finding of the basis for the $\mathbf{Z}$-module in step [3] can be re-expressed as asking which rows of a matrix are linearly independent, and this is easy to answer, although not easy to answer efficiently for large matrices. If this matrix is large, though, the integrand which gave rise to it is almost certainly far too large to integrate anyway, so this point should not concern us much.

The problem of finding the residue at a place requires the computation of the t^{-1} term in the Puiseux expansion (where t is a local parameter at the place in question). As described in Chapter 2, Puiseux expansions are really just Laurent expansions, and can be computed by the techniques described in Chapter 2, based on those in Norman (1975). Although a significant body of code is required to perform these expansions correctly and efficiently, the difficulties involved are technical rather than mathematical.

The algorithm DIVISOR__TO__FUNCTION is described in Chapter 3. FIND__ORDER is harder, and we will return to it in a later Chapter.

Proof of Theorem

Risch enunciated the theorem mentioned above as a consequence of his major theorem on integration (the proof of which was not included in the published work, though a proof can be deduced from (Risch, 1969, pp. 171 et seq)). It is therefore necessary to prove this theorem before we rely on it for the construction of a theory of integration. The theorem is stated as an "if and only if" theorem, but if w has the form stated, then it is clearly elementary. Hence the major task is to prove that all elementary differentials are of this form. Rather than adapt Risch's earlier proof (which was mostly concerned with transcendental functions), we shall give a new short one, relying on a recent result of Rosenlicht (1976).

We can observe intuitively that differentiation does not remove exponentials, or logarithms unless they occur as $\ldots + c \log f(x)$. Furthermore differentiation does not remove algebraics. Therefore it seems intuitively reasonable that the integral of an algebraic function, if it is elementary, can be expressed in the form $u(x) + \Sigma\, c_i \log u_i(x)$, where the c_i are constants. This remark was made by Laplace, and first proved, by an ingenious descent argument, by Liouville(1833c). This proof can be found in Ritt's book on Liouville's work (Ritt, 1948, p.20). We will base our attack on the more advanced methods of abstract algebra, since it seems likely that any generalisation will have to come from this source.

Lemma 1 For $f(X)$ in $L = \{K(X,Y,Z,\ldots) \mid F(X,Y) = 0 \quad G(X,Y,Z) = 0\ldots\}$ with K algebraically closed, $f\mathrm{d}X$ is elementary in the sense that there is an elementary extension of

L containing an element g with $f=g'$, iff there are constants $c_1,...,c_n$ linearly independent over the integers in L and elements $u_1,...,u_n,v$ in L such that $f = v' + \Sigma c_i \frac{u'_i}{u_i}$.

Proof: Apart from the part about linear independence of the c_i, this is a restriction of Theorem 3 of Rosenlicht(1976) to the case of one differentiation of a special form. And if the c_i are not linearly independent over the integers, we can combine terms to make them so, using the identity $D(u^k)/(u^k) = kDu/u$.

Corollary 2 The elementary integral of an algebraic function can contain no algebraics depending on the variable of integration other than those which occur in the integrand*.

Proof: If the integral is elementary, then it has the form specified in the Lemma above, and so its integral is $v + \Sigma c_i \log u_i$ where v and the u_i belong to the differential field in which the original integrand lay, and hence are expressible in terms of the same algebraics.

Note the importance of the phrase "depending on the variable of integration" in the above statement of Laplace's Principle. The Principle is not true without this restriction, as can be seen from Risch (1969, Proposition 1.1) who considers the integral of $1/(X^2-2)$, which necessarily contains $\sqrt{2}$.

Lemma 3 The field of power series in one variable $K((t))$ is a differential field with $\Sigma a_i t^{i\prime} = \Sigma iat^{i-1}$.

Proof: Addition is obvious, and multiplication follows from comparing the left and right hand sides.

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* This is Laplace's Principle (Hardy, 1916, pp. 9-10) quoting Laplace, "L'intégrale d'une fonction différentielle (algébrique) ne peut contenir d'autres quantités radicaux (sic) que celles qui entrent dans cette fonction" [I have been unable to find precisely this form of words in any of Laplace's works, but I did discover the following: "l'intégrale d'une fonction différentielle ne peut renfermer d'autres quantités exponentielles et radicales que celles qui sont contenues dans cette fonction" (Laplace, 1820, livre I, p. 5).] Note that Laplace did not state explicitly that this only relates to algebraics depending on the variable of integration. Another proof can be found in Ritt's work (1948, pp. 28-31).

Lemma 4 If x is the variable of differentiation, and $dz=y\ dt$, then $y=z'$, where ' is the "power-series" differentiation of the previous lemma.

Proof: This is Theorem 8 of (Lang, 1957, p.247).

Corollary 5 If t is a local variable, the Puiseux expansion of $f'\ dx/dt$ is the derivative (in the sense of Lemma 3) of the Puiseux expansion of f.

Corollary 6 If F has a purely algebraic integral, and the Puiseux expansion of F is $\Sigma\, a_i t^i$, then the Puiseux expansion of its integral is $\Sigma\, \frac{a_i t^{i+1}}{i+1}$.

Lemma 7 The Puiseux expansion of the derivative of log $F(X)$ is that of $F'(X)/F(X)$, in particular, if $F(X)$ has a zero (or pole) of order n at the place P, then $(\log F(X))'$ has a residue of n at this place.

Proof: The first part follows from the definition of logarithms given earlier, which characterises them by precisely this property. The second part follows from a consideration of Puiseux expansions: F has Puiseux expansion $a_n t^n + .$ and F' has expansion $n a_n t^{n-1} + \ldots$, giving a quotient of $nt^{-1} + \ldots$, as required.

Proof of Theorem. If w has an expression of the form stated, then it is clearly integrable. Conversely, suppose w is integrable. Then by Lemma 1, we can write w as $v dx + \Sigma\, c_i(\log u_i)' dx$, and we can assume that none of the u_i are constant, for if they were then they cannot contribute to the derivative of the integral. Choose such an expression with n minimal. Now, by Lemma 7, the residue of w at P is $\Sigma\, c_i b_{iP}$, where b_{iP} is the order of u_i at P. Also, since each u_i has at least one pole and the c_i are linearly independent, each c_i appears in at least one residue of w. Hence the c_i form a basis for the set of residues of w (or possibly of a larger module, but certainly one which only includes fractional multiples of the residues). So, if r_i is our basis for the set of residues of w, considered as a Z-module, then there are integers j_i such that the set $\{r_i/j_i\}$ is a basis for the set of c_i, so that we can write c_i as $\Sigma\, \frac{n_{ij} r_j}{j_j}$. And then we can re-express the sum of logarithms to have coefficients r_i/j_i and the theorem is proved.

Corollary 8 An integrand with no residues can have no logarithmic part in its integral.

The Algebraic Part

At step 5.2 in the integration algorithm outlined above, we are left with a function to integrate whose integral, if it is elementary, is purely algebraic. This is a much easier problem than the general one, and was considered, and essentially solved, more than 150 years ago (Liouville, 1833a, 1833b). We describe here a way of solving this problem which interacts closely with the rest of this work, and which is frequently very efficient since many of the computations have been made in the process of finding the logarithmic part of the integral. There are several ways in which this problem can be solved, and the one described here is not necessarily the most efficient,# it is merely the one which is closest to our algebraic-geometric view of the problem of integration.

This algorithm is based on the observation that the poles of an algebraic function $f(X,Y)$ and its derivative $f'(X,Y)$ are closely related (see Corollary 6 above). In fact, where f' has a pole of order n, f has a pole of order $n-1$ (or $n+1$ if the place lies over infinity). This is a generalisation of the usual rule for integrating powers of X, and can easily be proved by appealing to Puiseux expansions. We cannot say much about the zeros of f, because the Puiseux expansion of f could have a constant term which will disappear on differentiating.

There is one difficult point in this process. We cannot completely determine the integral this way, since the integral is indeterminate up to a constant of integration. Initially I tried various strategies for determining whether or not the indeterminacy in the linear combination of the functions with appropriate poles was due to this constant of integration or not, but I could not find a reasonable method of doing this. I therefore adopted the strategy of choosing a point P (not a pole of the integrand) and demanding

In the case of simple radical extensions, the theory simplifies substantially (and is essentially due to Chebyshev (1853)). Trager (1979) has recently published an algorithm for finding the algebraic part in this case which appears to be very efficient, since it only relies on polynomial algorithms, but I have been unable to find an implementation of this in order to obtain some comparative timings.

that the algebraic part of the integral have the value 0 there, i.e. making a definite choice for the value of the constant of integration. This leads to some slightly curious integrals at times, but there is clearly scope for a heuristic post-pass to choose more "appealing" values of the constant of integration after the complete integral has been found.

FIND_ALGEBRAIC_PART

This algorithm is stated for the simple cases; there are complications when $(X-a)$ is not a local parameter near $X=a$, but the principle is the same.

Input:

F(X,Y): the equation of an algebraic curve

f(X,Y): a function of X and Y

(w will then be $f(X,Y)\mathrm{d}X$)

Output:

FUNCTIONS : The integral of $f(X,Y)\mathrm{d}X$,

(or NOT_ALGEBRAIC if there is no algebraic integral)

[1] POTENTIAL_POLES:= "INFINITY"

and all factors of the denominator of f(X,Y) (which is a polynomial in X).

[2]

DIVISOR:=for each U in POTENTIAL_POLES

for each place V lying over U

Collect (V,order of f(X,Y) at V).

This order is computed from the Puiseux expansion of f(X,Y) at V.

[3] For each U in DIVISOR

ORDER:=minimum(0,ORDER+1).

Or ORDER−1 if the place lies over infinity. The + and − signs have changed since the description of the theory above, since the order of a pole is always negative.

[4] FUNCTIONS:=COATES(F(X,Y),DIVISOR).

We actually need a version of COATES which will deal with places at infinity, but, as explained in Chapter 3, this is not particularly difficult.

[5] For P = 1,2, ... do:
if there is no element of DIVISOR lying over P
then go to 6.

This loop must eventually terminate, because DIVISOR is a finite list of poles. P is then the point at which we can insist that the integral must have the value 0, in order to fix the value of the constant of integration.

[6] Let A[i] be the value of FUNCTIONS[i] at P.
Eliminate one of the FUNCTIONS
by using the constraint $\Sigma A[i] = 0$.

[7] For each U in DIVISOR with ORDER¬=0:
For each At^{-k} term in the Puiseux expansion of $f(X,Y)$:
Ensure that the integral has an $-At^{-k'}/k'$ term, where k' is $k+1$ (or $k-1$ if J lies over infinity). This is done by applying a linear constraint to FUNCTIONS as produced in step 4, and in practice these linear constraints are not dense for positive k. We cannot do this for $k'=0$.

[8] When FUNCTIONS is reduced to one possibility,
differentiate it, and then we have $f(X,Y)$ (in which case this is the integral) or not, in which case $f(X,Y)$ has no algebraic integral, and the answer is NOT_ALGEBRAIC.

This algorithm is sufficient to prove that many standard integrands are unintegrable, e.g. $1/\sqrt{(x^2-1)(x^2-k)}$, which has no poles.

A Partial Algorithm

Even without an effective realisation of FIND_ORDER the above techniques give us a partial algorithm for integrating algebraic functions: partial in the sense that if the algorithm is presented with an integrable algebraic function, then it will terminate with the integral; if the function is not integrable then it may return NOT_ELEMENTARY or it may run for ever.

We recall that the order of a divisor on an algebraic curve is defined as the smallest power of it such that there is a function corresponding to it, i.e. such that Coates' Algorithm succeeds on it. Hence we can produce a place holder for the FIND_ORDER procedure which works by trying each power of Di in turn until it finds such a power. This procedure will of course run for ever if the divisor is rationally inequivalent to zero.

This approach does demonstrate that we need only have a bound on the order of a divisor, rather than its actual order. If we assume that all divisors have order 1, then we will be correct for curves of genus 0, i.e. those integrands that could be solved by a trigonometric substitution. The more elaborate order-finding processes discussed below are a reflection of the difficulty of identifying elementary integrals among apparently elliptic and hyper-elliptic integrals.

We note that this partial algorithm is complete for all integrals that do not contain a logarithmic part, since these integrands will have no residues.

Efficiency Points

This algorithm is not necessarily efficient, even in the case of integrals which correspond to divisors of order 1. For example, consider integrating the function

$$\frac{1}{\sqrt{x^2+1}} + \frac{100}{\sqrt{x^2+10000}}.$$

This has residues of 99,101,−99,−101 at the four places lying over infinity. The corresponding function found from the call of DIVISOR_TO_FUNCTION in step 4.4 of the algorithm is

$$\frac{x^{101}}{(\sqrt{x^2+1}-1)(\sqrt{x^2+10000}-100)^{100}},$$

and, as explained in the previous chapter, finding this function by naive methods would involve the solution of a 200 by 200 set of dense linear equations. Even using the method of DIVISOR_TO_FUNCTION, we still need to compute the answer from DIVISOR_TO_FUNCTION, and its denominator is a dense even polynomial of degree 100 in X. Fortunately this function does have structure and can be computed in a non-expanded form, so this is not as much of a problem as might appear. Some examples on these lines are quoted as Example 8 in Appendix 2. Integrating each term separately involves almost no work, however, so it is clear that we should do this wherever possible, for we shall reduce the average running time of the algorithm, even when we do not decrease the maximum running time. In other words, there is significant scope for heuristics to improve the performance of the system, even if they do not increase the range of soluble problems.

An earlier version of the program described in this monograph, and the transcendental integration package of Norman & Moore (1977), have been incorporated with a pattern matcher and set of rules for user defined integrals into the algebra system REDUCE by Harrington(1977, 1979b), and his techniques, which rely on ours or Norman & Moore's as appropriate, are probably the best approach to a practical integration system, since they combine the elegance and certainty of an algorithmic approach with the speed of a semi-heuristic approach.