VIII. RECURSIVELY DEFINED FUNCTIONS

1. Recursive Equations Represented by Rule Schemata

Recursive equations provide a convenient language for defining computable functions. Recent studies by Cadiou [Cad72], Vuillemin [Vu74], Downey and Sexthi [DS76] and Berry and Lévy [BL77] separate computational steps involving recursively defined functions from those involving given primitive functions, and are based on continuous interpretations of primitive functions with fixpoint semantics for computed functions. The SRS approach in this section does not depend on fixpoint semantics, and follows Rosen [Ro73] in unifying the sequencing problems related to defined and given functions.

The strong syntactic restrictions on the forms of recursive equations in [Ca72], [Vu74], [DS76] allow simpler proofs of confluence, termination and optimality properties than those in Chapters II, III and IV, but the SRS approach has the virtue of demonstrating that some of these restrictions are not essential. A short discussion of the fixpoint approach appears in Section 4. Section 5 relates the schemata and continuous semantics versions of primitive functions.

Let $\mathcal{C}$ be a set of constants, $\mathcal{V}$ a set of variable symbols, $\mathcal{G}_n$ a set of n-ary function constant symbols for each $n \varepsilon \mathcal{P}$, $\mathcal{F}_n$ a set of n-ary function symbols to be defined recursively. Assume that $\forall n\ \mathcal{F}_n \cup \mathcal{G}_n = \phi$.

Let $\Sigma = \bigcup\{\mathcal{C}, \mathcal{G}_n, \mathcal{F}_n | n \varepsilon \mathcal{P}\}$. For each $f \varepsilon \mathcal{F}_n$, consider a single recursive equation $f(\underline{X}_1,\ldots,\underline{X}_n)=A$ where $f \varepsilon \mathcal{F}_n$, $\underline{X}_1,\ldots,\underline{X}_n \varepsilon \mathcal{V}$, $A \varepsilon (\Sigma \cup \mathcal{V})_{\#}$ and A contains no variables other than $\underline{X}_1,\ldots,\underline{X}_n$.
$[f(\underline{X}_1,\ldots,\underline{X}_n)=A]$ is a rule schema. Let

$\mathcal{S}_f = \{[f(\underline{X}_1,\ldots,\underline{X}_n)=A]\}$, $\mathcal{S}_{\mathcal{F}} = \bigcup\{\mathcal{S}_f | f \varepsilon \mathcal{F}_n, n \varepsilon \mathcal{P}\}$.

Suppose that, for each $g \varepsilon \mathcal{G}_n$, $n \varepsilon \mathcal{P}$, we also have a nonoverlapping consistent set $\mathcal{S}_g$ of rule schemata of the form $[g(a_1,\ldots,a_n)=b]$ with $a_1,\ldots,a_n, b \varepsilon \mathcal{C} \cup \mathcal{V}$. $\mathcal{S}_g$ must be valid in the sense that whenever $\langle A=B \rangle$ is an instance of a schema in $\mathcal{S}_g$, the equation $A=B$ is true for the intended interpretation of g. In addition, it is desirable that $\mathcal{S}_g$ represent all available information about g. In Sections 3 and 5 we will see several ways to generate such a set given a semantic interpretation of g.

Let $\mathcal{S}_{\mathcal{G}} = \bigcup\{\mathcal{S}_g | g \varepsilon \mathcal{G}_n, n \varepsilon \mathcal{P}\}$ and $\mathcal{S} = \mathcal{S}_{\mathcal{F}} \cup \mathcal{S}_{\mathcal{G}}$. Now let $\mathcal{S}$ generate A and r. Using Lemma 23 (p. 70), we see that $\mathcal{S}$ is nonoverlapping, consistent and outer, so the SRS Rec = $\langle \Sigma, \Sigma_{\#}, \rightarrow, A \rangle$ is commutative with respect to (innermost preserving) r, and has the outer property.

Consider any $f \varepsilon F_n$ and $c_1,\ldots,c_n,d\varepsilon C$. The singleton Σ-tree $d\varepsilon\Sigma_{\#}$ is in normal form, and $f(c_1,\ldots,c_n)\to^*d$ iff $f(c_1,\ldots,c_n)=d$ follows from the rule schemata S generating the equations A (see Th. 5, p. 29). Also $f(c_1,\ldots,c_n)\to^*d$ iff $\tilde{f}(c_1,\ldots,c_n)=d$ where $\tilde{f}$ is the meaning of f given by fixpoint semantics [Ro73] (Th. 8.4, p. 184). Reduction sequences form a rich class of computations of recursively defined functions, mimicking as special cases the types of computations considered by Cadiou [Cad72] (1.4.2, pp. 12-15), Vuillemin [Vu74] (1.3, 1.4, pp. 336-343) and Downey and Sethi [DS76] (Section 3, pp. 382-384). So, all of the following results about reduction sequences may be interpreted as statements about computations of recursively defined functions.

<u>Theorem 19</u> (Recursive Equation Theorem).
Let Rec be a SRS constructed by the above methods to represent a set of recursive equations with primitive functions. Then Rec has the following properties:

(1) Normal Forms are unique (i.e., recursively defined functions are single valued).
(2) Eventually outermost reduction sequences terminate whenever possible.
(3) Strictly innermost sequences are infinite whenever possible.
(4) Some strictly outermost noncopying sequence is optimal.
(5) Strictly innermost noncopying sequences have maximal cost.

Proof:

(1) Theorem 6 (p. 38) and Theorem 17 (p. 70).
(2) Theorem 10 (p. 50) and Theorem 18 (p. 71).
(3) Theorem 11 (p. 53) and Theorem 16 (p. 67).
(4) Theorem 13 (p. 63) and Theorem 17 (p. 70).
(5) Theorem 12 (p. 60) and Theorem 17 again. □

(1) is well known. Vuillemin's "delay-rule" is related to (4).

2. <u>Rule Schemata for Primitive Functions</u>

In some cases elegant nonoverlapping sets of rule schemata for a function symbol will be suggested by popular sets of axioms for that symbol. For instance, the functions v and <u>if then else</u> may be represented by

$S_v = \{[v(T,\underline{X})=T],[v(\underline{X},T)=T],\ [v(F,F)=F]\}$

$S_{\underline{\text{if then else}}} = \{[\underline{\text{if then else}}\ (\underline{T},\underline{A},\underline{B})=\underline{A}],$
$[\underline{\text{if then else}}\ (F,\underline{A},\underline{B})=\underline{B}]\}.$

$T,F\varepsilon\mathcal{C}$ represent truth and falsehood. In general, the construction of S_g depends on what kind of knowledge we have about g.

Consider an interpretation $\langle D,v\rangle$ of $\mathcal{C} \cup \bigcup\{G_n | n\varepsilon P\}$ (Def. 6, p. 8). Assume that every constant symbol is assigned a different value by v. Given v, we may define

$$S_g = \{[g(c_1,\ldots,c_n)=d] \mid c_1,\ldots,c_n,d\varepsilon\mathcal{C} \text{ and } \hat{v}(g(c_1,\ldots,c_n))=\hat{v}d\}.$$

Rosen's rule schemata $\mathcal{R}[g]$ [Ro73] (p. 178) may be generated in this fashion. Each S_g is nonoverlapping and consistent by Lemma 23 (p. 70). Note that $[g(c_1,\ldots,c_n)=d]\ \varepsilon\ S_g$ iff $\langle D,v\rangle \models g(c_1,\ldots,c_n)=d$.

More powerful sets of rule schemata may be obtained by using schema variables. Given $\langle D,v\rangle$ above, suppose that

$(vg)(vc_1,\ldots,\underline{a},\ldots,vc_n)=vd$ for all $\underline{a}\ \varepsilon\ D$.

Then, the entire set of schemata

$\{[g(c_1,\ldots,b,\ldots,c_n)=d] \mid b\varepsilon\mathcal{C}\} \subseteq S_g$ may be replaced by the singleton $\{[g(c_1,\ldots,\underline{X},\ldots,c_n)=d]\}$. In general, for $g\varepsilon G_n$, $d\varepsilon\mathcal{C}$, let

$\underline{\text{schemata}}\ (g,d)=\{[g(a_1\ldots,a_n)=d] \mid a_i\varepsilon\mathcal{C}\cup\{\underline{X}_i\}$
and $\langle D,v\rangle \models g(a_1,\ldots,a_n)=d$
and $a_i \neq \underline{X}_i \Rightarrow \langle D,v\rangle \not\models g(a_1,\ldots,\underline{X}_i,\ldots,a_n)=d\}$

Then, an alternate set of nonoverlapping consistent schemata for g is defined by

$S'_g = \bigcup\{\underline{\text{schemata}}\ (g,d) \mid d\varepsilon\mathcal{C}\}.$

Note that $\langle D,v\rangle \models g(c_1,\ldots,c_n)=d$ iff $\langle g(c_1,\ldots,c_n)=d\rangle$ is an instance of a schema in S'_g.

3. Dominance Orderings for Recursive Equations

Given a SRS E generated by a set of rule schemata representing recursive equations, we may characterize a large class of dominance orderings d for which E is d-outer.

Lemma 24

Let $S = \bigcup\{S_f, S_g \mid f \in F_n,\ g \in G_n,\ n \in P\}$

be a set of rule schemata representing a set of recursive equations, let A be generated by S, and let $E = <\Sigma, F, \to, A>$ be a SRS constructed by the methods of Section 1.

Let d be a dominance ordering satisfying the following for all $A \in F$ and $x,y,z \in$ domainA:
If $Ax = g \in G_n$ and $x\cdot(i)$ anc y, $x\cdot(j)$ anc z, $i \neq j$, and $y(dA)z$, then S contains no schema of the form $[g(\ldots,\underline{X}_i,\ldots,c_j,\ldots)=d]$ (if $i<j$) or the form $[g(\ldots,c_j,\ldots,\underline{X}_i\ldots)=d]$ (if $j<i$).
Then E is d-outer.

Proof: Straightforward from Definition 41 (p. 63). ☐

Definition 51

A set of rule schemata S representing recursive equations is sequential iff the following holds:
S does not contain two different schemata of the forms
$[g(\ldots,\underline{X}_i,\ldots,c_j,\ldots)=e]$
$[g(\ldots,c_i,\ldots,\underline{X}_j,\ldots)=f]$.

Vuillemin's sequential functions [Vu74] (3.3, pp. 351-353) generate sequential sets of rule schemata.

Theorem 20

Let S be a sequential set of rule schemata representing recursive equations and let $E = <\Sigma, F, \to, A>$ be the associated SRS.
Define d as follows:

Let x,y be in domainA.
If x anc y, then x(dA)y.
If y anc x, then y(dA)x.

If $x \perp y$, let z be the least common ancestor of x and y, that is $z\cdot(i)$ anc x and $z\cdot(j)$ anc y where $i \neq j$.
If $Az = g \in G^n$ and S contains $[g(\ldots,c_i,\ldots,\underline{X}_j,\ldots)=e]$, then x(dA)y.
If S contains $[g(\ldots,\underline{X}_i,\ldots,c_j\ldots)=f]$, then y(dA)x.

In case neither form of schema is in S, the relationship of x and y is irrelevant, so let

x(dA)y if $i<j$, y(dA)x if $j<i$.

Then E is d-outer and, for all $A\varepsilon F$, dA is a total ordering.

Proof: By Lemma 24 (p. 75) and Definition 51 (p. 75) □

Note that, when d is preorder, Theorem 20 is a special case of clause (2) of Theorem 18 (p. 71).

Theorem 20 (with Theorem 15, p. 65) guarantees an easy way to find optimal noncopying reduction sequences for sequential recursive systems by simply computing the unique strictly d-outermost sequence for some total dominance ordering d. Vuillemin's "generalized delay rule" [Vu74] (3.1, p. 352) is a variation on d-outermost sequences.

*4. Continuous Semantics for Recursive Equations

Given C, V, G_n, F_n, Σ as in Section 1, assume that we have a continuous interpretation <D,v> of Σ, with $(vf)(\underline{d}_1,\ldots,\underline{d}_n) = \omega$ for $f\varepsilon\mathsf{F}_n$, $\underline{d}_1,\ldots,\underline{d}_n\varepsilon D$. v represents the initial information we have about the primitive symbols in C and G_n, with no information at all about defined symbols in F_n. Consider the SRS $<\Sigma,\Sigma_\#,\rightarrow,A>$ where A is a set of recursive equations.

Theorem 21

Let A be a set of recursive equations containing exactly one equation of the form $f(\underline{X}_1,\ldots,\underline{X}_n)=A$ for each $f\varepsilon\mathsf{F}_n$, $n\varepsilon\mathsf{P}$. Let <D,v> be a continuous interpretation of $\Sigma = \mathsf{C}\cup\bigcup\{\mathsf{F}_n,\mathsf{G}_n \mid n\varepsilon\mathsf{P}\}$, with $(vf)(\underline{d}_1,\ldots,\underline{d}_n)=\omega$ for all $f\varepsilon\mathsf{F}_n$. Then v^a(Def. 14, p. 16) exists and $\hat{v}^a A = \underline{\text{def}}A$, for all $A\varepsilon\Sigma_\#$.

Proof: Define a function

$$\tau:(\Sigma\rightarrow(D^*\rightarrow D))\rightarrow(\Sigma\rightarrow(D^*\rightarrow D))$$

which maps valuations of Σ to valuations of Σ as follows:

$(\tau w)c = vc$ for $c\varepsilon\mathsf{C}$

$(\tau w)g = vg$ for $g\varepsilon\mathsf{G}_n$

$((\tau w)f)(\underline{d}_1,\ldots,\underline{d}_n) = \hat{w}(A((A^{-1}\underline{X}_i)\leftarrow\underline{d}_i \mid 1\le i\le n))$

for $f\varepsilon\mathsf{F}_n$, $\underline{d}_1,\ldots,\underline{d}_n\varepsilon D$.

$\underline{\text{lfp}}\,\tau = \sqcap \{w \mid \tau w = w\}$ exists by Lemma 4 (p. 14).
It is straightforward to show that $\tau w = w$ iff $\langle D,w\rangle \models A$ and $v \sqsubseteq w$,
So, $v^a = \underline{\text{lfp}}\,\tau$.

$\underline{\text{def}}$ A exists by Theorem 7 (p. 38).
$\underline{\text{def}}\ A \sqsubseteq \hat{v}^A A$ by clause (1) of Theorem 2 (p. 17).
To show that $\hat{v}^A A \sqsubseteq \underline{\text{def}}\ A$, consider the alternate form of $\underline{\text{lfp}}\,\tau$:

$v^a = \underline{\text{lfp}}\,\tau = \bigsqcup \{\tau^i\Omega \mid i\in\mathbb{N}\}$ (Lemma 4).

(Ω is the valuation such that

$(\Omega a)(\underline{d}_1,\ldots,\underline{d}_n)=\omega$ for all $a\in\Sigma$, $\underline{d}_1,\ldots,\underline{d}_n\in D$).

First, show by induction on the structure of A that $\hat{v}^A A = \bigsqcup \{(\tau^i\Omega)A \mid i\in\mathbb{N}\}$. Then, show by induction on i that $(\tau^i\Omega)A \sqsubseteq \hat{v}B$ for some B where $A \to^* B$. So,
$\hat{v}^A A \sqsubseteq \bigsqcup \{\hat{v}B \mid A\ {}^*B\} = \bigsqcup \{\hat{v}B \mid A \vdash A = B\} = \underline{\text{def}}\ A$ □

Corollary 4

(1) In $\langle\Sigma,\Sigma_\#,\to A\rangle$ described above, $(A_i)\in\mathcal{C} \Rightarrow \hat{v}^A A_0 = \bigsqcup \{\hat{v}A_i \mid i\in\mathbb{N}\}$.

(2) If, in addition, for each equation $[f(\underline{X}_1,\ldots,\underline{X}_n)=A]\in\varepsilon$, A is of the form $g(B_1,\ldots,B_m)$, then $(A_i)\in\mathcal{O}^e \Rightarrow \hat{v}^A A_0 = \bigsqcup \{\hat{v}A_i \mid i\in\mathbb{N}\}$.

Proof:

(1) Direct from Theoroms 9 (p. 43) and 21.

(2) With such strong restrictions on the equations, it is straightforward to show that $\mathcal{O}^e = \mathcal{C}$. □

A detailed comparison of this chapter with the work of Cadiou [Cad72], Vuillemin [Vu74], Downey and Sethi [Ds76] and Berry and Lévy [BL77] reveals multitudinous small, and sometimes subtle, differences, too intricate to describe fully here. Roughly, the authors above treat $\hat{v}^A A_0$ as the "denotational" meaning of a term A_0. They (implicitly) consider a SRS such as $\langle\Sigma,\Sigma_\#,\to,A\rangle$ of this section. Given a reduction sequence (A_i), the computed meaning of A_0 is $\bigsqcup \{\hat{v}A_i \mid i\in\mathbb{N}\}$. A computation rule is correct ([Vu74] Section 2, p. 343; [DS76] Def. 3.1, p. 385; [BL77] III. 2.5 Definition, p.223) when it produces sequences

(A_i) such that $\bigsqcup\{\hat{v}A_i \mid i\varepsilon N\} = \hat{v}^A A_0$, that is, the denotational and computed meanings agree. Cadiou (1.7.1, p. 20), and Vuillemin ((6), p. 230; Prop. 4, Th. 3, p. 231) show that the special complete sequences described in V.3 (p. 43) are correct, essentially a special case of Corollary 4, clause (1). Vuillemin ((4), p. 230; Prop. 4, Th. 3, p. 231) and Downey and Sethi (Th. 4.1, p. 386) go on to show that the eventually outermost sequences described in V.5 are correct under strong syntactic restrictions. Downey and Sethi use a notion of syntactic dominance to essentially characterize $\mathcal{O}^e$ and prove correctness of all eventually outermost sequences. The syntactic restrictions correspond to those in Corollary 4, clause (2), so $\mathcal{O}^e = \mathcal{C}$ in cases covered by the above work. Vuillemin's result on the optimality of the delay rule (3.1, p. 232) corresponds roughly to Theoroms 13 (p. 63) and 20 (p. 75). Berry and Lévy generalize the approach of Cadiou, Vuillemin, Downey and Sethi by relaxing the syntactic restrictions on recursive equations.

*5. Continuous Semantics for Primitive Functions

The relationship between rule schemata and continuous interpretations may become quite complex if the CPO domains involved are allowed to be very rich. To illustrate the general nature of this relationship, we consider the very simplest of domains, the flat or discrete domains. Given the set of constant symbols $\mathcal{C}$, let $D = \{\underline{a} \mid a\varepsilon\mathcal{C}\}\cup\{\omega\}$. Let $\sqsubseteq$ be the simplest partial ordering of D, that is $\underline{a} \sqsubseteq \underline{b}$ iff $\underline{a} = \omega$ or $\underline{a} = \underline{b}$. Let v be the continuous interpretation defined by

$$va = \underline{a} \quad \text{if} \quad a\varepsilon\mathcal{C}$$

$$(va)(\underline{d}_1,\ldots,\underline{d}_n)=\omega \qquad \text{in all other cases.}$$

Now, given a nonoverlapping consistent set A of schemata for the primitive functions $\mathcal{G}_n$, all of the form $g(a_1,\ldots,a_n) = b$ with $a_1,\ldots,a_n,b\,\varepsilon\,\mathcal{C}\cup\mathcal{V}$, v^A exists and satisfies

$$(v^A g)(\underline{a}_1,\ldots,\underline{a}_n) = \begin{cases} b & \text{if } g(a_1,\ldots,a_n) = b \text{ is an instance of a rule in } A \\ \omega & \text{otherwise} \end{cases}$$

Conversely, given a continuous interpretation $\langle D,w\rangle$ with $v\sqsubseteq w$, let

$$\underline{\text{schemata}}'(g,d)=\{[g(a_1,\ldots,a_n)=d] \mid a_i\varepsilon\mathcal{C}\cup\{\underline{X}_i\} \text{ and } \langle D,w\rangle \models g(a_1,\ldots,a_n)=d \text{ and } a_i \neq \underline{X}_i \Rightarrow \langle D,w\rangle \not\models g(a_1,\ldots,a_{i-1},\omega,a_{i+1},\ldots,a_n)=d\}$$

This definition is essentially the same as the definition of <u>schemata</u> (g,d) Section 2.

$A = \bigcup\{\underline{\text{schemata}'}\ (g,d) \mid d \in C\}$ is a nonoverlapping consistent set of equations such that $v^a = w$. Berry and Lévy describe special domains, richer than the flat domains, which correspond to the Herbrand domains in classical logic [BL77] (III.1.2 Def., p. 222). As an exercise, the interested reader may work out a similar relation between rule schemata and interpretations using those richer domains.