APPLICATION OF THE OPTIMAL CONTROL THEORY
WITH DISTRIBUTED PARAMETERS ON A SEARCHING
PROBLEM

Olavi Hellman
 University of Turku
 Turku, Finland

1§ The free motion of the target

We shall assume that the target is moving randomly in such a way that $u(x, t)$ the probability density of the location of the target at time t , satisfies - if no search is going on - the parabolic equation

$$(1) \quad \frac{\partial u}{\partial t} = \sum_{i=1}^n \frac{\partial}{\partial x_i} \left[\sum_{j=1}^n a_{ij}(x, t) \frac{\partial u}{\partial x_j} - b_i(x, t) u \right]$$

valid for $(x, t) \in Q_T = \{(x, t): x \in \Omega \subset R^n, t \in (0, T)\}$,
 the initial condition

$$(2) \quad u(x, 0) = \gamma_0(x),$$

and the boundary condition

$$(3) \quad \sum_{i=1}^n n_i(x, t) \left[\sum_{j=1}^n a_{ij}(x, t) \frac{\partial u}{\partial x_j} - b_i(x, t) u \right] \Big|_{S_T} = 0,$$

where $S_T = \{(x, t): x \in S', t \in [0, T]\}$ and where $n_i(x, t)$ is the normal to S_T at point (x, t) . The initial-and boundary value problem (1)...(3) has a unique solution such that $u(x, t) \geq 0$ for all $x \in \Omega$ and $t \geq 0$ and such that $\int_{\Omega} u(x, t) dx = 1$ for all $t \geq 0$. The functions $a_{ij}(x, t)$, $b_i(x, t)$, and $\gamma_0(x)$ may be of a fairly general nature. Equation (1) is the most general parabolic equation of second order which has these properties.

2§ Mathematical formulation of a search

The search for a target is usually carried out by some system of searching equipment which is distributed in Ω and which in general changes as a function of time. - We shall assume that the searching system may be represented by a function $\lambda(x, t)$ with the following properties:

$$i) \lambda(x, t) \geq 0 \text{ for } (x, t) \in \Omega \times [0, T]$$

$$ii) \int_{\Omega} \lambda(x, t) dx = L_0 = \text{const. for all } t \in [0, T]$$

iii) $\lambda(x, t) \Delta t + o(\Delta t)$ is the probability that the target will be located, during time $(t, t + \Delta t)$, given that it is at $x \in \Omega$ (for other ways of defining a search see for instance refs. [3] and [4])

Within the theory of generalized parabolic equations applied here, $\lambda(x, t)$ must be a measurable function in $\Omega \times [0, T]$.

3§ Equation for the a posteriori probability density of the location of the target

With the search going on, the probability density of the location of the target will no longer change according to equation (1). Let $\hat{u}(x, t; \lambda)$ denote the said probability density, given that search during time $[0, t]$ has not been successful. Then [5]

$$(4) \quad \frac{\partial \hat{u}}{\partial t} = L(\hat{u}) + \hat{u} \left[\int_{\Omega} \hat{u}(\bar{x}, t; \lambda) \lambda(\bar{x}, t) d\bar{x} - \lambda(x, t) \right],$$

where we wrote more briefly

$$(5) \quad L(u) = \sum_{i=1}^n \frac{\partial}{\partial x_i} \left[\sum_{j=1}^n a_{ij}(x, t) \frac{\partial u}{\partial x_j} - b_i(x, t) u \right].$$

Equation (4) may be replaced, through the substitution

$$(6) \quad \hat{u}(x, t; \lambda) = y(x, t; \lambda) \exp \left(\int_0^t \int_{\Omega} u(\bar{x}, \tau; \lambda) \lambda(\bar{x}, \tau) d\bar{x} d\tau \right)$$

by the simpler parabolic equation

$$(7) \quad \frac{\partial y}{\partial t} = L(y) - \lambda y$$

while the conditions (2) and (3) become, respectively,

$$(8) \quad y(x, 0; \lambda) = \gamma_0(x)$$

and

$$(9) \quad \sum_{i=1}^n n_i(x, t) \left[\sum_{j=1}^n a_{ij}(x, t) \frac{\partial y}{\partial x_j} - b_i(x, t) y \right] \Big|_{S_T} = 0$$

The probability that the target will be detected, during time $[0, T]$, as a result of the search is [2]

$$(10) \quad P(T, \lambda) = 1 - \int_{\Omega} y(x, T; \lambda) dx$$

4§ The optimal search

This far nothing has been said about the choice of search density $\lambda(x, t)$. It is natural to try to find a $\lambda(x, t)$ which maximizes $P(T, \lambda)$, the probability of detection of the target during time $[0, T]$. One has now the following optimal control problem with distributed variables:

$$(11) \quad \int_{\Omega} y(x, T; \lambda) dx = \text{Min}!$$

$$(12) \quad \frac{\partial y(x, t; \lambda)}{\partial t} = \mathcal{L}(y(x, t; \lambda)) - \lambda(x, t) y(x, t; \lambda)$$

$$(13) \quad y(x, 0; \lambda) = \gamma_0(x)$$

and

$$(14) \quad \sum_{i=1}^n n_i(x, t) \left[\sum_{j=1}^n a_{ij}(x, t) \frac{\partial y(x, t; \lambda)}{\partial x_j} - b_i(x, t) y(x, t; \lambda) \right] \Big|_{S_T} = 0.$$

The condition of optimality for $\lambda(x, t)$ will now be obtained by following closely the approach by Lions [6]. Let $\lambda_0(x, t)$ be the optimal search density. Then

$$(15) \quad \int_{\Omega} [y(x, T; \lambda) - y(x, T; \lambda_0)] dx \geq 0,$$

for all $\lambda(x, t)$. Condition (15) may now be expressed (cf. ref. [7] sec. 16.2.1) in the form

$$(16) \quad \int_0^T d\tau \int_{\Omega} p(x, t; \lambda_0) (\lambda_0(x, t) - \lambda(x, t)) y(x, t; \lambda) dx d\tau \geq 0,$$

where $p(x, t; \lambda_0)$, the adjoint of $y(x, t; \lambda_0)$, is the solution of the boundary value problem

$$(17) \quad \begin{aligned} \frac{\partial p(x, t; \lambda)}{\partial t} = & - \sum_{i,j=1}^n \frac{\partial}{\partial x_i} \left(a_{ij}(x, t) \frac{\partial p(x, t; \lambda)}{\partial x_j} \right) + \sum_{i=1}^n b_i(x, t) \frac{\partial p(x, t; \lambda)}{\partial x_i} + \\ & + \lambda(x, t) p(x, t; \lambda) \quad \text{for } (x, t) \in Q_T, \end{aligned}$$

$$(18) \quad p(x, T; \lambda) = 1$$

$$(19) \quad \sum_{i,j=1}^n n_i(x, t) a_{ij}(x, t) \frac{\partial p(x, t; \lambda)}{\partial x_j} \Big|_{S_T} = 0.$$

It is a well known fact of the theory of search, that there will be a region $A(t) \subset \Omega$ such that $\lambda(x, t) > 0$ for all $x \in A(t)$ and such that $\lambda(x, t) \equiv 0$ for $x \in \Omega \setminus A(t)$. Furthermore, $\lambda(x, t)$ will very likely be discontinuous across the border of $A(t)$. - By using essentially the technique of [3], one now obtains, from condition (16), the following condition of optimality:

$$(20) \quad p(x, t; \lambda) y(x, t; \lambda) = \Lambda(t), \text{ for } x \in A(t)$$

and

$$(21) \quad p(x, t; \lambda) y(x, t; \lambda) \leq \Lambda(t), \text{ for } x \in \Omega \setminus A(t).$$

Here $\Lambda(t)$ is a function which will remain to be determined later.

5§ Application of the condition of optimality

It may be shown, through a fairly straightforward calculation, that equations (12), (17), and condition (20) imply, for $(x, t) \in A(t)$, that

$$(22) \quad \sum_{i,j=1}^n \frac{\partial}{\partial x_i} \left[a_{ij}(x, t) \frac{\partial z}{\partial x_j} \right] = -\frac{1}{2} \frac{d}{dt} (\ln \Lambda(t)) + \sum_{i=1}^n \frac{\partial b_i(x, t)}{\partial x_i}$$

where

$$(23) \quad z(x, t) = \ln y(x, t; \lambda).$$

It is of interest that equation (22) contains no derivatives with respect to time.

The optimal $\lambda_0(x, t)$ is finally obtained from (12) as follows:

$$(24) \quad \lambda_0(x, t) = \sum_{i,j=1}^n a_{ij}(x, t) \frac{\partial z}{\partial x_i} \frac{\partial z}{\partial x_j} - \sum_{i=1}^n b_i(x, t) \frac{\partial z}{\partial x_i} - \frac{\partial z}{\partial t} - \frac{1}{2} \frac{d}{dt} (\ln \Lambda(t)).$$

6§ An example

In order to illustrate the way the theory given in previous sections would be applied in practical cases, let us consider the perhaps simplest case, the one dimensional diffuse motion of the target on the real axis. Therefore, now,

$$(25) \quad L(u) = a^2 u_{xx}(x, t).$$

Furthermore, let $u(x, 0) = f(x)$, where $f(-x) = f(x)$. The region $A(t)$ of sec. 5 becomes now the interval $[-x(t), x(t)]$, where $x(t)$ is an unknown function of time which remains to be determined later. We have now for $x \in [-x(t), x(t)]$, the equation

$$(26) \quad (\ln y(x, t))_{xx} = -2C_0(t)$$

obtained from equation (22), where we wrote more briefly

$$C_0(t) = \frac{1}{4a^2} \frac{d}{dt} (\ln A(t)).$$

Equation (26) is solved at once to give, for $x \in [-x(t), x(t)]$,

$$(27) \quad y_1(x, t) = \exp(-C_0(t)x^2 + C(t)),$$

where $C(t)$ is an unknown function of time. Solution (27) satisfies the obvious requirement that $y_1(-x, t) = y_1(x, t)$. Outside interval $[-x(t), x(t)]$ function $y(x, t)$ satisfies equation

$$(28) \quad y_t'(x, t) = a^2 y_{xx}(x, t),$$

while $y(x, t)$ and $y_{xx}(x, t)$ must be continuous at points $-x(t)$ and $x(t)$. Our problem of optimizing the search has now been reduced to an initial- and boundary value problem with a moving boundary. - We shall now proceed in the well known manner. First let

$$(29) \quad w(x, t) = y(x, t) - \bar{y}(x, t)$$

where

$$\bar{y}(x, t) = (4\pi a^2 t)^{-\frac{1}{2}} \int_{-\infty}^{+\infty} f(\xi) \exp\left\{-\frac{(x-\xi)^2}{4a^2 t}\right\} d\xi,$$

so that

$$(30) \quad w_t(x, t) = a^2 w_{xx}(x, t)$$

$$(31) \quad w(x, 0) = 0$$

$$(32) \quad w(\pm x(t), t) = y_1(\pm x(t), t) - \bar{y}(\pm x(t), t)$$

and

$$(33) \quad w_x(\pm x(t), t) = y_1'(\pm x(t), t) - \bar{y}_x(\pm x(t), t)$$

Next we substitute, for $x \geq x(t)$ and for $x \leq -x(t)$

$$(34) \quad w(x, t) = (4a^2 \pi)^{-\frac{1}{2}} \int_0^t \frac{K(t')}{\sqrt{t-t'}} \left[\exp\left(-\frac{(x-x(t'))^2}{4a^2(t-t')}\right) + \exp\left(-\frac{(x+x(t'))^2}{4a^2(t-t')}\right) \right] dt'$$

Expression (34) satisfies equation (30) and condition (31), but contains the unknown-functions $x(t)$ and $K(t')$. For their determination one has conditions (32) and (33). Function $y_1(x, t)$ contains, however, the unknown functions $C_0(t)$ and $C(t)$ so that two more

conditions will be needed. One obtains now, from equation (24), that

$$\lambda(x, t) = (a^2 C_0^2(t) + \dot{C}_0(t)) x^2 - (a^2 C_0(t) + \dot{C}(t))$$

It will be required that

$$(35) \quad \lambda(x, t) \geq 0$$

Since finally

$$\int_{-x(t)}^{+x(t)} \lambda(x, t) dx = L_0$$

we obtain the condition that

$$(36) \quad \frac{2}{3} (a^2 C_0^2(t) + \dot{C}_0(t)) x^3(t) - 2(a^2 C_0(t) + \dot{C}(t)) x(t) = L_0.$$

Condition (35) can be satisfied by $C_0(t) \equiv 0$, provided that $\dot{C}(t) \leq 0$, in which case condition (36) reduces to

$$-2x(t)\dot{C}(t) = L_0$$

It is easy to see, that the determination of unknown functions $x(t)$, $C(t)$, and $K(t)$ still is a fairly complicated problem. By assuming a^2 of (25) to be small, perturbation calculations associated with an asymptotic treatment of function $w(x, t)$, enable one to obtain the unknown functions as power series of a^2 . This seems to indicate that the assumption $C_0(t) \equiv 0$ leads to a function $y(x, t)$ which satisfies our initial- and boundary value problem. Since the said problem has a unique solution, our solution would be the solution of the problem. We shall not reproduce the lengthy perturbation calculations here.

The case $a^2 = 0$, that of a stationary target, is well known [3]. However, since a^2 appears as the coefficient of the second partial derivative u_{xx} , the limiting process $a^2 \rightarrow 0$ would not be a trivial one. When considering the case $a^2 = 0$ one has to give up condition (33). The situation is very much related by nature to the corresponding situation in the theory of fluid flow in the limit of a vanishing viscosity.

References

- [1] O.A.Ladyzenskaya, V.A. Solonnikov, and N.N.Ural'tseva, Linear and quasi-linear parabolic equations, AMS, Trans.Math.Monogr. 23 (1968)
- [2] O.Hellman, On the optimal search for a randomly moving target. SIAM J.Appl.Math.22, 545 - 552
- [3] B.O.Koopman, Search and screening, OEG Rep. 56, Washington D.C., (1946)
- [4] L.O.Saretsalo, On stochastic models of search for stationary and moving objects, Publication of the institute for applied mathematics, University of Turku, Finland, (1971) (Dissertation).
- [5] O.Hellman, On the effect of search upon the probability distribution of a target whose motion is a diffusion process, Ann.Math.Statist. 41 (1970), pp. 1717 - 1724.
- [6] J.L.Lions, Controle optimal de systems gouvernes par des equations aux derivees partielles, Dunod Gauthier-Villars, Paris (1968).