FINDING A FEASIBLE CONTROL
FOR REAL PROCESS UNDER UNCERTAINTY

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Abstract

In the paper the problem of finding a feasible control for real process is discussed. It is assumed that the available mathematical model of the process differs from its real mathematical description but there is some consistence between them. Under such assumption the feasible control for real process is found by using the process mathematical model only. To show that this approach makes sense the existence theorems are given.

From the point of view of the control the problem is considered in two aspects. First, we want only to find the feasible control for the process; a numerical algorithm with convergence analysis is given. Second, we want to generate on-line a feasible control for the real process when its constraints are violated. In this case numerical algorithms with convergence analysis are also given. Finally, numerical example is presented.

1. Introduction

One of the important aspects of the practice of control of real processes is connected with the problem of finding control satisfying the real process constraints /feasible control/.

In the practical situation the mathematical model of the process differs from its exact but unknown mathematical description for several reasons, e.g. simplifying assumptions made when the model is being built, disturbances which cannot be exactly foreseen, etc. It means that the control for which the model constraints are fulfilled/feasible control for the model/ is not feasible for the real process in general. In many practical cases we are not allowed to apply non-feasible control to the real process. The possibilities of adaptation of the model cannot therefore be considered at this stage.

It means that for generating the feasible control for the real process its mathematical model can only be used.

In the paper this problem of finding feasible control for real process is discussed and numerical algorithms are proposed.

2. Problem formulation and concept of the solution

Let us suppose that the mathematical model of the process has the following form:

process equation -

$$F : C \times B \longrightarrow Y, \quad (1)$$

$$F(c, \beta) = y,$$

constraints -

$$P : C \times Y \times A \longrightarrow Z, \quad (2)$$

$$P(c, y, \alpha) \in S \subset Z,$$

$$c \in G \subset C, \quad y = F(c, \beta).$$

Variables α, β which are arguments of mappings F, P represent the unknown parameters or disturbances in the process, the sets of their values being estimated by A, B .

Variables c, y are respectively control and output of the process.

Definition 1

We say that control $\bar{c}$ is $(\bar{\alpha}, \bar{\beta})$ - feasible control for the model if and only if the following relations are fulfilled

$$P(\bar{c}, \bar{y}, \bar{\alpha}) \in S, \quad (3)$$

$$\bar{c} \in G, \quad \bar{y} = F(\bar{c}, \bar{\beta}).$$

and $\bar{\alpha}, \bar{\beta}$ are fixed in sets A and B .

This means that the control $\bar{c}$ satisfies constraints in the model when variables α, β take the values $\bar{\alpha}, \bar{\beta}$ in sets A, B .

As it was mentioned above the real mathematical description of the process is different from the process model and it is assumed that this real process description has the following yet unknown form:

process equation -

$$F_* : C \times B_* \longrightarrow Y, \quad (4)$$

$$F(c, \beta_*) = y,$$

constraints -

$$\begin{aligned} P_* &: C \times Y \times \mathcal{A}_* \longrightarrow Z, \\ P_*(c, y, \alpha_*) &\in S \subset Z, \\ c &\in G \subset C, \quad y = F_*(c, \beta_*) \end{aligned} \quad (5)$$

In the deterministic case the sets $\mathcal{A}_*$, $\mathcal{B}_*$ are single-point ones and mappings P_* , F_* do not depend on variables α_* , β_* .

Such situation takes place for example when the values of some parameters in the process are not given but it is known that they belong to sets $\mathcal{A}$ and $\mathcal{B}$.

In the presence of disturbances, α_* and β_* are stochastic processes and sets $\mathcal{A}_*$, $\mathcal{B}_*$ are the realization sets of them. In this case sets $\mathcal{A}$, $\mathcal{B}$ in the model approximate $\mathcal{A}_*$, $\mathcal{B}_*$.

Definition 2

We say that control $\bar{c}$ is a feasible control for the process if and only if the following relations are fulfilled:

$$\begin{aligned} P_*(\bar{c}, \bar{y}, \bar{\alpha}_*) &\in S, \\ \bar{c} &\in G, \quad \bar{y} = F_*(\bar{c}, \bar{\beta}_*), \end{aligned} \quad (6)$$

where $\bar{\alpha}_*$, $\bar{\beta}_*$ are the actual values of variables α_* , β_* in the process. This means that control $\bar{c}$ satisfies constraints in the process when variables α_* , β_* take values $\bar{\alpha}_*$, $\bar{\beta}_*$.

Because we do not know the actual values of variables α_* , β_* in the process and only the model is given, some connection between the mathematical model and real mathematical description of the process have to be assumed.

Definition 3

We say that the real mathematical description of the process is consistent with the process mathematical model if and only if the following condition is satisfied:

$$\forall (\beta_* \in \mathcal{B}_*) \exists (\beta \in \mathcal{B}) \forall (c \in G) \quad F_*(c, \beta_*) = F(c, \beta) \quad (C)$$

and

$$\forall (\alpha_* \in \mathcal{A}_*) \exists (\alpha \in \mathcal{A}) \forall (c \in G) \forall (y \in F_*(G \times \mathcal{B}_*)) \quad P_*(c, y, \alpha_*) = P(c, y, \alpha)$$

It is very easy to see that the control which is (α, β) -feasible for the model for every $(\alpha, \beta) \in \mathcal{A} \times \mathcal{B}$ is also feasible for the process for every $(\alpha_*, \beta_*) \in \mathcal{A}_* \times \mathcal{B}_*$ if the consistency condition (C) is satisfied. Thus feasible control for the process could be found as a control which is (α, β) -feasible for the model, independently on what are the values of variables α, β (we will denote such control as $(\mathcal{A}, \mathcal{B})$ -feasible control). To show that this approach makes sense the existence theorem for such control under reasonable assumptions should be given. That is the subject of our considerations in the next section.

3. An existence theory

We start with the definition of convexlike and concavelike mappings.

Definition 4

We say that the mapping $f : X \times Y \rightarrow Z$ is convexlike (concavelike) on set X (Y) with respect to set S if and only if the following relation is fulfilled:

$$\begin{aligned} & \forall (x_1, x_2 \in X) \forall (t \in [0, 1]) \exists (x \in X) \forall (y \in Y) \quad tf(x_1, y) + (1 - t)f(x_2, y) - f(x, y) \in S \\ & (\forall (y_1, y_2 \in Y) \forall (t \in [0, 1]) \exists (y \in Y) \forall (x \in X) \quad f(x, y) - tf(x, y_1) - (1 - t)f(x, y_2) \in S) \end{aligned}$$

The definition of a convexlike (concavelike) function (that is the case when $Z = \mathbb{R}$ and $S = \{z \in Z : z \geq 0\}$) is given in [6].

Proposition 1

If the mapping f is convexlike on X and concavelike on Y with respect to S , Z is the linear space, then the function μ defined as follows:

$$\begin{aligned} \mu : X \times Y &\rightarrow \mathbb{R} \\ \mu(x, y) &\triangleq z'(f(x, y)), \end{aligned}$$

where z' is linear functional on Z such that

$$\forall (z \in S) \quad z'(z) \geq 0;$$

is convexlike on X and concavelike on Y .

The proof of this proposition instantly results from the Definition 4 and the property of z' .

Let us define mapping $\tilde{P}$ as follows:

$$\begin{aligned} \tilde{P} : C \times \Gamma &\rightarrow Z, \\ \tilde{P}(c, \gamma) &\triangleq P(c, F(c, \theta), \alpha), \end{aligned} \tag{7}$$

where

$$\Gamma = A \times B, \quad \gamma = (\alpha, \beta).$$

Now we can formulate the basic existence theorem.

Theorem 1

Let us assume that G, Γ are compact topological Hausdorff spaces,

Z is Banach space, S is a closed convex cone and

1) mapping $\tilde{P}$ is convexlike on G and concavelike on G with respect to S ,

2) $\forall (z^* \in \tilde{S}) \forall (\gamma \in \Gamma)$ function $G \ni c \rightarrow z^*(\tilde{P}(c, \gamma)) \in R$ is upper semicontinuous,

where $\tilde{S} = S^* \cap K(\theta; 1)$,

$S^* \subset Z^*$ is a cone conjugate to S , where Z^* is a space conjugate to Z ,

by $K(\theta; 1)$ we denote the closed unit ball in Z^* .

3) $\forall (z_1^*, z_2^* \in \tilde{S}) \forall (c \in G) \forall (t \in [0, 1]) \inf_{\gamma \in \Gamma} [t z_1^*(\tilde{P}(c, \gamma)) + (1-t) z_2^*(\tilde{P}(c, \gamma))] =$
 $= t \inf_{\gamma \in \Gamma} z_1^*(\tilde{P}(c, \gamma)) + (1-t) \inf_{\gamma \in \Gamma} z_2^*(\tilde{P}(c, \gamma)),$

4) $\forall (\gamma \in \Gamma) \exists (c \in G) \tilde{P}(c, \gamma) \in S$,

5) $\forall (z^* \in \tilde{S}) \forall (c \in G)$ function $\Gamma \ni \gamma \rightarrow z^*(\tilde{P}(c, \gamma)) \in R$ is lower semicontinuous.

Then there exists the (A, B) - feasible control.

Proof

To simplify the notation we denote:

$$\inf_{z^* \in \tilde{S}} = \inf_{z^*}, \quad \inf_{\gamma \in \Gamma} = \inf_{\gamma}, \quad \sup_{c \in G} = \sup_c$$

On the base of the assumption 4) and the definition of S^* we have

$$\forall (\gamma \in \Gamma) \exists (c \in G) \forall (z^* \in S^*) z^*(\tilde{P}(c, \gamma)) \geq 0,$$

or (what is equivalent)

$$\inf_{\gamma} \sup_c \inf_{z^*} z^*(\tilde{P}(c, \gamma)) \geq 0. \quad (8)$$

Because for any function $f: X \times Y \rightarrow R$ there is ([2])

$$\sup_{x \in X} \inf_{y \in Y} f(x, y) \leq \inf_{y \in Y} \sup_{x \in X} f(x, y),$$

hence from (8) it follows that

$$\inf_{z^*} \inf_{\gamma} \sup_c z^*(\tilde{P}(c, \gamma)) \geq 0. \quad (9)$$

By Proposition 1 and assumption 1) the function

$\mu_1: G \times \Gamma \rightarrow R$, $\mu_1(c, \gamma) \triangleq z^*(\tilde{P}(c, \gamma))$, where z^* is any element from $\tilde{S}$, is convexlike on Γ and concavelike on G . Besides by assumption 5) for every $c \in G$ the function $\mu_1(c, \cdot)$ is lower semicontinuous on Γ .

Thus we can apply Kneser-Fan Theorem (see [6], [12]), by virtue of which the following equality is fulfilled:

$$\sup_c \inf_{\gamma} z^*(\tilde{P}(c, \gamma)) = \inf_{\gamma} \sup_c z^*(\tilde{P}(c, \gamma)). \quad (10)$$

Combining relations (9) and (10) we have

$$\inf_{z^*} \sup_c \inf_{\gamma} z^*(\tilde{P}(c, \gamma)) \geq 0 \quad (11)$$

Let us consider the function μ_2 , where

$$\mu_2: G \times \tilde{S} \rightarrow R, \quad (12)$$

$$\mu_2(c, z^*) \triangleq \inf_{\gamma} z^*(\tilde{P}(c, \gamma)).$$

By assumption 2) the function $\mu_2(\cdot, z^*)$ is upper semicontinuous on G , for every $z^* \in \tilde{S}$.

Taking into account assumption 1) and the property of $z^* \in \tilde{S}$ we conclude what follows:

$$\forall (c_1, c_2 \in G) \forall (t \in [0, 1]) \exists (c \in G) \forall (z^* \in \tilde{S}) \inf_{\gamma} z^*(\tilde{P}(c, \gamma)) \geq t \inf_{\gamma} z^*(\tilde{P}(c_1, \gamma)) + (1-t) \inf_{\gamma} z^*(\tilde{P}(c_2, \gamma)). \quad (13)$$

It means that the function μ_2 is concavelike on G .

Further from assumption 3) it directly follows that the function μ_2 is convexlike on $\tilde{S}$.

Now we can again apply Kneser-Fan Theorem with respect to the function μ_2 and conclude what follows:

$$\inf_{z^*} \sup_c \mu_2(c, z^*) = \sup_c \inf_{z^*} \mu_2(c, z^*). \quad (14)$$

Combining relations (11) and (14) we have

$$\sup_c \inf_{\gamma} \inf_{z^*} z^*(\tilde{P}(c, \gamma)) \geq 0. \quad (15)$$

Since the function

$$G \ni c \quad \inf_{\gamma} \inf_{z^*} z^*(\tilde{P}(c, \gamma)) \in R,$$

is upper semicontinuous (see assumption 2)) and G is a compact set, the supremum on the left side of inequality (15) is attained.

It means that

$$\exists (c \in G) \forall (\gamma \in \Gamma) \forall (z^* \in \tilde{S}) z^*(\tilde{P}(c, \gamma)) \geq 0,$$

and this is, by the property of $\tilde{S}$, equivalent to

$$\exists (c \in G) \forall (\gamma \in \Gamma) \tilde{P}(c, \gamma) \in S, \quad (16)$$

and the proof is completed.

Remark 1

Assumption 4) seems to be a natural one. It is strictly connected with the quality of the mathematical model. A mathematical model is better than another one when its sets $\mathcal{A}, \mathcal{B}$ approximate more exactly the unknown constant values of the parameters in the process (when the deterministic case takes place) or the sets $\mathcal{A}_*, \mathcal{B}_*$ (in the stochastic case).

Remark 2

Fulfillment of assumption 3) mainly depends on the character of the mapping $\tilde{P}(c, \cdot)$, where $c \in G$. We will give below the example of such situation where this assumption is fulfilled.

Example 1

Assume that $Z = \mathbb{R}^n$, $S = \{z \in \mathbb{R}^n : z \leq 0\}$,

$\forall (z^* \in \tilde{S}) \forall (c \in G)$ number $\inf_{\gamma} z^*(\tilde{P}(c, \gamma))$ is finite, mapping $\tilde{P}$ has the following structure:

$$\tilde{P}(c, \gamma) = (\tilde{P}_1(c, \gamma_1), \dots, \tilde{P}_n(c, \gamma_n)) \quad (17)$$

where $\Gamma = \Gamma_1 \times \dots \times \Gamma_n$, $\gamma = (\gamma_1, \dots, \gamma_n)$.

By direct computation we can test that the assumption 3) of Theorem 1 is satisfied here.

Remark 3

If $Z = \mathbb{R}^n$, $S = \{z \in \mathbb{R}^n : z \leq 0\}$, sets $G, \mathcal{A}, \mathcal{B}$ are convex, mapping $\tilde{P}$ has the following structure

$$\tilde{P}(c, \gamma) = (\tilde{P}_1(c, \gamma), \dots, \tilde{P}_n(c, \gamma)) \quad (18)$$

and for every i , $1 \leq i \leq n$ functions $\tilde{P}(\cdot, \gamma)$ and $\tilde{P}_i(c, \cdot)$ are convex on G and concave on Γ respectively, then the assumption 1) of Theorem 1 is satisfied. Truthfulness of the above fact can be easily verified by direct computation.

To apply Theorem 1 we must test whether G, A, B are compact sets. However, especially the set G is not always compact in practice. So our important task is to omit this requirement. To do it we will only consider the mappings $\tilde{P}$ which have the following form:

$$\tilde{P}(c, \gamma) = \phi(\tilde{P}_1(c), \tilde{P}_2(\gamma)) \quad (19)$$

where $\tilde{P}_1: G \rightarrow Z$, $\tilde{P}_2: \Gamma \rightarrow Z$, $\phi: Z \times Z \rightarrow Z$.

For mappings $\tilde{P}$ which belong to the class defined by (19) we can formulate.

Theorem 2

Assume that Z is Banach space, S is a closed convex cone in Z and

- 1) mapping $\tilde{P}$ has the form given by (19), where mappings $\phi(\cdot, z)$ and $\phi(z, \cdot)$ are weakly continuous on Z for arbitrary $z \in Z$,
- 2) mapping $\tilde{P}$ is convexlike on Γ and concavelike on G with respect to S ,
- 3) sets $\tilde{P}_1(G)$ and $\tilde{P}_2(\Gamma)$ are weakly compact,
- 4) $\forall (z_1^*, z_2^* \in S) \forall (c \in G) \forall (t \in [0, 1]) \inf_{\gamma \in \Gamma} [t z_1^*(\tilde{P}(c, \gamma)) + (1 - t) z_2^*(\tilde{P}(c, \gamma))] =$
 $= t \inf_{\gamma \in \Gamma} z_1^*(\tilde{P}(c, \gamma)) + (1 - t) \inf_{\gamma \in \Gamma} z_2^*(\tilde{P}(c, \gamma))$,
- 5) $\forall (\gamma \in \Gamma) \exists (c \in G) \tilde{P}(c, \gamma) \in S$,

then there exist the control which is (A, B) -feasible.

The proof of this theorem is given in [3].

Remark 4

Compactness requirements for the sets G, A, B were in fact omitted. They were substituted by assumption 3). Interpretation of the other assumptions of Theorem 2 are the same as in Theorem 1 (see Remarks 1, 2, 3).

From the above existence theorems it follows that the class of problems in which the control $(\mathcal{A}, \mathcal{B})$ - feasible exists is quite wide.

It means that our approach to the problem of finding feasible control for real process using only its mathematical model makes sense.

5. Numerical methods for feasible control generation

In this section the algorithms for finding feasible control are presented. The feasible control is found as a control which is $(\mathcal{A}, \mathcal{B})$ -feasible. It is assumed in this section that the model is consistent with the real mathematical description of the process, mapping $\tilde{P}$ has the representation given by (19), $S = \{z \in \mathbb{R}^N : z \leq 0\}$, C is Hilbert space, $\mathcal{A}$ and $\mathcal{B}$ are compact topological Hausdorff spaces, G is defined by a finite set of functional inequalities which are considered to be included in the set of mappings $\tilde{P}_i$ and for all $1 \leq i \leq N$, $\tilde{P}_i$ is continuous on $C \times \Gamma$, $\tilde{P}_i(\cdot, \gamma)$ is convex on C for each fixed $\gamma \in \Gamma$, $C \times \Gamma \ni (c, \gamma) \rightarrow \tilde{P}_i'(c, \gamma; e) \in \mathbb{R}$ is continuous function for each fixed $c \in G$,

where

$$\tilde{P}_i'(c, \gamma; e) = \lim_{t \rightarrow 0^+} \frac{\tilde{P}_i(c + te, \gamma) - \tilde{P}_i(c, \gamma)}{t}.$$

As it was mentioned above our task of finding a feasible control is equivalent under this representation of $\tilde{P}$ to finding $\bar{c}$ such that

$$P_{\max}(\bar{c}) \leq 0 \quad (24)$$

where $P_{\max} : C \rightarrow \mathbb{R}$, $P_{\max}(c) \triangleq \max_{1 \leq i \leq N} \bar{P}_i(c)$

where for all i , $1 \leq i \leq N$, $\bar{P}_i(c) = \max_{\gamma \in \Gamma} \tilde{P}_i(c, \gamma)$.

Under above assumptions the following hold (see [7], [9], [13]):

- (i) $P_{\max}$ is convex and continuous on C ,
- (ii) for each point $c \in G$ and direction $e \in C$ the directional derivative $P_{\max}'(c; e)$ of $P_{\max}$ exists and is given by the formula

$$P_{\max}'(c; e) = \max_{i \in I(c)} \max_{\gamma \in \Gamma_i(c)} \tilde{P}_i'(c, \gamma; e) \quad (25)$$

where

$$I(c) = \{i : P_{\max}(c) = \bar{P}_i(c)\}, \quad \Gamma_i(c) = \{\gamma \in \Gamma : \bar{P}_i(c) = \tilde{P}_i(c, \gamma)\}.$$

In order to solve problem (24) a descent technique is applied.

More precisely, let us consider the following algorithm.

Algorithm 1

Let be given an accuracy parameter of the solution ε , $\varepsilon > 0$, a starting point c_0 , $c_0 \in C$ and a sequence $\{g_n\}_{n \in N}$ such that

$\forall (n \in N) \quad g_n > 0$, $\lim_{n \rightarrow \infty} g_n = 0$, $\sum_{n=1}^{\infty} g_n = +\infty$. Let be given also c_n such that $P_{\max}(c_n) > \varepsilon$.

Step 1. Find a direction e_n as a solution of the problem

$$\min_{e \in K(0;1)} P_{\max}^i(c, e), \quad (26)$$

where

$K(0; 1) = \{e \in C : \|e\| \leq 1\}$, and $P_{\max}^i(c; e)$ is given by (25).

Step 2. Set $c_{n+1} = c_n + g_n e_n$.

Step 3. If $P_{\max}(c_{n+1}) \leq \varepsilon$ then stop. In the opposite case return to Step 1.

We will start to analyse the properties of Algorithm 1 with the following propositions.

Proposition 2

Let be given a positively homogeneous functional f on a linear normed space C . If $\bar{c}$ is a solution of the problem

$$\min_{c \in X} f(c),$$

where

$$X = \{c \in C : \|c\| \leq 1, \varphi_k(c) \leq 0 \text{ for } k, 1 \leq k \leq r\},$$

where

$\forall (1 \leq k \leq r)$, φ_k is positively homogeneous

and

$$f(\bar{c}) < 0, \text{ then } \|\bar{c}\| = 1$$

Proof

Assume the contrary, i.e., $\|\bar{c}\| < 1$ and set $\tilde{c} = \frac{\bar{c}}{\|\bar{c}\|}$. Since

$$\forall (1 \leq k \leq r) \quad \psi_k(\tilde{c}) = \frac{1}{\|\tilde{c}\|} \psi_k(\bar{c}) \leq 0 \quad \text{and} \quad \|\tilde{c}\| = 1, \text{ hence } \tilde{c} \in X.$$

On the other hand $f(\bar{c}) \leq f(\tilde{c})$ which implies that $f(\bar{c}) \geq 0$ what contradicts the assumption. Thereby $\|\bar{c}\| = 1$ and the proof is completed.

Proposition 3

Let be given a positively homogeneous, convex functional f on Hilbert space C . If $c_1, c_2 \in C$ are such that

$$f(c_1) = f(c_2) = \min_{c \in \{c \in C : \|c\|=1\}} f(c) = d,$$

$$\text{and } \|c_1\| = \|c_2\| = 1, \quad d < 0$$

then $c_1 = c_2$.

The proof of this proposition is given in [3].

We can formulate now the following lemma.

Lemma 1

If there exists $\bar{c}$ such that $P_{\max}(\bar{c}) \leq 0$ (i.e., a problem (1) has a solution) then the problem (3) has a unique solution e_n and $\|e_n\| = 1$.

Proof

By (i), $P_{\max}^1(c_n, \cdot)$ is weakly lower semicontinuous on C (see [7], [5]). $K(\theta; 1)$ is a weakly compact set ([5]).

Therefore, the solution e_n of the problem (3) exists. Further, from the convexity of $P_{\max}$ it follows

$$P_{\max}(\bar{c}) \geq P_{\max}(\tilde{c}) + P_{\max}^1(\tilde{c}; \bar{c} - \tilde{c}),$$

which implies that

$$P_{\max}^1(\tilde{c}; e) < 0, \text{ where } e = \bar{c} - \tilde{c}.$$

Hence, the fact that e_n is unique and $\|e_n\| = 1$ follows from the above Propositions.

The task solved in Step 1 of Algorithm 1 can sometimes be formulated in more suitable way. If for example $C = R^k$, $\Gamma_1(c_n) = \{\gamma_i\}$ for all i , $1 \leq i \leq N$ and we get $\|e\| = \max |e^i|$, where $e = (e^1, \dots, e^k) \in R^k$ then the problem (26) is equivalent to the

following one

$$\min_{(e, x)} x \quad (27)$$

where

$$\tilde{P}_i^1(c_n, \gamma_i; e) \leq x, \\ -1 \leq e^i \leq 1, \quad i = 1, \dots, k.$$

When in addition $\tilde{P}_i^1(c_n, \gamma_i; \cdot)$ are linear functions for all i ,

$1 \leq i \leq N$, then the problem (27) is a linear programming one.

The coefficients β_n were chosen in Step 2 in such a way as to guarantee that the sequence $\{c_n\}$ generated by Algorithm 1 converge to the solution of the problem (24).

The choice β_n by directional minimization seems to be more attractive since it guarantees a bigger progress in every iteration. However, on the other hand it is well known that the steepest descent procedure can generate a sequence which does not converge to the solution of the minimization problem with a cost function which is not differentiable in the Fréchet sense, when a directional minimization is used.

Because of that and in view of the fact that $P_{\max}$ is in general not Fréchet differentiable functional the directional minimization in Step 2 of the Algorithm should be avoided. Now we will investigate the properties of the direction e_n generated in Step 1. A satisfactory information about it is given by the following lemma.

Lemma 2

Let be given a convex, continuous functional f on Hilbert space G and an element $c \in G$ such that $0 \in \partial f(c)$.^{1/}

Then the solution $\bar{e}$ of the problem

$$\min_{e \in K(0;1)} f^1(c; e) \quad (28a)$$

exists and $\bar{e} = -\frac{c^*}{\|c^*\|}$, where $c^* \in \partial f(c)$.

^{1/} $\partial f(c)$ is a subdifferential of f at c .

Proof

The first part of the Lemma results from the facts that $f^1(c; \cdot)$ is a weakly lower semicontinuous function and $K(0; 1)$ is a weakly compact set. Consider now the following problem

$$\max_{c^* \in \partial f(c)} \min_{e \in K(\theta; 1)} \langle e, c^* \rangle^{2/} \quad (28)$$

Since $\forall (e \in K(\theta; 1)) \forall (c^* \in \partial f(c)) \langle e, c^* \rangle \geq -\|c^*\|$,

$$\langle e, c^* \rangle \Big|_{e = -\frac{c^*}{\|c^*\|}, c^* \neq 0} = -\|c^*\| \text{ and by the assumption}$$

that $0 \notin \partial f(c)$, it follows that any solution of the problem (28) has the form $(c^*, -\frac{c^*}{\|c^*\|})$.

A functional f is convex and continuous from the assumption, what implies that $\partial f(c)$ is weakly compact ([7], [8]). Hence by Ky Fan-Sion Theorem (see [4]) a saddle point $(\hat{e}, \hat{c}^*)$ of a function

$$G \times G \ni (e, c^*) \rightarrow \langle e, c^* \rangle \in \mathbb{R} \text{ on}$$

$K(\theta; 1) \times \partial f(c)$ exists. Since this saddle point solves a problem (28), then $\hat{e} = -\frac{\hat{c}^*}{\|\hat{c}^*\|}$. From the saddle point existency and by the property of a saddle point we have that $(\bar{e}, \bar{c}^*)$, where $\bar{c}^*$ is a part of the solution of the following problem

$$\min_{e \in K(\theta; 1)} \max_{c^* \in \partial f(c)} \langle e, c^* \rangle, \quad (29)$$

is also a saddle point of $\langle \cdot, \cdot \rangle$ on $K(\theta; 1) \times \partial f(c)$.

Hence and by the expression ([8], [9])

$$\min_{e \in K(\theta; 1)} f^1(c; e) = \min_{e \in K(\theta; 1)} \max_{c^* \in \partial f(c)} \langle e, c^* \rangle, \quad (30)$$

we have $\bar{e} = -\frac{\bar{c}^*}{\|\bar{c}^*\|}$ what completed the proof.

The convergence property of Algorithm 1 is formulated in the following theorem.

Theorem 3

Assume that the solution set of the problem (1) is not empty. Then

2/ $\langle \cdot, \cdot \rangle$ is a scalar product.

- (a) for every sequence $\{c_n\}_{n \in \mathcal{N}}$ generated by Algorithm 1 there exists the natural number M such that $P_{\max}(c_M) \leq \varepsilon$.
- (b) if, in addition, there exists $\bar{c}$ such that $P_{\max}(\bar{c}) < 0$ then there exists the natural number J for which $P_{\max}(c_J) < 0$, i.e., $(A, \mathcal{B})$ - feasible control after a finite number of steps is attained.

Proof

Note that $P_{\max}$ satisfies the assumptions of Lemma 2. Hence by the property of e_n we can look at Algorithm 1 as at the well known subdifferential minimization technique which is applied to $P_{\max}$. Thereby the proof is completed by [1], [10].

Remark 5

For given c we can treat $P_{\max}(c)$ as a certain measure of feasibility of c . The open problem is to find another measures.

The very important property of Algorithm 1 is that the constructive method of finding a subgradient is used in it. Bertsekas and Mitter presented in [2] so called ε -Subgradient Method for optimization problems with nondifferentiable cost functionals. Since the directional minimization in ε -subgradient direction can be used in this method then it could be more efficient than Algorithm 1 when applied to problem (1). But on the other hand constructive possibilities of the ε -subgradient generation seem to be more limited than the possibility of subgradient generation by formula (28a).

Suppose now that the constraints of the process are violated because of some reasons. The following task then arises: starting from the actual nonfeasible control, generate a sequence $\{c_n\}_{n \in \mathcal{N}}$ convergent to a feasible control such that every element in this sequence is better for the real constraints from the previous one. The important property of such sequence is that all its elements can be applied in the process successively during their generation. This task is the subject of our further considerations. First we have to define more

precisely what it means that a certain control is better from another one. It is of course not possible to give a universal definition. However it seems that the following one will be useful in many practical situations.

Definition 5 (Control Quality Definition)

We say that a control c_1 nonfeasible for the process is better from c_2 which is also nonfeasible if and only if the following relations are satisfied:

$$\forall (i \in \tilde{I}_1(c_2)) \tilde{P}_i(c_1, \gamma_a) \leq \tilde{P}_i(c_2, \gamma_a), \quad (31)$$

and

$$\forall (i \in \tilde{I}_2(c_2)) \tilde{P}_i(c_1, \gamma_a) \leq 0,$$

where

$$\forall (c \in C) \tilde{I}_1(c) = \{i: 1 \leq i \leq N, \tilde{P}_i(c_2, \gamma_a) > 0\} \text{ and } \tilde{I}_2(c) = \{i: 1 \leq i \leq N, \tilde{P}_i(c_2, \gamma_a) \leq 0\},$$

and γ_a is the actual value of the variable γ in the process.

By the essential assumption of this paper the adaptation procedure for recognizing γ_a is excluded. Hence we are not able to generate such sequence as it was mentioned above if the Definition 5 is applied. However the possibility of doing it by using the model only, would if the Control Quality Definition (31) is modified as follows:

$$\forall (i \in \bar{I}_1(c_2)) \bar{P}_i(c_1) \leq \bar{P}_i(c_2) \text{ and } \forall (i \in \bar{I}_2(c_2)) \bar{P}_i(c_1) \leq 0, \quad (32)$$

where

$$\forall (c \in C) \bar{I}_1(c) = \{i: 1 \leq i \leq N, \bar{P}_i(c_2) > 0\} \text{ and } \bar{I}_2(c) = \{i: 1 \leq i \leq N, \bar{P}_i(c_2) \leq 0\}.$$

This modified Definition will be valid in the next part.

Further, the set of the assumption from page 14 will be broadened by supplementing the following one:

$$\exists (\bar{c} \in C) P_{\max}(\bar{c}) < 0. \quad (33)$$

We are now starting construction of an algorithm for the generation of the above sequence with some set definitions:

$$\forall (c \in C) \bar{I}_3(c) \triangleq \{i: 1 \leq i \leq N, \bar{P}_i(c) \geq 0\}, \quad (34)$$

and

$$\bar{K}(c) = \{e \in C: \|e\| \leq 1, \exists (t(c) \in R) \forall (t \in (0, t(c))) \forall (i \in \bar{I}_3(c)) \quad \bar{P}_i(c+te) < \bar{P}_i(c)\} \quad (35)$$

Roughly speaking, for a given control c , $\bar{K}(c)$ consist of all improvement directions for these constraints $\bar{P}_i$ which are violated by c or can be violated by $c + \Delta c$, where Δc is arbitrarily small. Consider the possibility of choosing a direction from $\bar{K}(c)$ by solving the following problem:

$$\min_{e \in \bar{K}(c)} \max_{i \in \bar{I}_3(c)} \bar{P}_i'(c; e). \quad (36)$$

An exhaustive information about it is given by the following lemma.

Lemma 3

Suppose that the set $\bar{I}_3(c)$ is not empty. Then the problem (36) is equivalent to the following one:

$$\min_{e \in K(\theta; 1)} \max_{i \in \bar{I}_3(c)} \bar{P}_i'(c; e) \quad (37)$$

The problem (37) has a unique solution with a unit norm.

Proof

If $e \in K(\theta; 1) - \bar{K}(c)$ then $\exists (i \in \bar{I}_3(c)) \bar{P}_i'(c; e) \geq 0$.

If $e \in \bar{K}(c)$ then $\forall (i \in \bar{I}_3(c)) \bar{P}_i'(c; e) \leq 0$. Thus we conclude that every solution of the problem (36) solves the problem (37).

Suppose now that $\bar{e}$ solves (37). Since $\bar{K}(c) \subset K(\theta; 1)$ then

$\forall (e \in \bar{K}(c)) \max_{i \in \bar{I}_3(c)} \bar{P}_i'(c; \bar{e}) \leq \max_{i \in \bar{I}_3(c)} \bar{P}_i'(c; e)$ and in order to show

that $\bar{e}$ solves (36) it is enough to show that $\bar{e} \in \bar{K}(c)$.

Assume the contrary, i.e., $\bar{e} \notin \bar{K}(c)$. Then $\max_{i \in \bar{I}_3(c)} \bar{P}_i'(c; \bar{e}) \geq 0$.

On the other hand, from convexity of the functions $\bar{P}_i$, assumption

(33), definition of the set $\bar{I}_3(c)$ and assumption of Lemma it follows that $\forall (i \in \bar{I}_3(c)) \bar{P}_i'(c; \frac{\bar{c} - c}{\|\bar{c} - c\|}) < 0$, which is not possible.

Hence the first part of Lemma is proved. Further, functional

$C \ni e \rightarrow \max_{i \in \bar{I}_3(c)} \bar{P}_i'(c; e) \in R$ is a positively homogeneous, convex and

continuous one and from the above consideration it follows that

$$\min_{e \in K(\theta; 1)} \max_{i \in \bar{I}_3(c)} \bar{P}_i'(c; e) < 0.$$

Thus the second part of Lemma results immediately from Proposition 2 and Proposition 3. This completes the proof.

Lemma 3 provides that the problem (36) makes sense and enables us to substitute it by a simpler one.

Now we can formulate an algorithm.

Algorithm 2

Let be given an accuracy of the solution ε , $\varepsilon > 0$ and a starting nonfeasible point c_0 , $c_0 \in C$. Let be given also c_n such that $P_{\max}(c_n) > \varepsilon$.

Assumption: $\{c \in C: \bar{P}_i(c) \leq \bar{P}_i(c_0)\}$ are bounded sets for all i , $1 \leq i \leq N$.

Step 1. Find a direction e_n as the solution of the problem

$$\min_{e \in K(0;1)} \max_{i \in \bar{I}_3(c_n)} \bar{P}_i'(c; e).$$

Step 2. For all $i \in \bar{I}_3(c_n)$ determine numbers $\bar{\rho}_i$ as the solutions of the problems

$$\min_{\rho \in [0, \infty)} \bar{P}_i(c_n + \rho e_n).$$

Step 3. Set $\tilde{c}_{n+1} = c_n + \tilde{\rho}_n e_n$, where $\tilde{\rho}_n = \inf_{i \in \bar{I}_3(c_n)} \{\bar{\rho}_i\}$.

Step 4. If $\forall (i \in (1:N) - \bar{I}_3(c_n)) \bar{P}_i(\tilde{c}_{n+1}) \leq 0$ then 1/
set $\rho_n = \tilde{\rho}_n$ and go to Step 6. In opposite case go to Step 5.

Step 5. Select the greatest number ρ_n such that

$$\forall (i \in (1:N) - \bar{I}_3(c_n)) \bar{P}_i(c_n + \rho_n e_n) \leq 0.$$

Step 6. Set $c_{n+1} = c_n + \rho_n e_n$

Step 7. If $P_{\max}(c_{n+1}) \leq \varepsilon$ then stop. In the opposite case return to Step 1.

$$1/ (1 : N) = \{i: 1 \leq i \leq N\}$$

Theorem 4

Algorithm 2 is well defined and generates the sequence which satisfies the quality requirement expressed by inequalities (32)

Proof

The proof directly follows from the Algorithm construction and Lemma 3. Unfortunately Algorithm 2 can sometimes generate a sequence which is not convergent to the point satisfying relation (24). This phenomenon is strictly connected with the fact mentioned already in this section, i.e., that $\max_{i \in \bar{I}_3(c)} \bar{P}_i(c)$ is not Fréchet differentiable in general. In spite of such disadvantage this heuristic algorithm can be useful in practise. The algorithm which has a convergence property is formulated below.

Algorithm 3

Let be given an accuracy of solution ε , $\varepsilon > 0$, a starting nonfeasible point c_0 , $c_0 \in C$ and a sequence $\{\varrho_n\}_{n \in \mathbb{N}}$ such that $\forall (n \in \mathbb{N}) \varrho_n > 0$,

$\lim_{n \rightarrow \infty} \varrho_n = 0$, $\sum_{n=1}^{\infty} \varrho_n = +\infty$. Let be given also c_n such that

$$P_{\max}(c_n) > \varepsilon.$$

Step 1. Find a direction e_n as solution of the problem

$$\min_{e \in K_1(c_n)} \sum_{i \in \bar{I}_1(c_n)} \bar{P}'_i(c_n; e), \quad (38)$$

where

$$\forall (c \in C) K_1(c) = \{e \in C: \|e\| \leq 1, \forall (i \in \bar{I}_3(c)) \bar{P}'_i(c; e) \leq 0\}. \quad (39)$$

Step 2. Set $c_{n+1} = c_n + \varrho_n e_n$.

Step 3. If $P_{\max}(c_{n+1}) \leq \varepsilon$ then stop. In the opposite case return to Step 1.

Theorem 5

Algorithm 3 is well defined.

Proof

Only Step 1 requires the proof. Since $\forall (n \in \mathbb{N}) \{e \in C: \bar{P}_1'(c_n; e) \leq 0\}$ are a convex, closed sets, $\{e \in C: \|e\| \leq 1\}$ is a weakly compact set then $K_1(c_n)$ is a weakly compact one.

Besides $\forall (c \in C) \exists e \rightarrow \sum_{i \in \bar{I}_1(c)} \bar{P}_1'(c; e) \in \mathbb{R}$ is a weakly lower semicon-

tinuous functional. Then the proof is completed by application of Weierstrass Theorem.

The convergence property of that Algorithm is expressed by the following theorem.

Theorem 6

Assume that

$$\forall (c \in C) \forall (i \in \bar{I}_3(c)) \forall (c^* \in K_2(c)) \bar{P}_1'(c; -c^*) \leq 0. \quad (40)$$

where

$$(c^* \in K_2(c)) \leftrightarrow \exists (e \in K_1(c)) \quad (e, c^*) \text{ is a saddle point of a functional } \langle \cdot, \cdot \rangle \text{ on } K_1(c) \times K_3(c),$$

where

$$K_3(c) = \left\{ c^* \in C: c^* = \sum_{i \in \bar{I}_1(c)} c^{*i}, \forall (i \in \bar{I}_1(c)) c^{*i} \in \partial \bar{P}_1(c) \right\} \quad (41)$$

Then for every sequence $\{c_n\}_{n \in \mathbb{N}}$ generated by Algorithm 3 there exists the natural number M such that $P_{\max}^{(c_n)} \leq \varepsilon$.

Proof

Suppose that $P_{\max}^{(c_n)} > \varepsilon$. Note that $\sum_{i \in \bar{I}_1(c_n)} \bar{P}_1'(c_n; \cdot)$ is a

positively homogeneous convex functional and the inequality (33) holds. Hence from the above and by definitions of the sets

$$K_1(c_n), \bar{I}_1(c_n), \bar{I}_3(c_n)$$

we have

$$\exists (\bar{c} \in C) \sum_{i \in \bar{I}_1(c_n)} \bar{P}_1'(c_n; \frac{\bar{c} - c_n}{\|\bar{c} - c_n\|}) < 0. \quad (42)$$

Making use of the property of the subgradient we can write the following

$$\begin{aligned}
\min_{e \in K_1(c_n)} \sum_{i \in \bar{I}_1(c_n)} \bar{P}_i^1(c_n; e) &= \min_{e \in K_1(c_n)} \sum_{i \in \bar{I}_1(c_n)} \max_{c_n^{*i} \in \partial \bar{P}_i(c_n)} \langle e, c_n^{*i} \rangle = \\
&= \min_{e \in K_1(c_n)} \max_{c_n^* \in K_3(c_n)} \langle e, c_n^* \rangle. \quad (43)
\end{aligned}$$

Since $\forall (i \in (1:N)) \partial \bar{P}_i(c_n)$ are weakly compact, convex sets

([7], [8]), then $K_3(c_n)$ is also weakly compact and convex([5]). As it was noted in the proof of Theorem 5, $K_1(c_n)$ is a weakly compact convex set too. Hence by application of Ky Fan-Sion Theorem we conclude that a saddle point of $\langle \cdot, \cdot \rangle$ on $K_1(c_n) \times K_3(c_n)$ exists. Let $(\hat{e}_n, \hat{c}_n^*)$ be this point. It also solves the following problem:

$$\max_{c_n^* \in K_3(c_n)} \min_{e \in K_1(c_n)} \langle e, c_n^* \rangle. \quad (44)$$

Note that $c_n^* \neq 0$, since (42) holds. Now, by application of Schwartz inequality and from the inequality (40) the saddle point representation is received in the form:

$$(\hat{e}_n, \hat{c}_n^*) = \left(-\frac{\hat{c}_n^*}{\|\hat{c}_n^*\|}, \hat{c}_n^* \right), \quad (45)$$

where

$$\hat{c}_n^* \in K_3(c_n).$$

Since $(e_n, \bar{c}_n^*)$ where $\bar{c}_n^*$ is a part of any solution of the problem

(44) is the saddle point then

$$e_n = \frac{-\bar{c}_n^*}{\|\bar{c}_n^*\|}, \quad (46)$$

where c_n^* is a certain element from the set $K_3(c_n)$.

Consider now a functional $\bar{\Psi}$ where

$$\bar{\Psi}(c) \triangleq \sum_{i=1}^N \max \{0, \bar{P}_i(c)\}. \quad (47)$$

By the well known rule of computing of a subgradient ([9]) we conclude that an element $\sum_{i \in \bar{I}_1(c_n)} c_n^{*i}$ where $c_n^{*i} \in \partial \bar{P}_i(c_n)$, belong to $\partial \bar{\Psi}(c_n)$.

Hence, taking into account the definition of $K_3(c)$ and (46) we have:

$$e_n = - \frac{c_n^*}{\|c_n^*\|}, \quad (48)$$

where $c_n^* \in \partial \bar{\Psi}(c_n)$.

The important conclusion from the above is that Algorithm 3 generates a minimizing sequence for $\bar{\Psi}([1], [10])$. This together with the definition of $\bar{\Psi}$, completed the proof.

Remark 6

Not all elements of the sequence generated by Algorithm 3 satisfy the quality requirement. There are two reasons of such situation. First, the direction e_n is not the direction of improvement for every $\bar{P}_i$ in c_n . We have only that $\bar{P}_i'(c_n; e_n) \leq 0$, for all $i, i \in \bar{I}_3(c_n)$. Second, even if e_n is a direction of improvement for all $\bar{P}_i$ in c_n then the choice of a coefficient ρ_n in Algorithm 3 would not guarantee satisfaction of the relations (32). On the other hand, as it was shown in the proof of Theorem 6, every sequence which is generated by Algorithm 3, minimizes the functional $\bar{\Psi}$. From the form of $\bar{\Psi}$ then follows that these drawbacks of Algorithm 3 do not considerably limit the possibility of its application. The positive property of Algorithm 3 is that we can favour some constraints, depending on their practical validity. It should be done by introducing weight coefficients w_i and substituting the problem (38) in Step 1 by the following one:

$$\min_{e \in K_1(c_n)} \sum_{i \in \bar{I}_1(c_n)} w_i \bar{P}_i'(c_n; e).$$

It seems that a good idea might be to construct on the base of Algorithm 2 and Algorithm 3 a new one. Numerical experiments relating to this concept were done in [11].

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An numerical example

The following example of the process has been considered:

$$\tilde{P}_1(c, \gamma) = c_1^2 + c_2^2 - \gamma_1,$$

$$\tilde{P}_2(c, \gamma) = c_1 - \gamma_1 c_2^2 + \gamma_2,$$

$$\tilde{P}_3(c, \gamma) = -c_2 + \gamma_2 c_1,$$

where

$$\gamma = (\gamma_1, \gamma_2) \in \Gamma = [2, 3] \times [-1, 1], \quad \gamma_1 \in [2, 3], \quad \gamma_2 \in [-1, 1], \quad c = (c_1, c_2).$$

(A, B) -feasible controls set for this example is shown on Fig.1.

Three possibilities of a sequence $\{\rho_n\}_{n \in N}$ choosing in Algorithm 2 and Algorithm 3 have been tested. It has been assumed in general that

$\forall (n \in N) \rho_n = \frac{a}{n}$, where a has depended on a variable KROK as follows:

$$a = \begin{cases} \|c_0\|, & \text{where KROK} = 1, \\ \hat{\rho}, & \text{where } \hat{\rho} \text{ solves a problem: } \min_{\rho \in [0, \infty)} P_{\max}(c_0 + \rho e_1); \\ & \text{where KROK} = 0, \\ P_{\max}(c_0), & \text{where KROK} = -1. \end{cases}$$

The best numerical results have been attained for $a = \hat{\rho}$.

Next, two possibilities of choosing of a starting value of a step coefficient ρ_0 in a directional minimization procedure used in Step 1 of Algorithm 2, have been tested.

The value of a coefficient ρ_0 has depended on a variable KROK as follows:

$$\rho_0 = \begin{cases} \|c_n\|, & \text{where KROK} = 1 \text{ or } 0, \\ P_{\max}(c_n), & \text{where KROK} = -1. \end{cases}$$

On the base of numerical results we can say that one and the other possibility is useful. Finally, two norms have been considered in Step 1 of all Algorithms: $\|e\| = \left(\sum_{i=1}^k e_i^2\right)^{1/2} (NL)$ and $\|e\| = \max_{1 \leq i \leq k} |e_i| (IN)$, where $e = (e_1, \dots, e_k)$.

It seems that the proper choice of a norm is strictly depended on the problem.

The computational results are given in Table 1.

The sequence generated by Algorithm 1 when $c_0 = (5, -7)$ is shown on Fig. 1.

Table 1

Starting point	Algorithm	K R O K							
		1		0		-1			
		LN	NL	LN	NL	LN	NL		
-0.5, 3.0	1	3	-	2 145	3 -	2 145	3 72	1	30
	2	2	-	2 -	2 -	2 -	3 70	2	89
	3	6	-	4 127	3 -	2 -	10 68	7	77
5.0, -7.0	1	-	-	5 -	- -	5 186	- -	2	61
	2	-	-	- -	- -	5 138	- -	-	-
	3	9	-	7 -	3 44	3 61	- -	11	118
10.0, 0.0	1	3	52	- -	3 -	- -	5 136	3	74
	2	2	19	12 652	2 -	7 -	5 57	4	112
	3	12	62	9 89	5 -	11 -	31 176	11	114
-9.5, 5.0	1	4	75	- -	4 87	5 206	- -	3	102
	2	3	36	3 94	3 41	4 165	4 74	2	77
	3	-	-	- -	5 56	11 131	- -	11	123
1.0, -0.5	1	2	36	1 18	2 37	4 18	1 -	2	-
	2	-	-	- -	- -	- -	- -	2	-
	3	-	-	21 231	21 150	21 246	21 -	4	-
-1.2-0.4	1	1	9	2 53	1 10	2 53	1 -	1	-
	2	1	10	- -	1 10	- -	1 -	1	-
	3	1	10	2 21	1 13	3 47	1 -	1	-
0.0, 8.0	1	3	63	- -	3 63	- -	3 -	2	60
	2	4	47	1 41	4 48	1 41	2 -	1	45
	3	10	56	7 57	3 37	7 69	21 -	5	63

The first number indicates a number of iterations,
the second one - a computational time / in sec/.

Fig. 1

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