- Minimum Variance Control of Discrete - Time Linear Stochastic System, using instantaneous Output Feedback.

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Abstract

This paper considers the problem of determining the linear output feedback control which minimizes a quadratic performance index, for a linear, discrete - time stochastic system. Both the finite and infinite - time versions of this problem are solved. For the finite terminal time case, the two-point boundary value problem that specifies the optimal feedback gain matrices is derived, and an algorithm is proposed for solving it. For the infinite-time case, two coupled non linear matrix equations must be solved to realize the optimal control ; an algorithm is also proposed for solving those equations. A numerical example is treated comparing this control policy to the optimal Kalman - type control policy.

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I - Introduction -

The object of this paper is to solve a linear regulator problem for which the control is constrained to be, at any instant, a linear transformation of the measured system output.

As it is well known, [1] , [2] , the optimal control policy, when the admissible control is not constrained, is provided by the celebrated separation theorem [3] . However, the optimal controller, composed in this case of a Kalman filter together with the optimal control gains for the associated deterministic problem, might be much too complex to be implemented in practice.

It is thus interesting to look for a simple control policy, such as the one proposed, if we agree with the following statements :

- We don't require state estimation, but we rather want a good regulation
- We are ready to trade optimality for engineering simplicity.

Several authors have taken this approach, in a deterministic as well as in a stochastic framework. Athans and Levine [4] and Kosut [5] are to be mentioned for the deterministic case, Mac Lane [6] , Axaster [7] , and more recently Ben Zurion Kurtaran [8] , are to be mentioned for the stochastic problem. However, whereas noise - free measurements were considered in [6] , [7] , [8] , we will consider in this paper, that the measures are corrupted by additive white noise. Also we will not directly penalize the control, but rather use a minimum variance criterion on the state of the plant.

The organization of the paper is as follows : in section II the finite terminal - time problem is solved and an algorithm is proposed for solving the associated two point boundary value problem ; in section III the infinite terminal-time problem is solved and an algorithm is proposed for solving the associated non linear matrix equations. In section IV a numerical example is presented.

II - Finite terminal - time problem -

Consider the following linear, discrete - time stochastic system :

$$\begin{aligned} x_{k+1} &= A_k x_k + B_k u_k + W_k \\ y_k &= C_k x_k + V_k \end{aligned} \quad k = 0, 1, \dots, N-1 \quad (1)$$

where :

x the state , is an n - dimensional vector
 u the input , is an r - dimensional vector
 y the output , is an m - dimensional vector

and where A_k, B_k, C_k are time - varying matrices of appropriate dimensions.
 The following assumptions are also made on the initial state of the plant and the white - noise sequences w_k and v_k :

$$\begin{aligned} x_0 &\text{ is } N(0, X_0) \quad ; \quad E \{ v_k \} = 0 \quad ; \quad E \{ w_k \} = 0 \\ E \{ w_k w_l' \} &= \sum_k \delta_{(k-l)} \quad ; \quad E \{ v_k v_l' \} = \theta \delta_{(k-l)} \quad (2) \\ E \{ w_k v_l' \} &= 0 \quad ; \quad E \{ w_k x_0' \} = 0 \quad ; \quad E \{ v_k x_0' \} = 0 \end{aligned}$$

The control law to be implemented is the following linear transformation on the output y_k :

$$u_k = L_k y_k \quad (3)$$

where L_k is an $r \times m$ time - varying gain matrix to be optimized.

The closed - loop system becomes accordingly :

$$\begin{aligned} x_{k+1} &= \Gamma_k x_k + B_k L_k v_k + w_k \\ \Gamma_k &= A_k + B_k L_k C_k \end{aligned} \quad k = 0, 1, \dots, N-1 \quad (4)$$

The problem is then to find the sequence $\left\{ L_k \right\}_{k=0}^{k=N-1}$

that minimizes the following cost criterion :

$$J = E \left\{ x_N' S x_N \right\} + \sum_{k=1}^{N-1} E \left\{ x_k' Q_k x_k \right\} \quad (5)$$

where $S > 0$ and $Q_k > 0 \quad k = 1, \dots, N-1$

Defining now $\Lambda_k = E \left\{ x_k x_k' \right\}$, the second moment of x_k , the cost can then be rewritten :

$$J = \text{tr} (S \Lambda_N) + \sum_{k=1}^{N-1} \text{tr} (Q_k \Lambda_k) \quad (6)$$

Now from (4), Λ_k is known to satisfy :

$$\Lambda_{k+1} = \Gamma_k \Lambda_k \Gamma_k' + B_k L_k \Theta_k L_k' B_k' + \sum_k \Lambda_0 = X_0 \quad (7)$$

so that we can formulate our problem as follows :

Problem statement : Given the plant parameters $A_k, B_k, C_k, \sum_k, \Theta_k, X_0$, and the weighting matrices S and Q_k , find the time-varying gain matrix L_k^*

such that :

$$J \left[L_k^* \right] \leq J \left[L_k \right] \quad (8)$$

for all real $r \times m$ time - varying matrices L_k , where J is given by (6) subject to the dynamic constraint (7) on Λ_k .

This constrained dynamic optimization problem may be solved via the matrix minimum principle [9]. The results are given in the following theorem :

Theorem 1 - (Necessary conditions) - Any matrix of time-varying gains L_k^* which satisfies (8) for the cost functional (6) subject to the dynamic constraint equation (7), also satisfies the following equations :

$$L_k^* = - (B_k' \Lambda_{k+1}^* B_k)^{-1} B_k' \Lambda_{k+1}^* A_k \Lambda_k^* C_k' (C_k \Lambda_k^* C_k' + \Theta_k)^{-1} \quad (9)$$

$$\Lambda_{k+1}^* = \Gamma_k^* \Lambda_k^* \Gamma_k' + B_k L_k^* \Theta_k L_k' B_k' + \sum_k, \Lambda_0^* = X_0 \quad (10)$$

$$\Lambda_k^* = \Gamma_k' \Lambda_{k+1}^* \Gamma_k + Q_k, \Lambda_N^* = S \quad (11)$$

$$\Gamma_k^* = A_k + B_k L_k^* C_k \quad k = 0, 1, \dots, N-1. \quad (12)$$

Proof : The derivation of this non - linear two - point boundary value problem is given in appendix A.

It is interesting to note the particular structure of the optimal gain L_k^* : namely the first part $(B_k' \Lambda_{k+1}^* B_k)^{-1} B_k' \Lambda_{k+1}^*$ has a typical control - type form, whereas the second part $A_k \Lambda_k^* C_k' (C_k \Lambda_k^* C_k' + \Theta_k)^{-1}$ has a typical filtering-type form.

Also, since the conditions are only necessary, it may exist a sequence $\left\{ \hat{L}_k \right\}_{k=0}^{N-1}$ satisfying (9) - (12) which is not globally optimal.

Finally, existence and uniqueness of solutions of (9) - (10) are still largely open questions.

A feasible algorithm is now proposed, which has proven to have reasonable convergence properties.

Notation : L_k^n

n = number of iteration

k = index of time

1 - Make an initial guess on $\left\{ L_k^0 \right\}_{k=0}^{N-1}$ which determines also $\left\{ \Gamma_k^0 \right\}_{k=0}^{N-1}$.

2 - Compute forward in time, for $n = 0$ and $k = 0, 1, \dots, N-2$

$$\Lambda_{k+1}^{n+1} = \Gamma_k^n \Lambda_k^{n+1} \Gamma_k'^n + B_k L_k^n \Theta_k L_k'^n B_k' + \sum_k \Lambda_0^{n+1} = X_0 \quad (13)$$

giving the sequence $\left\{ \Lambda_k^1 \right\}_{k=0}^{N-1}$

3 - Compute backwards in time, for $n = 0$ and $k = N-1, \dots, 1$

$$\Pi_k^{n+1} = \Gamma_k'^n \Pi_{k+1}^{n+1} \Gamma_k^n + Q_k, \quad \Pi_N^{n+1} = S \quad (14)$$

giving the sequence $\left\{ \Pi_k^1 \right\}_{k=1}^N$

4 - Compute for $n = 0, k = 0, \dots, N-1$

$$L_k^{n+1} = - (B_k' \Pi_{k+1}^{n+1} B_k)^{-1} B_k' \Pi_{k+1}^{n+1} A_k \Lambda_k^{n+1} C_k' (C_k \Lambda_k^{n+1} C_k' + \Theta_k)^{-1} \quad (15)$$

$$\Gamma_k^{n+1} = A_k + B_k L_k^{n+1} C_k$$

giving the sequences $\left\{ L_k^1 \right\}_{k=0}^{N-1}$ and $\left\{ \Gamma_k^1 \right\}_{k=0}^{N-1}$

5 - Iterate steps 2, 3, 4 for $n = 1, \dots$ giving

$$\left\{ \Lambda_k^n \right\}_{k=0}^{N-1}, \left\{ \Pi_k^n \right\}_{k=1}^N, \left\{ L_k^n \right\}_{k=0}^{N-1}, \left\{ \Gamma_k^n \right\}_{k=0}^{N-1} \quad \text{until}$$

the desired degree of convergence is reached.

III - Infinite terminal - time problem -

Consider now the following linear, discrete - time, time - invariant, stochastic system :

$$\begin{aligned}x_{k+1} &= A x_k + B u_k + w_k \\ y_k &= C x_k + v_k\end{aligned}\quad (16)$$

where x , y , u are as in section II, and A , B , C are constant matrices of appropriate dimension. We will also make the following assumptions on the initial state of the plant and the stationary white - noise sequences w_k and v_k :

$$\begin{aligned}x_0 \text{ is } N(0, X_0) ; E\{v_k\} &= 0 ; E\{w_k\} = 0 \\ E\{w_k w'_l\} &= \Sigma \delta(k-l) ; E\{v_k v'_l\} = \Theta \delta(k-l) \\ E\{w_k v'_l\} &= 0 ; E\{w_k x'_0\} = 0 ; E\{v_k x'_0\} = 0\end{aligned}\quad (17)$$

The control law to be implemented is the following linear transformation on the output y_k :

$$u_k = L y_k \quad (18)$$

where L is an $r \times m$ constant gain matrix to be optimized.

The closed - loop system becomes accordingly :

$$\begin{aligned}x_{k+1} &= \Gamma x_k + B L v_k + w_k \\ \Gamma &= A + B L C\end{aligned}\quad (19)$$

The following cost is to be minimized :

$$J = \lim_{N \rightarrow \infty} \frac{1}{N} \sum_{k=0}^N E \left\{ \mathbf{x}'_k Q \mathbf{x}_k \right\} \quad (20)$$

or equivalently, assuming the system has reached steady - state :

$$J = E \left\{ \mathbf{x}'_k Q \mathbf{x}_k \right\} \quad (21)$$

Important remark - Note that we are tacitly assuming that the initial system (16) is output stabilizable (i. e. : there exists a constant $r \times m$ matrix L_0 such that $A + B L_0 C$ is stable). If it was not so, our problem would be meaningless since J would become infinite.

Defining now $\Lambda = E \left\{ \mathbf{x}'_k \mathbf{x}_k \right\}$ the steady - state covariance of the system state, J can simply be rewritten as :

$$J = \text{tr} (\Lambda Q) \quad (22)$$

Now from (19), Λ is known to satisfy :

$$\Lambda = \Gamma \Lambda \Gamma' + B L \Theta L' B' + \Sigma \quad (23)$$

so that we can formulate our problem as follows :

Problem statement : Given the output stabilizable plant (A, B, C) , the noise statistics Σ and Θ , and the weighting matrix Q , find the constant gain matrix L^* such that :

$$J(L^*) \leq J(L) \quad (24)$$

for all real $r \times m$ constant matrices L , where J is given (22).

This parameter optimization problem may be solved using Lagrange multipliers technique. The results are given in the following theorem :

Theorem 2 - (Necessary conditions) - Any constant gain matrix L which satisfies (24) for the cost criterion (22), also satisfies the following equations :

$$L^* = -(B' \Lambda^* B)^{-1} B' \Lambda^* A \Lambda^* C' (C \Lambda^* C' + \Theta)^{-1} \quad (25)$$

$$\Lambda^* = \Gamma^* \Lambda^* \Gamma'^* + B L^* \Theta L'^* B' + \Sigma \quad (26)$$

$$\Lambda^* = \Gamma'^* \Lambda^* \Gamma^* + Q \quad (27)$$

$$\Gamma^* = A + B L^* C \quad (28)$$

Proof : The simple proof of this theorem is omitted. Λ is an $n \times n$ Lagrange multiplier matrix.

Again note the particular structure of L^* . Also, since the conditions are only necessary, it may exist a matrix $\hat{L}$ satisfying (25) - (28) which is not globally optimal. Finally, note that plugging (25) and (28) in (26) and (27) reduces the problem to solving two coupled non linear equations in Λ^* and Λ^* .

We will now propose a feasible algorithm, for solving (25) - (28) which has proven to have reasonable convergence properties.

1 - Choose a stabilizing L^0 , for $\Gamma^0 = A + B L^0 C$

2 - Solve the linear algebraic matrix equation :

$$\Lambda^{n+1} = \Gamma^n \Lambda^{n+1} \Gamma'^n + B L^n \Theta L'^n B' + \Sigma$$

for $n = 1$ giving Λ^1 .

3 - Solve the linear algebraic matrix equation :

$$\text{for } n = 1 \quad \text{giving } \Lambda^1, \quad \Lambda^{n+1} = \Gamma^{n+1} \Lambda^{n+1} \Gamma^n + Q$$

4 - Compute :

$$L^{n+1} = - (B' \Lambda^{n+1} B)^{-1} B' \Lambda^{n+1} A \Lambda^{n+1} C' (C \Lambda^{n+1} C' + Q)^{-1}$$

$$\Gamma^{n+1} = A + B L^{n+1} C$$

5 - Iterate steps 2, 3, 4 for $n = 2, \dots$ giving sequences $\{\Lambda^n\}$, $\{\Gamma^n\}$, $\{L^n\}$ until the desired degree of convergence is reached.

Remark - Consider a time - invariant system and weighting matrix over a finite - horizon $[0, N]$, but allow as in section II time varying gains. It does not follow that, as $N \rightarrow \infty$, the time - varying gains of (9) - (12), go to the steady - state solution (25) - (28). From what we have previously seen, it is clear that it will be so if and only if the system is output stabilizable. Now, if this condition is not satisfied, section II gives a method of finding time-varying gains to stabilize the given system.

IV - A Numerical Example -

The results of sections II and III are applied to a fifth order academic example, simulated on an IBM 370/168 .

The actual values of A, B, C, Σ , Θ , Q, S and X_0 are :

$$A = \begin{bmatrix} 0.9 & 0. & -0.66 & 0. & 0.071 \\ 0. & 0.82 & -0.078 & 0. & 0.016 \\ 0. & 0. & 0.74 & 0. & 0.027 \\ 0. & 0. & 0. & 0.67 & 0. \\ 0. & 0. & 0. & 0. & 0.6 \end{bmatrix}$$

$$B = \begin{bmatrix} 0.62 & 0. \\ 0.086 & 0.27 \\ 0.17 & 0. \\ 0. & 0.5 \\ 0.4 & 0. \end{bmatrix}$$

$$C = \begin{bmatrix} 1.5 & 0. & -3. & 0. & 2.1 \\ 0. & 1. & -1. & 2. & 0.2 \end{bmatrix}$$

$$\Sigma = \begin{bmatrix} 0.11 & 0.0027 & -0.027 & 0. & 0.0024 \\ 0.0027 & 0.083 & -0.0032 & 0. & 0.0005 \\ -0.027 & -0.0032 & 0.076 & 0. & -0.0008 \\ 0. & 0. & 0. & 0.071 & 0. \\ 0.0024 & 0.0005 & -0.0008 & 0. & 0.067 \end{bmatrix}$$

$$\mathbb{Q} = \begin{bmatrix} 20. & 0. \\ 0. & 10. \end{bmatrix}$$

$$Q = 0.1 \quad I \ 5 \quad , \quad S = I \ 5 \quad , \quad X_0 = I \ 5$$

1 - Finite - time case (T = 10 s) : Time Varying Gains :

$$\text{We chose } \left\{ L_k^o \right\}_{k=0}^{N-1} = 0$$

Convergence of the algorithm is obtained when the whole sequence $\left\{ L_k^n \right\}_{k=0}^{N-1}$ reaches some stationary state.

As it was noted in section II, the optimal gain L_k has a typical behaviour as described by Fig. 1.

Figure 1

Fig. 2 (a. b. c. d) shows the different components

L_{ij}^* ($i = 1, 2 ; j = 1, 2$) of $\left\{ L_k^* \right\}$.

2 - Infinite - time case : Constant gains :

Figure 2

We chose for the initialisation : $L^0 = 0$ that stabilizes

$$\Gamma^0 = A + B L^0 C.$$

The results are :

$$L^* = \begin{bmatrix} -0.149 & -0.048 \\ 0.017 & -0.037 \end{bmatrix}$$

$$\Lambda^* = \begin{bmatrix} 1.36 & 0.068 & -0.35 & 0.0066 & -0.18 \\ 0.068 & 0.25 & -0.025 & -0.0075 & -0.0057 \\ -0.035 & -0.025 & 0.33 & -0.015 & 0.19 \\ 0.0066 & -0.0075 & -0.015 & 0.13 & -0.019 \\ -0.18 & -0.0057 & 0.19 & -0.019 & 0.30 \end{bmatrix}$$

$$\Pi^* = \begin{bmatrix} 0.32 & -0.017 & -0.39 & -0.012 & -0.048 \\ 0.017 & 0.29 & -0.0086 & -0.015 & 0.00086 \\ 0.39 & -0.0086 & 1.25 & 0.012 & 0.074 \\ 0.012 & -0.015 & 0.012 & 0.17 & 0.0027 \\ 0.048 & 0.00086 & 0.074 & 0.0027 & 0.14 \end{bmatrix}$$

The eigenvalues of Π^* are :

$$\begin{bmatrix} 0.916 \\ 0.813 \\ 0.687 \\ 0.630 \\ 0.451 \end{bmatrix}$$

and we note that :

$$\sum_{i=1}^5 \lambda_{ii}^* = 2.37 \quad (29)$$

Let's compare now these results with the L Q G design :

$$(Q_{LQG} = 0.5 \text{ I5} , \quad R_{LQG} = 0. \text{ I2})$$

The separation theorem gives us :

- Filtering Part : Optimal gain G^*

$$\begin{cases} G = -A \tilde{\Lambda}_{LQG}^* C' (C \tilde{\Lambda}_{LQG}^* C' + M)^{-1} \\ \tilde{\Lambda}_{LQG}^* = (A + G^* C) \tilde{\Lambda}_{LQG}^* A' + \Sigma \end{cases}$$

- Control Part : Optimal gain K^*

$$\begin{cases} K^* = -(B' \tilde{\Lambda}^* B + R_{LQG})^{-1} B' \tilde{\Lambda}^* A \\ \tilde{\Lambda}^* = A' \tilde{\Lambda}^* (A + BK^*) + Q_{LQG} \end{cases}$$

where $\tilde{\Lambda}^*_{LQG}$ is the steady - state error covariance matrix.

The obtained results were :

$$\tilde{\Lambda}^*_{LQG} = \begin{bmatrix} 0.94 & 0.05 & -0.16 & -0.008 & -0.01 \\ 0.05 & 0.25 & -0.016 & -0.007 & 0.0008 \\ -0.16 & -0.016 & 0.15 & 0.0024 & 0.0056 \\ -0.008 & -0.007 & 0.0024 & 0.12 & -0.00005 \\ -0.01 & 0.0008 & 0.0056 & -0.00005 & 0.10 \end{bmatrix}$$

Moreover the steady-state value of $\hat{\Lambda}^*_{LQG} = E[\hat{x} \hat{x}^t]$ is given by :

$$\hat{\Lambda}^*_{LQG} = (A+BK^*) \hat{\Lambda}^*_{LQG} (A+BK^*)' + G^* (C \tilde{\Lambda}^*_{LQG} C' + \Theta) G^{*}$$

It was obtained :

$$\hat{\Lambda}^*_{LQG} = \begin{bmatrix} 0.37 & 0.013 & -0.21 & 0.008 & -0.18 \\ 0.013 & 0.007 & -0.0003 & 0.002 & 0.007 \\ -0.21 & -0.0003 & 0.17 & -0.007 & 0.18 \\ 0.008 & 0.002 & -0.007 & 0.003 & -0.006 \\ -0.18 & 0.007 & 0.18 & -0.006 & 0.2 \end{bmatrix}$$

From where, for the LQG design, we have :

$$\Lambda^*_{LQG} = \hat{\Lambda}^*_{LQG} + \tilde{\Lambda}^*_{LQG}$$

$$\lambda_{LQG}^* = \begin{bmatrix} 1.31 & 0.06 & -0.38 & -0.0001 & -0.19 \\ 0.062 & 0.25 & -0.017 & -0.005 & 0.008 \\ -0.38 & -0.017 & 0.32 & -0.004 & 0.18 \\ -0.0001 & -0.005 & -0.004 & 0.13 & -0.006 \\ -0.19 & 0.008 & 0.18 & -0.006 & 0.31 \end{bmatrix}$$

$$\text{and } \sum_{i=1}^5 (\lambda_{ii}^*)_{LQG} = 2.32 \quad (30)$$

Moreover, the eigenvalues of $(A+G^*C)$ are :

$$\begin{bmatrix} 0.808 \\ 0.725 + 0.09j \\ 0.725 - 0.09j \\ 0.635 \\ 0.587 \end{bmatrix}$$

and the eigenvalues of $(A+BK^*)$ are :

$$\begin{bmatrix} 0.923 \\ 0.781 \\ 0.589 \\ 0.238 \\ 0.169 \end{bmatrix}$$

By comparison of the traces (29) and (30) we show that the performances of the linear output feedback control may be quite close to the optimum.

It is interesting to note that, as expected, the values for the infinite-time constant gain problem correspond to the "steady-state" values of the gains in the finite-time case.

V - Conclusion -

The control laws derived in this paper are to be thought as sub-optimal control laws for the linear regulator problem. Their interest relies on their easiness of implementation. An academic example has shown that their performances may be quite close to the optimum. Of course, this is not always the case and dynamic compensation may have to be used.

In case of a time - invariant, non output stabilizable system, section II provides a way of finding the time-varying gains that stabilizes the system.

Appendix A - Demonstration of theorem 1 -

From (6) J is given by :

$$J = \text{tr} \left(\delta \Lambda_N \right) + \sum_{k=1}^{N-1} \text{tr} \left(Q_k \Lambda_k \right) \quad (6) \quad (6)$$

Also from (7) Λ_{k+1} is given by :

$$\Lambda_{k+1} - \Lambda_k = \Gamma_k \Lambda_k \Gamma'_k - \Lambda_k + B_k L_k \Theta_k L'_k B'_k + \Sigma_k \quad (7\text{bis})$$

Consider now the scalar hamiltonian :

$$H(\Lambda_k, \Pi_{k+1}, L_k) = \text{tr}(\Lambda_k Q_k) + \text{tr} \left\{ (\Gamma_k \Lambda_k \Gamma'_k - \Lambda_k + B_k L_k \Theta_k L'_k B'_k + \Sigma_k) \Pi_{k+1} \right\}$$

From [9] we then obtain :

(i) Canonical equations :

$$\Lambda_{k+1}^* - \Lambda_k^* = \left. \frac{\partial H}{\partial \Pi_{k+1}} \right|_* \quad \text{which gives eq. (10)}$$

$$\Pi_{k+1}^* - \Pi_k^* = \left. -\frac{\partial H}{\partial \Lambda_k} \right|_* \quad \text{which gives eq. (11)}$$

(i i) Boundary conditions :

$$\text{At } k = 0 \quad \Lambda_0^* = X_0$$

$$\text{At } k = N \quad \Pi_N^* = S$$

(i i i) Minimization of the hamiltonian :

Since the L'_k s are not constrained :

$$\left. \frac{\partial H}{\partial L_k} \right|_* = 0 \quad \text{which yields eq. (9)}$$

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