OPTIMAL CONTROL PROBLEMS IN SOBOLEV SPACES
WITH WEIGHTS . NUMERICAL APPROACHES .APPLI-
CATIONS TO PLASMA OPTIMAL CONTROL AND TIME.DELAY PROBLEMS

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Abstract

We consider systems for which the state , $y(t,u)$, is given by the solution of the equation

$$(P(t,D)y,v)_H + a(t;y,v) = (g+Bu,v)_H \quad v \in V$$

in the sense of scalar distributions , where

$$P(t,D) = \sum_{r=1}^{s_1} B_r(t) D^r \quad s=1,2$$

A generalized class of optimal control problems with a quadratic performance criterion is considered and existence and unicity results are obtained.

The optimality system is studied, approximations for the solutions are given and applications to time-delay problems and plasma optimal control problems are shown .

Similar control problems for particular cases have been considered by the author in [11], [12], [13]. The definitions and notations are the same as in [1],

Since the technique of proving the existence and uniqueness of the solution in all cases is the same as the one we have been used in [11], [12], and [13], we shall resume ourselves to only give the results and comment them.

1. STATEMENT OF THE PROBLEM.

Let V and H be two Hilbert spaces, $V \subset H$, V dense in H and let the injection $V \rightarrow H$ be continuous. We identify H to its dual so that if V' denotes the dual of V , we have

$$V \subset H \equiv H' \subset V'$$

If in general we denote by $\mathcal{E}_t(\mathcal{L}(E, F))$ the space of all linear bounded operators from $E \rightarrow F$ which have continuous derivatives of every order with respect to t ; then, for $B_T(t) \in \mathcal{E}_t(\mathcal{L}(H, H))$ we shall be considering integro-differential operators of the form:

$$P(t, D) = \sum_{r=0}^k B_r(t) D^r \quad n \in \mathcal{N}^p \text{ (parabolic)} \quad (1)$$

and

$$P(t, D) = \sum_{r=0}^k B_r(t) D^r \quad n \in \mathcal{N}^h \text{ (hyperbolic)} \quad (2)$$

Let $a(t; u, v)$ be a sesqui-linear Hermitian continuous form on $V \times V$ such that for every $u, v \in V$:

$$|a(t; u, v)| \leq k \|u\| \|v\|, \quad k > 0 \quad (3)$$

$$a(t; u, u) \geq \alpha(t) \|u\|_V^2 \quad \alpha(t) > 0, \alpha(t) \in \mathcal{E}_t, t \in \mathcal{I} \quad (4)$$

We suppose that the mapping $t \rightarrow a(t; u, v)$ is measurable.

Let us consider now $\mathcal{U}$ - a hilbert space (the space of controls) , $\mathcal{U}_{ad}$ - a closed convex subset of $\mathcal{U}$, (admissible controls) $\mathcal{H}$ - another hilbert space and (S) a system governed by one of the integro- differential operators (1) or (2) , in the way that the state of the system , for each $u \in \mathcal{U}$ and $t \in \mathbb{R}$, is given by the solution $y=y(t, u)$ of the equation

$$(P(t, D)y, v)_H + a(t; y, v) = (g + Bu, v)_H, \quad v \in V \quad (5)$$

in the sense of scalar distributions.

OPTIMAL CONTROL PROBLEM. If $\mathcal{U}$ is the space of controls, $y(t, u)$ is the solution for (5) , C - the observation operator and $N \in \mathcal{L}(\mathcal{U}, \mathcal{U})$, find $u \in \mathcal{U}_{ad}$, such that

$$J(u) = \inf_{v \in \mathcal{U}_{ad}} J(v)$$

where $J(u)$ is a quadratic performance criterion:

$$J(u) = \|Cy - z_d\|_{\mathcal{H}}^2 + (Nu, u)_{\mathcal{U}} \quad (6)$$

2. RESULTS IN SOBOLEV SPACES WITH WEIGHTS.

We are going to establish a first result regarding this optimal control problem by considering Sobolev spaces with weights. This kind of spaces have been extensively used in obtaining existence and uniqueness results for boundary and mixt boundary value problems of parabolic or hyperbolic type in [3] , [5], [6], [16] .

We consider E a Hilbert space and $q(t)$ a real function with continuous derivatives , that satisfies the condition:

(A) there exists a constant $p_0 > 0$, such that $q'(t) \geq p_0, t \in \mathbb{R}$

For $k \in \mathbb{Z}$ (an integer), $\mathcal{D}^k(q; E)$ denotes the Hilbert space obtained by completing $\mathcal{D}(E)$ - (the space of functions defined on the real line and taking values in E , with derivatives of all orders and compact support), with respect to the structure defined by the Hermitian product

$$(\varphi, \psi)_{E; q, k} = (e^{-q(t)} D^k \varphi, e^{-q(t)} D^k \psi)_{L^2(E)} = \int_{-\infty}^{\infty} (e^{-q(t)} D^k \varphi, e^{-q(t)} D^k \psi)_E dt$$

The norm in $\mathcal{D}^k(q; E)$ is then:

$$\|\varphi\|_{E; q, k} = \|e^{-q(t)} D^k \varphi\|_{L^2(E)} = \left[\int_{-\infty}^{\infty} \|e^{-q(t)} D^k \varphi\|_E^2 dt \right]^{\frac{1}{2}}$$

For each integer $k \in \mathbb{Z}$ and each real function $q(t)$ verifying (A), $\mathcal{D}^k(q; E)$ is a space of distributions with values in E ,

$$\mathcal{D}^k(q; E) \subset \mathcal{D}'(E)$$

If $k \geq 0$, $\mathcal{D}^k(q; E)$ is the space of all (classes of) measurable functions from $\mathbb{R}$ into E such that $e^{-q(t)} D^h \varphi(t) \in L^2(E)$ for all $0 \leq h \leq k$.

If $k < 0$, $\mathcal{D}^{-k}(q; E)$ is a space of distributions having the property: for $T \in \mathcal{D}^{-k}(q; E)$, there exists $k+1$ functions $g_h \in L^2(E)$ such that

$$T = e^q g_0 + D(e^q g_1) + \dots + D^k(e^q g_k)$$

THEOREM 2.1. There exists a positive function $G(t)$, $G(t) \in C^1$, $(\forall) t \in R$, and for each integer k , the positive functions $g_k, g_k(t) \in C^1$, $t \in R$, such that if:

a) $p(t) \in C^1$, with $p'(t) \geq g_k(t)$ for all $t \in R$

and $p'(t) + G'(t) \geq p_0$ for all $t \in R$,

b) there exists a positive function $b(t)$, $b(t) \in C^1$ such that

$$(B_1(t)u, u)_H \geq b(t) \|u\|_H^2 \quad \text{for every } u \in H, t \in R$$

c) $B \in \mathcal{L}(\mathcal{U}, \mathcal{D}^k(p, H))$

d) $g \in \mathcal{D}^k(p; H)$

e) N is Hermitian and for every $u \in \mathcal{U}$, $(Nu, u)_{\mathcal{U}} \leq \|u\|_{\mathcal{U}}^2$

then, the stated optimal control problem for (1) has a unique solution $u^* \in \mathcal{U}_{ad} \subset \mathcal{U}$.

In this case, the state space of the problem is

$$\mathcal{D}^k(p+G; V) \cap \mathcal{D}^k(p; H)$$

with the norm $\| \cdot \|_{V; p+G, k}^2 + \| \cdot \|_{H; p, k}^2$

Also, for each integer $k \in \mathbb{Z}$ there exists a positive function $G_k(t) \in C^1$, such that if $p+G_k$ and $p+G+G_k$ are verifying (A), then, for $u \in \mathcal{U}_{ad}$ we have:

$$\|y\|_{V; p+G+G_k, k}^2 + \|y\|_{H; p+G_k, k}^2 \leq (g+Bu, y)_{H; p+G_k, k} \quad (7)$$

Since

$$C : \mathcal{D}^k(p+G; V) \cap \mathcal{D}^k(p; H) \longrightarrow \mathcal{H}$$

the adjoint :

$$C^* \in \mathcal{L}(\mathcal{H}'; [\mathcal{D}^k(p+G; V) \cap \mathcal{D}^k(p; H)])$$

or

$$C^* \in \mathcal{L}(\mathcal{H}'; \mathcal{D}^{-k}(p+G; V) \cup \mathcal{D}^{-k}(p; H))$$

We denote by $\tilde{P}(t, D)$ the formal adjoint of (1), with $\downarrow$ the canonical isomorphism $\mathcal{H} \rightarrow \mathcal{H}'$ and with $a^{\tilde{x}}(t; u, v)$ the adjoint defined by $a^{\tilde{x}}(t; u, v) = \overline{a(t; v, u)}$.

Thanks to the assumptions we made, the equation

$$(P(t, D)p(t, u^{\tilde{x}}), v)_H + a^{\tilde{x}}(t; p(t, u^{\tilde{x}}), v) = (C^{\tilde{x}} \downarrow (Cy(t, u^{\tilde{x}}) - z_d), v)_H; v \in V \quad (8)$$

has a unique solution

$$p(t, u^{\tilde{x}}) \in [\mathcal{D}^{\tilde{x}}(-(p+G)+F; V) \cap \mathcal{D}^{\tilde{x}}(-(p+G); H)] \cup [\mathcal{D}^{\tilde{x}}(-p+E; V) \cap \mathcal{D}^{\tilde{x}}(-p; H)]$$

where F and E are positive functions of class C^1 , such that $F-(p+G)$ and $-p+E$ are satisfying the condition a). Since $a(t; u, v)$ has been considered real and Hermitian, we can write then the optimality system in the following manner:

$$(g) \begin{cases} a(t; y(t, u^{\tilde{x}}), v) + (\sum_{r=-n}^1 B_r(t) D^r y(t, u^{\tilde{x}}), v)_H = (g + Bu^{\tilde{x}}, v)_H & v \in V \\ y(t; u^{\tilde{x}}) \in \mathcal{D}^{\tilde{x}}(p+G; V) \cap \mathcal{D}^{\tilde{x}}(p; H) \\ a(t; v, p(t, u^{\tilde{x}})) + (\sum_{r=-n}^1 \widetilde{B_r}(t) D^r p(t, u^{\tilde{x}}), v)_H = (C^{\tilde{x}} \downarrow (Cy(t, u^{\tilde{x}}) - z_d), v)_H; v \in V \\ p(t, u^{\tilde{x}}) \in [\mathcal{D}^{\tilde{x}}(-(p+G)+F; V) \cap \mathcal{D}^{\tilde{x}}(-(p+G); H)] \cup [\mathcal{D}^{\tilde{x}}(-p+E; V) \cap \mathcal{D}^{\tilde{x}}(-p; H)] \\ \left(\int_{\mathcal{U}} B^{\tilde{x}} p(t, u^{\tilde{x}}) + Nu^{\tilde{x}}, u - u^{\tilde{x}} \right)_{\mathcal{U}} \geq 0 \quad u \in \mathcal{U}_{ad} \end{cases}$$

THEOREM 2.2.

For each integer k , there exists a positive function $g_k(t)$, $g_k(t) \in C^1$ for $t \in R$, such that if:

a') $p(t) \in C^1$ and for every $t \in R$, $p'(t) \geq g_k(t)$

b') $(\exists) b(t) > 0$, $b(t) \in \mathcal{C}_t$ such that

$$(B_2(t)u, u)_H \geq b(t) \|u\|_H^2 \quad (\forall u \in H, t \in R)$$

c') $B \in \mathcal{L}(\mathcal{U}, \mathcal{D}^{\tilde{x}}(p; H))$

d') $g \in \mathcal{D}^{\tilde{x}}(p; H)$

e') N is hermitian and $(Nu, u)_{\mathcal{U}} \geq \alpha \|u\|_{\mathcal{U}}^2 \quad u \in \mathcal{U}$,

then we have a unique solution $u^{\mathbf{x}} \in \mathcal{U}_{ad}$ for the optimal control problem where $P(t, D)$ has the hyperbolic form.

The state space is given by $\mathcal{D}^k(p; V) \cap \mathcal{D}^{k+l}(p; H)$,
 $C \in \mathcal{L}(\mathcal{D}^k(p; V) \cap \mathcal{D}^{k+l}(p; H), \mathcal{H})$, $C^{\mathbf{x}} \in \mathcal{L}(\mathcal{H}', \mathcal{D}^{-k}(-p; V') \cup \mathcal{D}^{-k-l}(-p; H))$

Hence, the optimality system is going to take the form:

$$\begin{cases}
 a(t; y(t, u^{\mathbf{x}}), v) + \left(\sum_{r=-k}^l B_r(t) D^r y(t, u^{\mathbf{x}}), v \right)_H = (g + Bu^{\mathbf{x}}, v)_H \quad v \in V \\
 y(t, u^{\mathbf{x}}) \in \mathcal{D}^k(p; V) \cap \mathcal{D}^{k+l}(p; H) \\
 a(t; v, p(t, u^{\mathbf{x}})) + \left(\sum_{r=-k}^l B_r(t) D^r p(t, u^{\mathbf{x}}), v \right)_H = (C^{\mathbf{x}} \Delta (Cy(t, u^{\mathbf{x}}) - z_d), v)_V \quad v \in V \\
 p(t, u^{\mathbf{x}}) \in [\mathcal{D}^k(-p; V) \cap \mathcal{D}^{k+l}(-p; H)] \cup [\mathcal{D}^k(-p; V) \cap \mathcal{D}^{k+l}(-p; H)] \\
 \left(\Delta_{\mathcal{U}}^{-1} B^{\mathbf{x}} p(t, u^{\mathbf{x}}) + Nu^{\mathbf{x}}, u - u^{\mathbf{x}} \right)_{\mathcal{U}} \geq 0 \quad u \in \mathcal{U}_{ad}
 \end{cases}$$

3. OPTIMAL CONTROL IN DISTRIBUTIONS.

We denote with $\mathcal{D}'(A)$ the space of distributions on the real line having the values in H , $\mathcal{D}_-(A)$ - the space of indefinite differentiable functions on the real line with the support limited at right, having the values in H and with the usual Schwartz topology, $\mathcal{D}'_+(A)$ - the dual of $\mathcal{D}_-(A)$, $\mathcal{D}'_+(A) = \mathcal{L}(\mathcal{D}_-(A), H)$.

Because $a(t; u, v)$ satisfies to the conditions (3), (4), we can extend it to a sesqui-linear continuous form on $\mathcal{D}'(V) \times V$, $a(t; U, v)$. We can also extend canonically the Hermitian product $(\cdot, \cdot)_H$ to $\mathcal{D}'(H) \times H$.

Let be then a system for which the state is a distribution, solution for the equation

$$a(t; U, v) + (P(t, D)U, v)_H = (T + Bu, v)_H \quad v \in V$$

in the sense of scalar distributions.

THEOREM 3.1. If:

$$a'') \quad B \in \mathcal{L}(\mathcal{U}, \mathcal{D}'_+(H))$$

$$b'') \quad C \in \mathcal{L}(\mathcal{D}'_+(V), \mathcal{H})$$

$$c'') \quad N \in \mathcal{L}(\mathcal{U}, \mathcal{U}) \text{ Hermitian and } (Nu, u)_{\mathcal{U}} \geq k \|u\|_{\mathcal{U}}^2, u \in \mathcal{U}$$

$$d'') \quad (B_1(t)u, u) \geq b(t) \|u\|^2; u \in H, t \in \mathbb{R} \text{ for parab.}$$

$$\text{or } (B_2(t)u, u) \geq b(t) \|u\|^2; u \in H, t \in \mathbb{R} \text{ for hyperb.}$$

$$e'') \quad T \in \mathcal{D}'_+(H)$$

then the optimal control problem has a unique solution $u^{\mathbf{x}} \in \mathcal{U}$.

The state space is $\mathcal{D}'_+(V)$ and if $\text{supp}(T+Bu) \subset \{t; t \leq a\}$ then the same for U , $\text{supp}U(t, u) \subset \{t; t \geq a\}$.

Denoting by $X(t, u)$ the adjoint state, the optimality system gets the form:

$$(u) \left\{ \begin{array}{l} a(t; U(t, u^{\mathbf{x}}), v) + (P(t, D)U(t, u^{\mathbf{x}}), v)_H = (T+Bu^{\mathbf{x}}, v)_H; \quad v \in V \\ U(t, u^{\mathbf{x}}) \in \mathcal{D}'_+(V) \\ a^{\mathbf{x}}(t; X(t, u^{\mathbf{x}}), v) + (\tilde{P}(t, D)X(t, u^{\mathbf{x}}), v)_H = (C^{\mathbf{x}} \wedge (CU(t, u^{\mathbf{x}}) - z_d), v)_V; v \in V \\ X(t, u^{\mathbf{x}}) \in \mathcal{D}'_-(V) \\ (\int_{\mathcal{U}} B^{\mathbf{x}} X(t, u^{\mathbf{x}}) + Nu^{\mathbf{x}}, u - u^{\mathbf{x}})_{\mathcal{U}} \geq 0 \quad u \in \mathcal{U}_{ad} \subset \mathcal{U} \end{array} \right.$$

Another example of control problem in non normed spaces is given in ([6]). See also [7].

If $\mathcal{D}'_{a+}(A)$ has been a subspace of $\mathcal{D}'_+(A)$ and namely the space of distributions having the support limited at left by $a > 0$ then, we denote by $\mathcal{D}'_{a+}(A)$ the space of distributions of finite order, on the real line, with the values in A and having the support limited at left by $a > 0$.

This means that for $0 \leq h \leq m$, we can find a class of continuous functions $\{g_h(t)\}$ with the values in A , such that we can write:

$$T = g_0 + Dg_1 + \dots + D^m g_m \quad (g_1=0 \text{ for } t < a)$$

Obviously

$$\mathcal{D}^f_+(A) = \bigcup_{a \in R} \mathcal{D}^f_{a+}(A)$$

THEOREM 3.2.

If:

$$\text{aIV) } B \in \mathcal{L}(\mathcal{U}, \mathcal{D}^f_+(H))$$

$$\text{bIV) } C \in \mathcal{L}(\mathcal{D}^f_+(V), \mathcal{H})$$

$$\text{cIV) } N \in \mathcal{L}(\mathcal{U}, \mathcal{U}), \text{ Hermitian and}$$

$$(Nu, u)_{\mathcal{U}} \geq k \|u\|_{\mathcal{U}}^2 \quad ; \quad u \in \mathcal{U}, \quad k > 0$$

$$\text{dIV) } (B_i(t)u, u) \geq b(t) \|u\|^2 \quad u \in H, \quad t \in R, \quad i = 1 \text{ or } 2$$

$$\text{eIV) } T \in \mathcal{D}^f_+(H)$$

then the optimal control problem has a unique solution $u^{\mathbf{x}} \in \mathcal{U}$.

The state space is a space of distributions of finite order, $\mathcal{D}^f_+(V)$ and the support of $U(t, u)$ is on the half line $t \geq a$ a-real, if the support of $T+Bu$ is the same.

We obtaine the following optimality system:

$$(2) \left\{ \begin{array}{l} a(t; U(t, u^{\mathbf{x}}), v) + (P(t, D)U(t, u^{\mathbf{x}}), v)_H = (T + Bu^{\mathbf{x}}, v)_H \quad ; \quad v \in V \\ U(t, u^{\mathbf{x}}) \in \mathcal{D}^f_+(V) \\ a^{\mathbf{x}}(t; X(t, u^{\mathbf{x}}), v) + (\tilde{P}(t, D)X(t, u^{\mathbf{x}}), v)_H - (C^{\mathbf{x}} \wedge (CU(t, u^{\mathbf{x}}) - z_d), v)_V \quad ; \quad v \in V \\ X(t, u^{\mathbf{x}}) \in \mathcal{D}^f_-(V) \\ (A_{\mathcal{U}}^{-1} B^{\mathbf{x}} X(t, u^{\mathbf{x}}) + Nu^{\mathbf{x}}, u - u^{\mathbf{x}})_{\mathcal{U}} \geq 0 \quad u \in \mathcal{U}_{ad} \subset \mathcal{U} \end{array} \right.$$

Observation.

In general, if Ω is a domain in $\mathbb{R}^n$, $\Omega \subset \mathbb{R}^n$, the space of distributions of order zero on Ω is the space of Radon measures on Ω and we put $\mathcal{M}(\Omega) = \mathcal{D}'(\Omega) = \mathcal{X}'(\Omega)$, $\mathcal{X}(\Omega)$ being the vector space of all continuous functions on Ω whose support is contained in some compact subset of Ω , equipped with the finest locally convex topology for which the canonical injection $\mathcal{X}(\Omega) \rightarrow \mathcal{X}'(\Omega)$ is continuous ($K \subset \Omega$, K -compact). If $\varphi \in \mathcal{X}(\Omega)$ and $\mu \in \mathcal{M}(\Omega)$, then the value

$$\langle \mu, \varphi \rangle = \int_{\Omega} \varphi(x) d\mu(x)$$

is called the integral of φ with respect to the measure μ .

Hence if the state of a system is represented as a Radon measure on an interval, solution for the equation

$$a(t; \mu(t, u), v) + (P(t, D)\mu(t, u), v)_H = (T + Bu^x, v)_H \quad v \in V$$

in the distributional sense, then, by the THEOREM 3.2. we get an existence and uniqueness result for the optimal control problem and (11) becomes:

$$(13) \left\{ \begin{array}{l} a(t; \mu(t, u^x), v) + (P(t, D)\mu(t, u^x), v)_H = (T + Bu^x, v)_H \quad v \in V \\ a^x(t; \varphi(t, u^x), v) + (\tilde{P}(t, D)\varphi(t, u^x), v)_H = (C^x \wedge (C\mu(t, u^x) - z_d), v)_V \quad v \in V \\ \mu(t, u^x) \in \mathcal{M}_+(V) \\ \varphi(t, u^x) \in \mathcal{X}_c(V) \\ (\Delta_{26}^T B^x \varphi(t, u^x) + Nu^x, u - u^x)_{26} \geq 0 \quad u \in \mathcal{U}_{ad} \subset \mathcal{U} \end{array} \right.$$

4. INTERPRETATION OF RESULTS.

There are manyways of giving an interpretation to the system (7). All depends on the different assumptions regarding V, H, C, B . For instance for $V=H_0^1(\Omega)$ and $H=L^2(\Omega)$, or $V=H^1(\Omega)$ and $H=L^2(\Omega)$ we obtaine well known optimal control problems

We are going to consider here, that interpretation, which is more connected to our further applications.

Since $a(t; u, v)$ is continuous on $V \times V$ and (3) is satisfied, we can find an operator $\mathcal{A}(t) \in \mathcal{L}_t(\mathcal{X}(V, V))$, such that

$$a(t; u, v) = (\mathcal{A}(t)u, v)_V \quad (14)$$

From (4) there is obvious that we have

$$(\mathcal{A}(t)u, u)_V \geq \alpha(t) \|u\|_V^2, \quad \alpha(t) > 0, \quad \alpha(t) \in \mathcal{L}_t, t \in \mathbb{R} \quad (15)$$

We supposed $a(t; u, v)$ real and Hermitian; then $a(t; u, v) = a(t; v, u) = \overline{a(t; u, v)}$ and hence $\mathcal{A}(t)$ is autoadjoit $\mathcal{A}(t) = \mathcal{A}^*(t)$, $t \in \mathbb{R}$

We introduce a continuous linear operator $\mathcal{Y} \in \mathcal{L}(H, V)$ by the equality:

$$(f, v)_H = (\mathcal{Y}f, v)_V \quad f \in H, \quad v \in V$$

Then, the equations in (9) can be written as:

$$(\mathcal{A}(t)y, v)_V + \left(\sum_{r=-n}^i B_r(t) D^r y, v \right)_H = (\mathcal{Y}g, v)_V + (Bu^x, v)_H \quad v \in V$$

$$(\mathcal{A}(t)p, v)_V + \left(\sum_{r=-n}^i \widetilde{B}_r(t) D^r p, v \right)_V = (C^x / 1 (Cy(t, u^x) - z_d), v)_V \quad v \in V$$

and so , the system (9) becomes:

$$(9) \left\{ \begin{array}{l} \sum_{r=-n}^1 B_r(t) D^r y(t, u^x) = B u^x \quad \text{in } H \\ \mathcal{A}(t) y(t, u^x) = \gamma g(t) \quad \text{in } V \\ \mathcal{A}(t) p(t, u^x) + \sum_{r=-n}^1 \widetilde{B_r(t)} D^r p(t, u^x) = C^x \wedge (C y(t, u^x) - z_d) \quad \text{in } V \\ e^{-(p+G)_D k_y} \in L^2(V) \\ e^{-p_D k_y} \in L^2(H) \\ p(t, u^x) \in [\mathcal{D}^{\sim}(F-p-G; V) \cap \mathcal{D}^{\sim}(-p-G; H)] \cup [\mathcal{D}^{\sim}(E-p; V) \cap \mathcal{D}^{\sim}(-p; H)] \\ (\bigwedge_{\mathcal{U}}^i B^x p(t, u^x) + N u^x, u - u^x)_{\mathcal{U}} \geq 0 \quad u \in \mathcal{U}_{ad} \subset \mathcal{U} \end{array} \right.$$

and the same for (10) :

$$(10) \left\{ \begin{array}{l} \sum_{r=-n}^1 B_r(t) D^r y = B u^x \quad \text{in } H \\ \mathcal{A}(t) y(t, u^x) = \gamma g(t) \quad \text{in } V \\ \mathcal{A}(t) p(t, u^x) + \sum_{r=-n}^1 \widetilde{B_r(t)} D^r p(t, u^x) = C^x \wedge (C y(t, u^x) - z_d) \quad \text{in } V \\ e^{-p_D k_y} \in L^2(V) \\ e^{-p_D k_y + 1} \in L^2(H) \\ p(t, u^x) \in [\mathcal{D}^{\sim}(-p; V) \cap \mathcal{D}^{\sim}(-p; H)] \cup [\mathcal{D}^{\sim}(-p; V) \cap \mathcal{D}^{\sim}(-p; H)] \\ (\bigwedge_{\mathcal{U}}^i B^x p(t, u^x) + N u^x, u - u^x)_{\mathcal{U}} \geq 0 \quad u \in \mathcal{U}_{ad} \subset \mathcal{U} \end{array} \right.$$

We must mention that , as usual [L9], [10], by $D^r \varphi$, $r > 0$ we understand the derivative in the distributional sense , and for $\varphi \in \mathcal{D}_-(E)$,

$$\begin{aligned}
 D^{-r} \varphi(t) &= (Y^{(r)} \varphi)(t) = (Y_r \varphi)(t) = \frac{1}{\Gamma(r)} \left[\text{Pf}(y^{r-1})_{y>0} \right] \varphi(t) = \\
 &= \frac{1}{\Gamma(r)} \int_0^\infty y^{r-1} \varphi(t-y) dy \quad \text{in } E
 \end{aligned}$$

where Y is the Heaviside function, and

$$Y_m = \frac{1}{\Gamma(m)} \text{Pf}(x^{m-1})_{x>0} \quad m > 0$$

$$Y_{-1} = \delta^{(1)} = (-1) \varphi^{(1)}(0) \quad m = -1 \leq 0$$

$$Y^{(m)} = Y_m$$

$$I^m T = Y_m \times T$$

$$D^m T = Y_{-m} \times T$$

for $T \in \mathcal{D}'_+$

This way, if $B_r(t) \in \mathcal{C}_c(\mathcal{L}(H, H))$, for every $\varphi \in \mathcal{D}_-(H)$

we have:

$$\begin{aligned}
 P(t, D) \varphi(t) &= B_{-n}(t) \frac{1}{\Gamma(n)} \int_0^\infty y^{n-1} \varphi(t-y) dy + \dots + \frac{B_{-2}(t)}{2} \int_0^\infty y \varphi(t-y) dy + \\
 &+ B_{-1}(t) \int_0^\infty \varphi(t-y) dy + B_0(t) \varphi(t) + B_1(t) \frac{d\varphi}{dt}
 \end{aligned}$$

with the values in H .

In the same time, for $\varphi \in \mathcal{D}_-(V)$

$$\widetilde{P}(t, D) \varphi(t) = \sum_{r=-n}^1 (-1)^r D^r [B_r(t) \varphi(t)]$$

If we consider for instance $n = 2$, we obtaine the following form

for the optimality system:

$$\begin{cases}
 B_1(t) \frac{dy}{dt} + B_0(t)y + B_{-1}(t) \int_0^\infty y(t-x)dx + \frac{B_{-2}(t)}{2} \int_0^\infty xy(t-x)dx = Bu^x & \text{in } H \\
 A_0(t)y = \int g(t) & \text{in } V \\
 -A_1(t) \frac{dp}{dt} + (A_0(t) - A_1^{(1)}(t) + A_2(t))p - A_{-1}(t) \int_0^\infty p(t-x)dx + \\
 + d_1^{-1} \int_0^\infty dx \int_0^\infty A_{-1}^{(1)}(t-x)p(t-x-z)dz + \frac{A_{-2}(t)}{2} \int_0^\infty xp(t-x)dx + \\
 + d_1^{-1} \int_0^\infty dx \int_0^\infty A_{-2}^{(1)}(t-x)p(t-x-z)dz + \frac{d_2^{-2}}{2} \int_0^\infty x B_{-2}^{(2)}(t-x)dx \int_0^\infty zp(t-x-z)dz \\
 = 0^x \wedge (G(y(t, u^x) - z_d)) & \text{in } V \\
 e^{-(p+G)} D^k y \in L^2(V) \\
 e^{-p} D^k y \in L^2(H) \\
 p(t, u^x) \in [\tilde{\mathcal{D}}(F-p-G; V) \cap \tilde{\mathcal{D}}(-p-G; H)] \cap [\tilde{\mathcal{D}}(E-p; V) \cap \tilde{\mathcal{D}}(-p; H)] \\
 (\int_{\mathcal{U}} B^x p(t, u^x) + Nu^x, u - u^x)_{\mathcal{U}} \geq 0 \quad u \in \mathcal{U}_{ad} = \mathcal{U}
 \end{cases}
 \quad (10)$$

where d_i^j are integers, $\int B_r(t) = A_r(t)$ $t \in R$ and $A_r^{(i)}(t)$ means the i -th derivative of $A_r(t)$.

4. NUMERICAL APPROACH.

From all the numerical methods used in control theory, we choosed the one which seems to us that works the best; this is the Galerkin method, and we shall apply it to the systems (9) and (10).

Let V be separable and we consider an internal Hilbert approximation for V , $\{V_h, P_h, R_h\}_{h \in \mathcal{P}}$ which is stable and convergent ([5], [14]) and such that

$$\overline{\bigcup_{h \in \mathcal{P}} V_h} = V$$

For instance , we take $w_1, w_2, \dots$ a free and total sequence in V and we denote by V_h the subspace of V generated by $w_1, w_2, \dots, w_h$. P_h is the canonical injection $V_h \longrightarrow V$ and R_h is the orthogonal projection of V on V_h .

This way , to each V_h , $k \in \mathbb{Z}$, it corresponds the space $\mathcal{D}^k(p; V_h)$ where the weight p satisfies (A). Obvious that:

$$\mathcal{D}^k(p; V_h) \subset \mathcal{D}^k(p; V) \quad \text{and} \quad \bigcup_{h \in \mathcal{P}} \mathcal{D}^k(p; V_h) = \mathcal{D}^k(p; V)$$

For each h , we define y_h as the solution of the equation:

$$(P(t, D)y_h, v_h)_H + a(t; y_h, v_h) = (f + Bu, v_h)_H; \quad v_h \in V_h \quad (19)$$

If $f + Bu \in \mathcal{D}^k(p; H)$ then , we find a unique solution $y \in \mathcal{D}^k(p; G; V_h) \cap \mathcal{D}^k(p; H)$ such that:

$$\|y_h\|_{V_h, p+G+G_k, k}^2 + \|y_h\|_{H, p+G_k, k}^2 \leq (f + Bu, y_h)_{H, p+G_k, k}$$

where G can be choosen the same for all h (by taking the supremum) in the way that $p+G_k$, $p+G+G_k$ are satisfying (A).

Let us consider then , the sequence pf approximative solutions $\{y_h\}_{h \in \mathcal{P}}$. If y is the solution of the first equation in (9) then, in particular , we have:

$$(P(t,D)y, v_h)_H + a(t; y, v_h) = (f + Bu, v_h)_H \quad v_h \in V_h$$

and for each $h \in \mathcal{H}$, $y_h - y$ is satisfying the homogenous equation

$$(P(t,D)[y_h - y], v_h)_H + a(t; y_h - y, v_h) = 0 \quad v_h \in V_h$$

and

$$\|y_h - y\|_{V_h, p+G_k, k}^2 + \|y_h - y\|_{H, p+G_k, k}^2 \leq 0$$

Since P_h is a stable continuous injection from $V_h \rightarrow V$ the application $\varphi_h \rightarrow P_h \varphi_h$ from $\mathcal{D}(V_h) \rightarrow \mathcal{D}(V)$ can be extended to a continuous application of

$$\mathcal{D}^k(p; V_h) \rightarrow \mathcal{D}^k(p; V)$$

Also, the approximation $\{V_h, P_h, R_h\}_{h \in \mathcal{H}}$ was supposed convergent, so, we have:

$$\|y_h - y\|_{V, p+G_k, k}^2 + \|y_h - y\|_{H, p+G_k, k}^2 \leq 0$$

and from here $y_h \rightarrow y$ when $h \rightarrow \infty$ in the norm of

$\mathcal{D}^k(p+G_k, V) \cap \mathcal{D}^k(p+G_k, H)$ for the $P(t,D)$ parabolic and in the norm of $\mathcal{D}^k(p+G_k, V) \cap \mathcal{D}^{k''}(p+G_k, H)$ for the hyperbolic form of $P(t,D)$.

If $G_k(t)$ is such that $\exp(G_k(t))$ is bounded over the whole real line, then there is a continuous embedding of $\mathcal{D}^k(p+G_k, V)$ in $\mathcal{D}^k(p+G, V)$, of $\mathcal{D}^k(p+G_k, H)$ in $\mathcal{D}^k(p, H)$, of $\mathcal{D}^k(p+G_k, V)$ in $\mathcal{D}^k(p, V)$ and $\mathcal{D}^{k''}(p+G_k, H)$ in $\mathcal{D}^{k''}(p, H)$ for all $k \in \mathbb{Z}$.

Hence $y_h \rightarrow y$ in the norm of the state spaces.

The approximative adjoint state $p_h(t, u)$ is defined as the solution of

$$(\widetilde{P}(t, D)p_h, v_h)_H + a(t; p_h, v_h) = (C^T \Delta (Cy_h - z_d), v_h)_{V_h} \quad v_h \in V_h \quad (20)$$

and the convergence of $\{p_h\}_{h \in \mathcal{H}}$ is obtained in the same way.

Now, if

$$J_h(u) = \|Cy_h - z_d\|_{\mathcal{Z}}^2 + (Nu, u)_{\mathcal{U}} \quad u \in \mathcal{U}_{ad} \subset \mathcal{U}$$

we denote by $u_h \in \mathcal{U}_{ad} \subset \mathcal{U}$ the corresponding optimal control

$$J_h(u_h) = \min_{u \in \mathcal{U}_{ad}} J_h(u)$$

By an already classical result ([5], [7], [14]) we have, under our hypothesis that $u_h \rightarrow u$ strongly in $\mathcal{U}$.

Remarque. Discretization for $\mathcal{U}$ in $\{\mathcal{U}_h\}$ can also be considered.

6. APPLICATIONS.

There are many interesting control problems that are arising in different fields of research and to which the above results can be applied.

Let us consider first the general form of a linear bounded delay-differential equation of closed-cycle type:

$$\dot{x}(t) + \int_{-\infty}^t d_{\tau} S(t-\tau)x(\tau) + \int_{-\infty}^t d_{\tau} K(t-\tau)x(\tau) = f(t) \quad (21)$$

where $S(t-\tau)$ and $K(t-\tau)$ are the saltus matrix and the continuous kernel corresponding to the kernel $G(t-\tau)$. If $S \equiv 0$ and K is differentiable and Riemann integrable, then the equation (21) becomes an

integro-differential equation of the form:

$$\dot{x}(t) + \int_{-\infty}^t K_z(t-z) \cdot x(z) \, dz = f(t)$$

This is leading us to consider the equation

$$\dot{x}(t) + \frac{B_z(t)}{z} \int_0^{\infty} z \, x(t-z) \, dz = B(t)u(t) + f(t) \quad (22)$$

in H , with the initial condition $x(t) = \mathcal{H}^{-1}(t) \int g(t)$ in V , and the corresponding hereditary process of closed-cycle type described by it. Each pair $(\mathcal{H}^{-1}(t) \int g(t), u(t))$ where $\mathcal{H}(t)$, $\int$ and u are defined as in 2. is called optimal. The optimal control problem is then: find an admissible pair $(\mathcal{H}^{-1}(t) \int g(t), u(t))$, with the corresponding trajectory $x(t, u)$ such that the quadratic performance criterion (6) gets minimized.

Under the hypothesis a), c), d), e) the THEOREM 2.1. is giving us an existence and uniqueness result. There is evident that we can take $H = R^n$ and V a Hilbert subspace of R^n , dense in R^n and this way, already classical results are included in our statement. The optimality system is easy to be written.

6.2. Areas of most importance as plasma physics, radiative transfer, rarefied gas dynamics, neutron transport theory are employing a form of transport equation.

In the kinetic theory a basic problem concerns the rate and manner in which gas molecules approach their equilibrium distribution.

A general problem can be envisaged as a coupled time and space relaxation due to an initial perturbation.

If the problem is considered linear, then, discussing separately the time and the space relaxation due to an initial perturbation, we can consider the two equations:

$$\frac{\partial h(c,t)}{\partial t} = - \hat{I}_0 [h(c,t)] \quad (19)$$

$$c \nabla h(r,c) = - \hat{I}_0 [h(r,c)] \quad (20)$$

The equation (19) describes the time-dependent perturbation of a single gas due to spatially uniform anisotropic disturbances and equation (20) describes space-dependent perturbation due to non-uniform anisotropic stationary disturbances.

The time dependent problem is closely related to the pulsed neutron problem in an infinite medium. This arises when a small admixture of an alien gas is introduced into a host gas, the latter being in an equilibrium condition. However we should point out that deviation from the equilibrium neutron distribution are introduced by the injection of particles, whereas in the gas problem the deviation is brought about by the introduction of energy.

If in (19) we set:

$$\hat{I}_0(h) = V(c)h(x,c) - \int_0^\infty dc' e^{-c'^2} K(c,c')h(x,c') \quad (21)$$

$c = v(m/2kT)^{1/2}$ - the reduced velocity

and if we shall require now that the kernel $K(c,c')$ takes the form:

$$K(c,c') = \frac{c'e^{c'^2} A(c,t)h(t-c',t)}{h(c',t)} \quad (22)$$

such that $A(c,t) \in \mathcal{C}_c(R)$ for each c and every $t \in R$, we can consider the equation $\frac{\partial h(c,t)}{\partial t} + V(c)h(c,t) - \int_0^\infty e^{-c'^2} K(c,c')h(c',t)dc' = u$ in R (22'). If we want now to control even the injection of particles even the introduction of the energy, we can formulate the problem: for a kernel given by (22) find a control $u \in \mathcal{U}_\mu \cap \mathcal{U}$ such that if $h=h(c,t,u)$ is the solution in (22'), the quadratic performance criterion $J(u)$ is minimized.

The THEOREM 2.1. is giving us and in this case an existence and uniqueness result, for $H = R$.

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