MODELLING AND CONTROL FOR
DISTRIBUTED PARAMETER SYSTEMS

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1. INTRODUCTION

For most systems it is usual to first perform the identification in order to obtain a mathematical model, and then design the control action on the basis of this model. However, it may be that the performance index which is used to design the control is not particularly sensitive to variations which result from updating the model by identification. For distributed parameter systems the identification and control problems are particularly difficult because of the limitation on the number of sensors, and the places at which control can be applied. In this paper a theory is developed from which it is possible to determine whether the identification should continue, the class of controls expanded, or whether the existing model and controls are satisfactory.

Another application of the theory is to control problems which are subject to perturbations. The optimal control problem for linear systems with quadratic performance indices is well understood (Curtain and Pritchard (1), Lions (2)). If the control can be implemented it will be of the feedback variety and will be independent of the initial state. This is regarded as particularly desirable since the control although designed from the mathematical model depends in an important way on the actual state

of the system, and will, therefore take some account of a lack of perfect identification through perturbations to the system. For distributed parameter systems the class of controls which can be practically implemented is very small, and the best control within this class will in general depend on the initial state of the system. In our theory we will reformulate the performance index so that the optimal control from the limited class for the new index will not depend on the initial state, and will take some account of the perturbation to the system dynamics.

In order to develop our ideas we will assume that the basic model for the system is determined by a semigroup (Hille and Phillips (3)). Thus our results will also be valid for systems governed by ordinary differential equations, and certain differential-delay equations, although our main interest is in systems governed by partial differential equations. We could easily extend our results to systems determined by mild evolution operators (1), but for ease of presentation we will only consider the autonomous case.

2. PRELIMINARIES

The mathematical model of the control process is assumed to take the form

$$(2.1) \quad \dot{z} = A_0 z + Bu, \quad z(0) = z_0$$

where z is an element of a real Hilbert space H , and u is a control with $u(\cdot) \in L_2(0, T, \mathcal{U})$, where $\mathcal{U}$ is also a real Hilbert space. We assume that the operator A_0 is a closed linear operator, defined on a

dense domain $D(A_0) \subset H$, and A_0 is the infinitesimal generator of a strongly continuous semigroup T_t . We take $B \in \mathcal{L}(\mathcal{U}, H)$, although this is a very restrictive assumption and does not allow us to consider control action on the boundary. Our results can be extended to the case where B is unbounded following the results in Curtain and Pritchard (4).

The first problem is to define a solution of (2.1). We use the concept of a mild solution

$$(2.2) \quad z(t) = T_t z_0 + \int_0^t T_{t-s} B u(s) ds$$

It can be shown for $z_0 \in H$, $u \in L_2(0, T; \mathcal{U})$ that $z \in C(0, T; H)$, however we are not able to differentiate (2.2) to obtain a strict solution of (2.1) unless for example $z_0 \in D(A_0)$, and $u \in C'(0, T; \mathcal{U})$.

The controls will be restricted by practical considerations and from this limited class a feedback control u_0 is taken of the form

$$(2.3) \quad u_0 = -Fz$$

where $F \in \mathcal{L}(H, \mathcal{U})$. Again we can extend the results to F depending on time, or F unbounded (4), and this would allow us to consider point sensors.

With u_0 given by (2.3) we may write (2.1) in the form

$$(2.4) \quad \dot{z} = Az + B\bar{u}, \quad z(0) = z_0$$

where

$$A = A_0 - BF, \quad D(A) = D(A_0)$$

$$\bar{u} = u - u_0$$

Of course we must interpret (2.4) in the mild sense, that is

$$(2.5) \quad z(t) = S_t z_0 + \int_0^t S_{t-s} B \bar{u}(s) ds$$

where

$$(2.6) \quad S_t z_0 = T_t z_0 - \int_0^t T_{t-s} B F S_s z_0 ds$$

It is easily shown that S_t is well defined by (2.6) and S_t is a strongly continuous semigroup (1, 3).

Now we wish to consider unbounded perturbations of the operator A_0 , so that the perturbed system can be written formally as

$$(2.7) \quad \dot{z} = A_0 z + A_1 z + B u, \quad z(0) = z_0$$

or equivalently

$$(2.8) \quad \dot{z} = A z + A_1 z + B \bar{u}, \quad z(0) = z_0.$$

In order to explain what is meant by a solution of (2.7), (2.8), we need to impose conditions on the unbounded operator A_1 . We will assume

$$(2.9) \quad D(A_0) \subset D(A_1)$$

$$(2.10) \quad \|T_t A_1 z_0\| \leq \frac{M}{t^\alpha} \|z_0\| \quad \text{for all } t \in (0, \bar{T}], z_0 \in D(A_0)$$

where M and α are constants with $0 \leq \alpha < 1$. Then for each $t \in (0, \bar{T}]$, $T_t A_1 \in \mathcal{L}(D(A_0), H)$ and since $D(A_0)$ is dense in H we may extend the operator $T_t A_1$ to $\overline{T_t A_1}$ where $\overline{T_t A_1} \in \mathcal{L}(H)$. We show in the appendix that we are then able to define a strongly continuous semigroup U_t , by

$$(2.11) \quad U_t z_0 = T_t z_0 - \int_0^t T_{t-s} B F U_s z_0 ds + \int_0^t \overline{T_{t-s} A_1} U_s z_0 ds$$

or

$$(2.12) \quad U_t z_0 = S_t z_0 + \int_0^t \overline{S_{t-s} A_1} U_s z_0 ds$$

The mild solution of either (2.7) or (2.8) is then given by

$$(2.13) \quad z(t) = U_t z_0 + \int_0^t U_{t-s} \overline{B} u(s) ds$$

or

$$(2.14) \quad z(t) = S_t z_0 + \int_0^t \overline{S_{t-s} A_1} z(s) ds + \int_0^t S_{t-s} \overline{B} u(s) ds.$$

Associated with the control problem will be a cost functional

$$(2.15) \quad J(u) = \langle z(T), G z(T) \rangle_H + \int_0^T \langle z(s), W z(s) \rangle_H + \langle u(s), R u(s) \rangle_{\mathcal{U}} ds$$

where $R \in \mathcal{L}(\mathcal{U})$, $G, W \in \mathcal{L}(H)$. R, G, W are self adjoint and $G, W \geq 0$, $R > 0$.

3. PERTURBATION RESULTS

In this section we derive a number of perturbation results for the cost functional (2.15). We will distinguish between the cost of a control u for the mathematical model (2.2) and the cost for the perturbed system (2.13) or (2.14) by writing $J_M(u)$ and $J_P(u)$ respectively.

A fundamental role will be played by a strongly continuous linear operator $Q(t)$, defined by

$$(3.1) \quad Q(t) z_0 = S_{T-t}^* G S_{T-t} z_0 + \int_t^T S_{s-t}^* (W + F^* R F) S_{s-t} z_0 ds$$

where S_t^* is the dual semigroup of S_t and is known to be strongly continuous since H is a Hilbert space. It is shown in (1) that Q satisfies the differential equation

$$(3.2) \quad \frac{d}{dt} \langle Q(t)z_1, z_2 \rangle_H + \langle Az_1, z_2 \rangle_H + \langle z_1, Az_2 \rangle_H + \langle z_1, (W+F^*RF)z_2 \rangle_H = 0$$

with $Q(T) = G$, and $z_1, z_2 \in D(A_0)$.

Theorem 3.1

For the perturbed system defined by (2.14), the model defined by (2.2) and Q defined by (3.1)

a) the cost of the control u_0 for the model is

$$(3.3) \quad J_M(u_0) = \langle z_0, Q(0)z_0 \rangle_H,$$

b) the difference in the cost of the perturbed system and the model with control u_0 , is

$$(3.4) \quad J_p(u_0) - J_M(u_0) = 2 \int_0^T (\langle z(t), \overline{Q(t)A_1} z(t) \rangle_H) dt$$

c) the difference in the cost of the perturbed system with control $\bar{u} + u_0$, and the model with control u_0 is

$$(3.5) \quad J_p(\bar{u} + u_0) - J_M(u_0) = 2 \int_0^T (\langle z(t), \overline{Q(t)A_1} z(t) \rangle_H + \\ + \langle \bar{u}(t), B^*Q(t)z(t) \rangle_{\mathcal{U}} - \langle \bar{u}(t), RFz(t) \rangle_{\mathcal{U}}) dt$$

Proof

a) The solution of (2.2) with $u = u_0$ is given by (2.5) with $\bar{u} = 0$, that is

$$z(t) = S_t z_0$$

From (3.1)

$$\begin{aligned} \langle z_0, Q(0) z_0 \rangle &= \langle z(T), G z(T) \rangle_H + \int_0^T \langle z(s), (W + F^* R F) z(s) \rangle_H ds \\ &= \langle z(T), G z(T) \rangle_H + \int_0^T \langle z(s), W z(s) \rangle_H + \langle u_0(s), R u_0(s) \rangle_{\mathcal{U}} ds \\ &= J_M(u_0) \end{aligned}$$

b) This follows from part (c) with $\bar{u} = 0$.

c) It is easy to show that

$$\| Q(t) A_1 z_0 \| \leq \frac{\bar{M}}{(T-t)^\alpha} \| z_0 \| \quad \text{for all } t \in [0, T) \text{ } z_0 \in D(A_0)$$

Thus $\overline{Q(t)A_1}$ is the bounded extension of $Q(t)A_1$ for all $t \in [0, T)$. With this interpretation the proof is by direct substitution and is a special case of a similar result in Pritchard and Crouch (5) and so will not be repeated here.

We now use this theorem to obtain bounds for $J_p(u_0) - J_M(u_0)$, and $J_p(\bar{u} + u_0) - J_n(u_0)$. In particular we need to establish the conditions under which there exists positive constants β, γ , such that

$$(3.6) \quad | J_p(u_0) - J_M(u_0) | \leq \beta J_p(u_0)$$

$$(3.7) \quad J_p(\bar{u} + u_0) - J_M(u_0) \geq -\gamma J_p(\bar{u} + u_0).$$

Lemma 3.1

If either

$$a) \quad G \equiv 0, \quad \alpha < 1, \quad W + F^*RF > wI \quad \text{for } w > 0.$$

$$\text{or } b) \quad G > gI, \quad g > 0; \quad W + F^*RF > wI, \quad w > 0, \quad \alpha < \frac{1}{2}.$$

then there exists β such that (3.6) holds.

Proof

Assume (a) holds and let $W + F^*RF < kI$. Substituting for the expression (3.1) for $Q(t)$ with $G = 0$ in (3.4), yields

$$J_p(u_o) - J_M(u_o) = 2 \int_0^T \int_t^T < S_{\sigma-t} z(t), (W + F^*RF) \overline{S_{\sigma-t} A_1} z(t) >_H d\sigma dt.$$

Now S_t is a strongly continuous semigroup, hence there exists a constant M_s such that

$$\| S_t \| \leq M_s \quad \text{for } t \in [0, T].$$

In the appendix we have shown that

$$\| \overline{S_{\sigma-t} A_1} \| \leq \frac{M_1}{(\sigma-t)^\alpha}.$$

Thus

$$\begin{aligned} |J_p(u_o) - J_M(u_o)| &\leq 2 \int_0^T \int_t^T M_s M_1 k \frac{\|z(t)\|^2}{(\sigma-t)^\alpha} d\sigma dt \\ &\leq \frac{2T^{1-\alpha} M_s M_1 k}{1-\alpha} \int_0^T \|z(t)\|^2 dt \\ &\leq \beta J_p(u_o) \end{aligned}$$

where
$$\beta = \frac{2T^{1-\alpha} M_s M_1 k}{w(1-\alpha)}$$

Now assume (b) holds, then from (2.14) we have

$$(3.8) \quad z(T) = S_{T-t} z(t) + \int_t^T \overline{S_{T-s} A_1} z(s) ds$$

Substituting from (3.8) and (3.1) in (3.4) yields

$$\begin{aligned} J_p(u_o) - J_M(u_o) &= 2 \langle z(T), G \int_o^T \overline{S_{T-\sigma} A_1} z(\sigma) d\sigma \rangle_H \\ &- 2 \int_o^T \int_t^T \overline{S_{T-\sigma} A_1} z(\sigma) d\sigma, G \overline{S_{T-t} A_1} z(t) \rangle_H dt \\ &+ 2 \int_o^T \int_t^T \langle S_{\sigma-t} z(t), (W+F^*RF) \overline{S_{\sigma-t} A_1} z(t) \rangle_H d\sigma dt \end{aligned}$$

Hence

$$\begin{aligned} |J_p(u_o) - J_M(u_o)| &\leq 2 \|z(T)\| \|G\| \int_o^T \frac{M_1}{(T-\sigma)^\alpha} \|z(\sigma)\| d\sigma \\ &+ 2 \int_o^T \int_t^T \frac{M_1}{(T-\sigma)^\alpha} \|z(\sigma)\| d\sigma \|G\| \frac{M_1}{(T-t)^\alpha} \|z(t)\| dt \\ &+ 2 \int_o^T \int_t^T \frac{M_s k M_1}{(\sigma-t)^\alpha} \|z(t)\|^2 d\sigma dt \\ &\leq N_1 \|z(T)\| \|z(\cdot)\|_T + N_2 \|z(\cdot)\|_T^2 \end{aligned}$$

where
$$N_1 = \frac{2 \|G\| M_1 T^{\frac{1}{2}-\alpha}}{\sqrt{1-2\alpha}}, \quad N_2 = w\beta + \frac{2M_1^2 \|G\| T^{1-2\alpha}}{1-2\alpha}$$

and
$$\|z(\cdot)\|_T = \|z(\cdot)\|_{L_2(o, T; H)}$$

So that

$$|J_p(u_o) - J_M(u_o)| \leq \beta_1 J_p(u_o)$$

$$\text{if } N_1 \|z(T)\| \|z(\cdot)\|_T + N_2 \|z(\cdot)\|_T^2 \leq \beta_1 (g \|z(T)\|^2 + w \|z(\cdot)\|_T^2)$$

That is

$$4(\beta_1 w - N_2) \beta_1 g > N_1^2$$

Clearly such a β_1 can be found if $w > 0$.

Lemma 3.2

If either

$$\text{a) } G \equiv 0, \quad \alpha < 1, \quad W > wI$$

$$\text{or b) } G > gI, \quad \alpha < \frac{1}{2}, \quad W > wI$$

there then exists a γ such that (3.7) holds.

Proof

The condition that

$$J_p(\bar{u} + u_o) - J_M(u_o) \geq -\gamma J_p(\bar{u} + u_o)$$

is by (3.5) equivalent to

$$(3.9) \quad \gamma < z(T), \quad Gz(T)_{>H} + \int_0^T [2 < z(s), \overline{Q(s)A_1} z(s)_{>H} + (1+\gamma) < \bar{u}(s), Ru(s)_{>\mathcal{U}} \\ + 2 < \bar{u}(s), (B^*Q(s) - RF - \gamma RF)z(s)_{>\mathcal{U}} + \gamma < z(s), (W + F^*RF)z(s)_{>H}] ds \geq 0$$

From (2.14) we have

$$(3.10) \quad z(T) = S_{T-t} z(t) + \int_t^T [\overline{S_{T-t} A_1} z(s) + S_{T-s} \bar{B} u(s)] ds$$

a) Using (3.1) with $G = 0$, as in Lemma 3.1, we can show

$$\left| \int_0^T \langle z(s), \overline{Q(s)A_1} z(s) \rangle ds \right| \leq \beta w \|z(\cdot)\|_T^2$$

Now (3.9) with $G = 0$ may be written in the form

$$\begin{aligned} & \int_0^T \left\{ (1+\gamma) \langle \bar{u}(s) + \frac{1}{1+\gamma} R^{-1}(B^*Q(s) - RF - \gamma RF) z(s) \rangle, R[\bar{u}(s) + \frac{1}{1+\gamma} R^{-1}(B^*Q(s) - RF - \gamma RF) z(s)] \right. \\ & \left. + \gamma \langle z(s), (W+F^*RF) z(s) \rangle_H - \frac{1}{1+\gamma} \langle z(s), (B^*Q(s) - (\gamma+1)RF)^* R^{-1}(B^*Q(s) - (\gamma+1)RF) z(s) \rangle_H \right. \\ & \left. + 2 \langle z(s), \overline{Q(s)A_1} z(s) \rangle_H \right\} ds \geq 0. \end{aligned}$$

So that this will certainly hold if

(3.11)

$$\begin{aligned} & \gamma \int_0^T \langle z(s), (W+F^*RF) z(s) \rangle_H ds - \frac{1}{1+\gamma} \int_0^T \langle z(s), (B^*Q(s) - (\gamma+1)RF)^* R^{-1}(B^*Q(s) - (\gamma+1)RF) z(s) \rangle_H ds \\ & - \beta w \|z(\cdot)\|_T^2 \geq 0. \end{aligned}$$

Clearly we can find such a γ if w is sufficiently large.

b) Using (3.1) and (3.10) we obtain

$$\begin{aligned} \int_0^T \langle z(s), \overline{Q(s)A_1} z(s) \rangle_H ds &= \int_0^T \langle z(T), G \overline{S_{T-t}A_1} z(t) \rangle_H dt \\ &+ \int_0^T \langle z(t), \int_t^T S_{\sigma-t}^* (W+F^*RF) \overline{S_{\sigma-t}A_1} z(t) \rangle_H ds dt \\ &- \int_0^T \langle \int_t^T \overline{S_{T-\sigma}A_1} z(\sigma) d\sigma, G \overline{S_{T-t}A_1} z(t) \rangle_H dt \\ &- \int_0^T \langle \int_t^T S_{T-\sigma} B \bar{u}(\sigma) d\sigma, G \overline{S_{T-t}A_1} z(t) \rangle_H dt \end{aligned}$$

The terms in the above expression may be estimated by

$\frac{N_1}{2} \|z(T)\| \|z(\cdot)\|_T$, $\frac{N_2}{2} \|z(\cdot)\|_T^2$ as in Lemma 3.1, and the last term may be written $\int_0^T \langle \bar{u}(t), B^* r(t) \rangle_{\mathcal{U}} dt$

where

$$r(t) = \int_0^t S_{T-t}^* G \overline{S_{T-\sigma}^{-1} A} z(\sigma) d\sigma$$

Completing the square with respect to $\bar{u}$, we find that (3.9) will certainly hold if

$$\begin{aligned} & \|z(T)\|^2 - N_1 \|z(T)\| \|z(\cdot)\|_T - \frac{N_2}{2} \|z(\cdot)\|_T^2 + \gamma w \|z(\cdot)\|_T^2 \\ & - \frac{1}{1+\gamma} \int_0^T \langle \{ (B^* Q(t) - (1+\gamma) R F) z(t) - B^* r(t) \}, R^{-1} \{ (B^* Q(t) - (1-\gamma) R F) z(t) - B^* r(t) \} \rangle dt \geq 0 \end{aligned}$$

So we can find such a γ if w or g is sufficiently large.

The above lemmas are essentially technical. Their main interest lies in the conditions which must be imposed on α and this indicates the class of unbounded perturbation operators which it is possible to consider. In applications, where it is necessary to obtain sharp estimates for β , γ , the explicit forms of Q , A_1 etc should be used rather than the crude estimates carried out in the lemmas.

We now show how (3.6) and (3.7) can be used to answer some of the problems posed in the introduction. From (3.6) we have

$$(3.12) \quad \frac{1}{1+\beta} \leq \frac{J_p(u_o)}{J_M(u_o)} \leq \frac{1}{1-\beta}$$

and from (3.7)

$$(3.13) \quad \frac{J_p(\bar{u} + u_o)}{J_M(u_o)} \geq \frac{1}{1+\gamma}$$

Combining (3.11) and (3.12) we find

$$\frac{J_p(\bar{u} + u_o)}{J_p(u_o)} \geq \frac{1-\beta}{1+\gamma}$$

Since $\bar{u}$ is any control we may take $\bar{u} + u_o$ to be the optimal control u^* for the perturbed system, then from the above we have

$$(3.14) \quad 1 \geq \frac{J_p(u^*)}{J_p(u_o)} \geq \frac{1-\beta}{1+\gamma}$$

Now β, γ depend only on the model, the parameters M, α (which determine a class of perturbation operators), and the control u_o . They do not depend on any knowledge of u^* , or the initial state z_o . Moreover, if the perturbation is zero, then $\beta = 0$, and if then the limited class of controls is wide enough to include the optimal control u^* , then $\gamma = 0$ where $u_o = u^*$. We propose the lower bound $\frac{1-\beta}{1+\gamma}$ as a criteria for answering the following questions.

a) For a system which is imperfectly identified and for which there is limited control action, should the identification continue or the class of controls expanded? - If we think of the operator A_1 as being the unidentified part of the system, then the largest value of $\frac{1-\beta}{1+\gamma}$ as the controls vary over the limited class defines a number ρ which reflects the lack of perfect identification and limited control action. Obviously $\rho \leq 1$, and

$$J_p(u_o) \leq \frac{1}{\rho} J_p(u^*)$$

so that the cost is at most a factor $1/\rho$ of the best that can possibly be achieved assuming perfect identification and no constraints on the control

action. If this is not regarded as satisfactory the exact forms of β , and γ will indicate whether or not the identification should continue or the class of controls expanded.

b) For a system working in the presence of perturbations, and with limited control action, what bounds should be placed on the perturbation so that the system operates in a satisfactory manner? - If the system is regarded as working satisfactorily for $\rho > \bar{\rho}$, then the value of $\rho = \bar{\rho}$ will give an upper bound on the class of perturbations.

4. EXAMPLE - TRAFFIC FLOW PROBLEM

We will apply the theory of § 3 to a traffic flow problem for which the basic dynamics are governed by the Lighthill-Whitham model (6). If the flow is heavy we may treat a stream of traffic as a continuum so that the number of cars passing a position x per unit time is $q(x, t)$, and the number of cars per unit length is $\rho(x, t)$. Then the flow velocity $v = q/\rho$, and conservation requires that at points where there is no entry or exit

$$\rho_t + (\rho v)_x = 0$$

Lighthill and Whitham assume there is a relationship between q and ρ of the form

$$q = Q(\rho) - v\rho_x$$

where Q is convex, zero at $\rho = 0$, and $\rho = \rho_j$ with a maximum at ρ_m , although the exact form of Q is in general not known. Actual observations indicate

that the typical values of ρ_1 , ρ_m , $Q(\rho_m)$ are 225 vehicles per mile, 80 vehicles per mile, 1500 vehicles per hour. v is a diffusion parameter and the term $-v\rho_x$ is a way of introducing a driver's awareness of conditions ahead.

There is a steady state solution of the form

$$v = v_0 = \frac{Q(\rho_0)}{\rho_0}, \quad \rho = \rho_0$$

and if we wish to maximise the flow rate q we would set $\rho_0 = \rho_m$, $v_0 = \frac{Q(\rho_m)}{\rho_m}$. Now suppose that at a given time there is a deviation from the steady state, then the linearized equations governing the perturbation in velocity, are

$$(4.1) \quad v_t = -c_0 v_x + v v_{xx}, \quad v(x,0) = \bar{v}(x)$$

where $c_0 = Q'(\rho_0)$. We have attempted to choose $\rho_0 = \rho_m$ which maximizes $Q(\rho)$, and so we would expect $c_0 = 0$. However, since the exact form of $Q(\rho)$ is not known this is unlikely to be the case. We may therefore regard c_0 as an unidentified parameter with nominal value zero, and the term $c_0 v_x$ as a perturbation of the system.

The boundary conditions for (4.1) are difficult to formulate, but since our main aim is to illustrate the theory of § 3 we will assume the boundary conditions

$$v = 0 \quad \text{at } x = 0, \quad x = L$$

Thus the speed at entry and exit is fixed at v_0 . If we assume that the system can be controlled, we have

$$(4.2) \quad \begin{aligned} v_t &= -c_0 v_x + v v_{xx} + g \\ v &= 0 \text{ at } x = 0, L, \quad v(x, 0) = \bar{v}(x) \end{aligned}$$

The control action is to be designed so that the system operates close to the steady state with a minimum use of control energy. One way of expressing this requirement is to choose the control to minimize

$$J(g) = \int_0^T \int_0^L (v^2 + \lambda g^2) dx dt$$

The problem can be transformed into an equivalent problem

$$(4.3) \quad \begin{aligned} v_t &= a v_x + v_{xx} + u \\ v &= 0 \text{ at } x = 0, 1, \quad v(x, 0) = \bar{v}(x) \end{aligned}$$

$$J(u) = \int_0^T \int_0^1 (v^2 + b u^2) dx dt$$

$$\text{where} \quad a = -\frac{c_0 L}{v}, \quad b = \frac{v^2 \lambda}{L^4}, \quad u = \frac{L^2 g}{v}$$

If we take $H = L_2(0, 1) = \mathcal{U}$, then (4.3) is equivalent to the abstract evolution problem

$$(4.4) \quad \dot{z} = A_0 z + A_1 z + Bu, \quad z(0) = z_0$$

$$\text{where} \quad A_0 z = z_{xx}, \quad z \in D(A_0)$$

$$D(A_0) = \{z \in L_2(0, 1), Az \in L_2(0, 1), z = 0 \text{ at } x = 0, 1\}$$

$$A_1 z = a z_x, \quad z \in D(A_0)$$

$$B = I \quad (\text{identity in } \mathcal{L}(H))$$

$$J(u) = \int_0^T (||z||^2 + b||u||^2) dt$$

It is easy to show that A_0 generates a strongly continuous semigroup T_t , where

$$T_t z_0 = \sum_1^{\infty} 2 e^{-n^2 \pi^2 t} \sin n\pi x \int_0^1 \sin n\pi y z_0(y) dy$$

Hence

$$||T_t|| = ||T_t^*|| \leq e^{-\pi^2 t}$$

Moreover

$$||T_t A_1 z_0||^2 = a^2 \sum_1^{\infty} 2 e^{-2n^2 \pi^2 t} n^2 \pi^2 \left(\int_0^1 \cos \pi y z_0(y) dy \right)^2, \quad z_0 \in D(A_0)$$

and so

$$||T_t A_1 z_0|| \leq \frac{|a|}{\sqrt{2e}} t^{\frac{1}{2}}$$

Since (4.4) is asymptotically stable in the Liapunov sense, and control action is difficult to implement we will make comparisons with the case $u_0 = 0$, so that $F = 0$. Then from (3.1)

$$Q(t) z_0 = \int_t^T T_{s-t}^* T_{s-t} z_0 ds$$

It is easy to show that

$$\left| 2 \int_0^T \langle z(t), \overline{Q(t) A_1 z(t)} \rangle_H dt \right| \leq \frac{4|a|}{\sqrt{2e}} \left(\int_0^T e^{-\pi^2 y^2} dy \right) ||z(\cdot)||_T^2$$

Hence from (3.4)

$$\frac{1}{1-\beta} \geq \frac{J_p(o)}{J_M(o)} \geq \frac{1}{1+\beta}$$

where
$$= \frac{4|a|}{\sqrt{2e}} \int_0^T e^{-\pi^2 y^2} dy .$$

γ is obtained from (3.11) with $F = 0$, so that we require

$$\gamma ||z(.)||^2_T - \beta ||z(.)||^2_T - \frac{1}{(1+\gamma)b} ||(Qz)(.)||^2_T \geq 0 .$$

Now

$$\begin{aligned} ||(Qz)(t)|| &= || \int_t^T T_{s-t}^* T_{s-t} z(t) ds || \\ &= || \int_t^T T_{2(s-t)} z(t) ds || \\ &\leq \frac{1 - e^{-2\pi^2 T}}{2\pi^2} ||z(t)|| = \alpha ||z(t)|| \end{aligned}$$

Hence

$$\gamma = \frac{\beta+1 + \sqrt{(\beta+1)^2 + 4\alpha^2/b}}{2}$$

and

$$1 \geq \frac{J_p(u^*)}{J_p(o)} \geq \frac{1-\beta}{1+\gamma}$$

For some numerical results we take $b = 1$, and estimate α and β by their values for $T = \infty$, then

$$1 > \frac{J_p(u^*)}{J_p(o)} > \frac{1 - .484|a|}{1 + .484|a| + .00257}$$

We see therefore that if $a = 0$ so that the system is perfectly identified, then

$$1 \geq \frac{J_p(u^*)}{J_p(o)} \geq .9974$$

and it would seem that it is not worth introducing sensors and applying control action. On the other hand if the process is regarded as working satisfactorily for

$$J_p(o) \leq 2 J_p(u^*)$$

then this will allow $|a| < .687$. So that identification need only continue until this bound is met.

5. REFERENCES

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6. APPENDICES

Appendix I

If A_1 satisfies (2.5), (2.10) and S_t is defined by (2.6), that is

$$(6.1) \quad S_t z_0 = T_t z_0 + \int_0^t T_{t-s} BF S_s z_0 ds$$

where $BF \in \mathcal{L}(H)$, then

$$(6.2) \quad \|S_t A_1 z_0\| \leq \frac{M_1}{t^\alpha} \|z_0\|, \quad z_0 \in D(A_0).$$

Proof

Using the usual iterative scheme

$$\begin{aligned} S_t^0 &= T_t, \\ S_t^n z_0 &= \int_0^t T_{t-s} BF S_s^{n-1} z_0 ds, \\ S_t &= \sum_{n=0}^{\infty} S_t^n, \end{aligned}$$

it is easy to show that (6.1) defines, uniquely, a strongly continuous semigroup with $\|S_t\| \leq M_s \quad t \in [0, T]$. Moreover it can also be shown that S_t satisfies

$$S_t z_0 = T_t z_0 + \int_0^t S_{t-s} BF T_s z_0 ds.$$

Hence

$$S_t A_1 z_0 = T_t A_1 z_0 + \int_0^t S_{t-s} BF T_s A_1 z_0 ds \quad z_0 \in D(A) \quad t > 0$$

Then

$$\begin{aligned} \|S_{t-1} A_1 z_0\| &\leq \frac{M}{t^\alpha} \|z_0\| + \frac{M_s \|BF\| MT}{T^\alpha (1-\alpha)} \\ &\leq \frac{M_1}{t^\alpha} \quad \text{where} \quad M_1 = \frac{M + M_s \|BF\| MT}{1-\alpha} \end{aligned}$$

Appendix II

U_t defined by (2.12) uniquely determines a strongly continuous semigroup.

Proof

The proof is straightforward, using (6.2) and the iterative scheme

$$U_t^0 = S_t$$

$$U_t^n z_0 = \int_0^t \overline{S_{t-s} A_1} U_s^{n-1} z_0 \, ds \quad ,$$