ON LARGE SCALE LINEAR FRACTIONAL PROGRAMS

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1. In the present profit regulating system the objective of a Hungarian firm can be expressed [4] as the maximization of the ratio

$$H = \frac{P}{\sigma W + K} \quad /1/$$

where

P = profit

σ = coefficient given by the planning authorities

W = wage cost

K = capital (sum of the fixed and the average working capital)

In the case of a large multidivision company this suggests the consideration of the following programming problem:^{x)}

$$\begin{aligned} \sum_j A_j x_j &= b & /j=1, 2, \dots / \\ B_j x_j &= b_j \\ x_j &\geq 0 \\ \max \frac{\sum_j c_j x_j + \gamma}{\sum_j d_j x_j + \delta} & & /2/ \end{aligned}$$

Several decomposition procedures can be developed for solving this problem. Two such procedures, obtainable through transforming the above problem into a linear program to which the Dantzig-Wolfe principle is applied, are considered. The economic interpretations of the procedures are also discussed.

^{x)} Matrices are denoted by capital letters, vectors by small letters and scalars by greek letters.

2. In this section it is shown, how the Dantzig-Wolfe principle can be applied for solving /2/ linear fractional programming problem.

It is assumed that for each j the set $S_j = \{x_j \mid Bx_j = b_j, x_j \geq 0\}$ is non-empty.

This is a necessary condition for having a feasible solution to problem /2/. Checking whether these necessary conditions are satisfied easily can be built in the algorithm suggested below. Furtherly it is assumed that $\sum_j d_j x_j + \delta > 0$ for each $x = /...x_j.../$ satisfying the constraints in /2/. In the above economic problem this does not mean a restriction since the value of the capital used always can be regarded positive.

Denote $\bar{x}_{j1}, \bar{x}_{j2}, \dots, \bar{\bar{x}}_{j1}, \bar{\bar{x}}_{j2}, \dots$, the elements of the canonical decomposition of the polyhedron S_j , where one and two bars represent the bounded and unbounded components, respectively [3].

As it is in the linear programming case, problem /2/ is equivalent to the problem (with variables λ_{jk} and μ_{jk}) [2]

$$\sum_j \left(\sum_k \bar{a}_{jk} \lambda_{jk} + \sum_k \bar{\bar{a}}_{jk} \mu_{jk} \right) = b$$

$$\sum_k \lambda_{jk} = 1 \quad /j=1, 2, \dots/$$

$$\lambda_{jk}, \mu_{jk} \geq 0$$

$$\max \frac{\sum_j \left(\sum_k \bar{\delta}_{jk} \lambda_{jk} + \sum_k \bar{\bar{\delta}}_{jk} \mu_{jk} \right) + \gamma}{\sum_j \left(\sum_k \bar{\delta}_{jk} \lambda_{jk} + \sum_k \bar{\bar{\delta}}_{jk} \mu_{jk} \right) + \delta} \quad /3/$$

$$\text{where} \quad \begin{array}{ll} \bar{a}_{jk} = A_j \bar{x}_{jk} & \bar{\bar{a}}_{jk} = A_j \bar{\bar{x}}_{jk} \\ \bar{\delta}_{jk} = c_j \bar{x}_{jk} & \bar{\bar{\delta}}_{jk} = c_j \bar{\bar{x}}_{jk} \\ \bar{o}_{jk} = d_j \bar{x}_{jk} & \bar{\bar{o}}_{jk} = d_j \bar{\bar{x}}_{jk} \end{array} .$$

Following the algorithm suggested by Charnes and Cooper [1] for solving a linear fractional programming problem one can introduce the variable

$$\tau = \frac{1}{\sum_j (\sum_k \bar{\delta}_{jk} \lambda_{jk} + \sum_k \bar{\delta}_{jk} \mu_{jk}) + \delta}$$

$$\alpha_{jk} = \frac{\lambda_{jk}}{\sum_j (\sum_k \bar{\delta}_{jk} \lambda_{jk} + \sum_k \bar{\delta}_{jk} \mu_{jk}) + \delta}$$

$$\beta_{jk} = \frac{\mu_{jk}}{\sum_j (\sum_k \bar{\delta}_{jk} \lambda_{jk} + \sum_k \bar{\delta}_{jk} \mu_{jk}) + \delta}$$

and obtain the linear programming programming problem

$$\sum_j (\sum_k \bar{a}_{jk} \alpha_{jk} + \sum_k \bar{a}_{jk} \beta_{jk}) - b \tau = 0$$

$$\sum_k \alpha_{jk} - \tau = 0 \quad /j=1, 2, \dots/$$

$$\sum_j (\sum_k \bar{\delta}_{jk} \alpha_{jk} + \sum_k \bar{\delta}_{jk} \beta_{jk}) + b \tau = 1$$

$$\alpha_{jk}, \beta_{jk} \geq 0 \quad /4/$$

$$\max \left[\sum_j (\sum_k \bar{\delta}_{jk} \alpha_{jk} + \sum_k \bar{\delta}_{jk} \beta_{jk}) + \delta \tau \right]$$

This problem can be solved by using the Dantzig-Wolfe principle and the following algorithm is obtained:

a) Determine an optimal basic feasible solution to /4/. Denote $(f, \dots, f_j, \dots, f)$ a system of dual variables corresponding to this basic solution.

b) Consider the LP problems

$$B_j x_j = b_j$$

$$x_j \geq 0 \quad /j=1, 2, \dots/ \quad /5/$$

$$\max (c_j - f A_j - f d_j) x_j$$

c) If for each j

$$(c_j - fA_j - \rho d_j) \bar{x}_j \leq \rho_j \quad /6/$$

where $\bar{x}_j$ is an optimal solution of /5/, the procedure is terminated.

If $\tau \neq 0$ in the regarded basic feasible solution of /4/, $x = \frac{1}{\tau}$

$(\dots \sum_k \alpha_{jk} \bar{x}_k + \sum_k \beta_{jk} \bar{\bar{x}}_k \dots)$ is an optimal solution of /1/.

If $\tau = 0$, the actual objective function value in /4/ - which is the optimal value of /4/ - is an upper bound for the objective function values in /2/ and moving along the direction $\bar{\bar{x}} = (\dots \sum_k \beta_{jk} \bar{\bar{x}}_k \dots)$ one can get a solution for /2/ for which the objective is arbitrarily close to this value.

d) If for some j the optimal solution of /5/ does not satisfy the condition /6/, or /5/ does not have an optimal solution, new columns are introduced into the coefficient matrix of problem /4/, the same way as in the LP case. That is to say problem /4/ is augmented by the column $\{A_j \bar{x}_j, 0, \dots, 1, \dots, 0, d_j \bar{x}_j, c_j \bar{x}_j\}$, where $\bar{x}_j$ is an extremal solution of /5/ not satisfying condition /6/ and/or by the column $\{A_j \bar{\bar{x}}_j, 0, \dots, 0, 0, d_j \bar{\bar{x}}_j, c_j \bar{\bar{x}}_j\}$, where $\bar{\bar{x}}_j$ is an extremal direction obtained in the case where /5/ was found unbounded, for which

$$(c_j - fA_j - \rho d_j) \bar{\bar{x}}_j > 0.$$

Now the procedure is continued from a) with the augmented central program. If for this new matrix problem /4/ is unbounded, /2/ is also unbounded and the procedure is completed. Otherwise the procedure is continued from b).

If for problem /2/ it is also assumed that for each solution of /5/ $d_j x_j > 0$, then the sector problem can be formulated as

$$\begin{aligned} B_j x_j &= b \\ x_j &\geq 0 \\ \max \quad & \frac{(c_j - fA_j) x_j - \rho_j}{d_j x_j} \end{aligned} \quad /7/$$

and the optimality condition is that for each j the maximum value of /7/ is at most $\bar{f}$.

For the solution of problem /3/ the algorithm proposed by Martos [5] also can be applied. As it was shown by Wagner [6] for each solution obtained by /6/ corresponds a solution obtained by /2/ and vice versa, and starting from the same basic feasible solution generally in each step the same basic feasible solutions are obtained.

If we apply the Martos procedure the solution must be kept as long as possible in the bounded part of the convex polyhedron defined by the constraints. This can be obviously fulfilled if the constraints are given explicitly which is not the case for /3/. Naturally this problem does not arise if the convex polyhedron defined by the constraints of /3/ is bounded, which often can be assumed in a real-life economic model.

For describing the application of the Martos-algorithm consider the two LP problems, both with constraints /3/ and one by taking the numerator and the other the denominator of the objective function in /3/ to be objective function. Denote $(p \dots \pi_j \dots)$ and $(r \dots \rho_j \dots)$ the vectors of the provisional dual prices corresponding to a basic feasible solution of the LP problems above, respectively, and denote Δ and Γ the objective function values in this basic solution. (i. e. Δ is the value of the numerator and Γ is the value of the denominator of the objective function in /3/.) A solution is optimal if for each λ_{jk} ,

$$\Delta(\bar{\gamma}_{jk} - p\bar{a}_{jk} - \pi_j) - \Gamma(\bar{\delta}_{jk} - r\bar{a}_{jk} - \rho_j) \leq 0$$

and for each μ_{jk}

$$\Delta(\bar{\gamma}_{jk} - p\bar{a}_{jk}) - \Gamma(\bar{\delta}_{jk} - r\bar{a}_{jk}) \leq 0.$$

It follows that a sector problem of the form

$$B_{x_j} = b ;$$

$$x_j \geq 0 \quad /8/$$

$$\max \left\{ \Delta(c_j - pA_j) x_j - T [(d_j - rA_j) x_j - \delta_j] \right\}$$

can be defined, while the other steps of the decomposition procedure remain the same.

If problem /8/ is bounded and the optimum value is positive, the vector submitted to the central program always will have a positive element, for example the element corresponding to the $\sum_k \lambda_{jk} = 1$ constraint and so the new solutions will be kept in the bounded part of the convex polyhedral defined by the constraints of problem /3/.

If one of the sector problems is unbounded, an appropriate $\bar{x}_j$ can be chosen by solving the LP problem

$$B_j x_j = 0$$

$$ex_j = 1 \quad /9/$$

$$\Delta(c_j - pA_j) - T (d_j - rA_j) x_j \geq \epsilon$$

$$\max e B^{-1} A_j x_j$$

where B^{-1} is the appropriate part of the actual basic solution of /3/, e is a vector all elements of which are 1 and ϵ is a sufficiently small positive number.

3. Now we turn to the economic interpretations of the above procedures. The iterations in a Dantzig-Wolfe type decomposition can be described as steps in a two-way planning process:

- 1) The center gives the initial values of the centrally determined parameters for the divisions optimization problems
- 2) Each subdivision submits a feasible subdivisional program based on the centrally given parameters
- 3) The center examines whether the optimum of the whole problem is

reached as a combination of the subdivisional programs, and if this is not the case preassings a revised set of parameters and the process is repeated.

The procedures always have this general structure but differences are found in the objective functions of the subdivisional programs and in the centrally given parameters.

In the first case discussed above a subdivision maximizes the objective function

$$\left[c_j - fA_j - \rho (\sigma w_j + k_j) \right] x_j .$$

The following interpretation can be given to this objective function. The subdivision maximizes the profit, but the profit of each activity is reduced by the sum of the "taxes" paid for the use of the central resources ($f A_j x_j$) and for the use of the factors of production ($\rho \sigma w_j + k_j$) x_j .

The elements of f corresponding not to a central resource but to a central task will be negative (or zero) so beside the taxes there can be subsidies too.

This decomposition procedure defines a decentralized decision making process in which the center determines the taxes and subsidies after comparing the centrally computed ρ_j values and the optimal profit values of the subdivisions and solving a new central problem. The optimal profit values in the sectors generally will be positive, which appears to be preferable to a system in which the optimal values of the sectors' objective function would be zero.

In the second formulation - when it can be assumed that for each possible sector program $\sigma w_j + k_j$ will be positive - the objective function of the sector can be stated as

$$\max \frac{(c_j - fA_j) x_j - \rho_j}{(\sigma w_j + k_j) x_j} .$$

This objective function requires the maximization of the ratio of the sectors profit to the weighted value of the additional (new) resource and capital requirements. The profit of the sector in this case is reduced by the taxes

paid after the use of the central resources and by an individual profit tax. Accordingly in the corresponding decentralized decision making system the center determines the resource taxes and the individual sector taxes, after comparing the centrally computed φ value and the values of the objective functions of the sectors and solving a new master program. The objective function values of the sectors in this system will always be positive.

The third possibility discussed above defines the following objective function in the sectors

$$c_j x_j - p A_j x_j - \frac{T}{\Delta} [\theta w_j + k_j / x_j - r_j A_j x_j - \varphi_j] \quad .$$

It also can be regarded as a special profit function, in which the profit is reduced by the taxes paid for the use of the central resources and for the use of the factors of production. However now this latter tax is determined in several steps. First the factor use is modified and this modification is also made in two steps: there is an individual modification element (φ_j) and a modification element determined by the use of the central resources ($r_j A_j x_j$).

The sector pay a factor tax after this modified value of factor use and the tax rate is determined by the actual value of the objective function ($\frac{T}{\Delta}$). This objective function appears to be highly complicated, although there can be cases where this system can be preferred to a simpler one because it offers the possibility of a dual valuation of the scarce resources and takes into account the differences in the factor needs of the subdivisions.

In the corresponding decentralized decision making system the center determines two sets for resource taxes (p and r), the factor tax rate ($\frac{T}{\Delta}$) and the subdivisional factor use tax (φ_j).

It does not seem appropriate to rank the above decentralized decision making systems, the best suitable one can be chosen only after taking into account the local circumstances.

REFERENCES

- [1] CHARNES, A. - COOPER, W. W. : Programming with linear fractional functionals. *Naval Research Logistics Quarterly*, 9/2 (1962).
- [2] DANTZIG, G. B. - WOLFE, Ph. : The decomposition algorithm for linear programs. *Econometrica*, 29/4 (1961).
- [3] GOLDMANN, A. I. : Resolution and separation theorems for polyhedral convex sets. In: *Linear inequalities and related systems*. Edited by KUHN, H. W. and TUCKER, A. W., Princeton, 1956. University Press.
- [4] KOVÁCS, Á. : On the normative behaviour of the firm in the Hungarian probit system, in *Colloquia Mathematica Societatis János Bolyai*: 12. Progress in Operations Research, Eger 1974.
- [5] MARTOS, B. : Hyperbolic programming. *Naval Research Logistics Quarterly*, June-Sept. (1964).
- [6] WAGNER, H. M. - YUAN, J. S. C. : Algorithmic equivalence in linear fractional programming. *Management Science*, 14/5 (1968).