A REMARK ON MULTIPLIER METHODS FOR NONLINEAR PROGRAMMING

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ABSTRACT. This paper is concerned with certain aspects of multiplier methods where the solution of a constrained minimization problem is obtained by means of a sequence of unconstrained minimizations of an augmented Lagrangian $L(x,y,r)$, followed each by an iteration on the Lagrange multiplier vector y . In spite of the growing recognition that multiplier methods are among the most effective constrained minimization methods, the value to be given to the penalty parameter r does not seem yet to have received enough attention. This paper - related to work done recently by Bertsekas and Polyak - contains a result in such direction, namely the following one: if G and Q are given matrices and Q is positive definite on the kernel of G , then it is produced r^* such that for all $r > r^*$, $Q + r G^T G$ is positive definite on the whole space. Also we prove a lemma - related to a known one - about Hilbert space operators with uniformly bounded inverses, that may be useful in extending the result above to more general situations. To test the computational value of the estimate r^* arrived at here, a computer program is being tested and some numerical results are reported.

1. INTRODUCTION

Let $f, g_1, \dots, g_m$ be real valued functions defined on R^n , $n \geq m > 0$, and $g = (g_1, \dots, g_m)$. Consider the problem

(P) minimize $f(x)$, subject to $g(x) = 0$

and the sequence of unconstrained minimizations

$$(1.1) \quad L(x_k, y_k, r_k) = \min \{L(\cdot, y_k, r_k) \mid x \in R^n\}$$

where $\{y_k\}, \{r_k\}$ are to be chosen appropriately and the augmented Lagrangian L - cf. Arrow and Solow [1], Fletcher [5], Hestenes [6] and the work of Rockafellar, see [10,11] and their references - is defined by

$$(1.2) \quad L(x, y, r) = f(x) + (y, g(x)) + \frac{r}{2} \|g\|^2, \quad (x, y, r) \in \mathbb{R}^n \times \mathbb{R}^m \times]0, \infty[.$$

The penalty method for solving (P) - see for instance Fiacco and McCormick [4] - can be thought of as the minimizations (1.1) with the choice $y_k = 0$ and $r_k \nearrow \infty$. The numerical difficulties (instability) that may arise in such procedure for " r_k large" can in general be avoided or reduced by using the iteration

$$(1.3) \quad y_{k+1} = y_k + r_k y(x_k)$$

for which the condition $r_k \nearrow \infty$ is not necessary: this choice of y_k characterizes the multiplier methods - see Hestenes [6] and Powell [9] - that are receiving growing recognition as one of the most effective class of methods for constrained minimization.

In the actual process of solving a specific problem (P) with such a scheme one has of course to decide which values the penalty parameter r_k is to be given; it may be easily found out that haphazard guesses may lead to a substantial waste of time and effort: indeed, roughly speaking, when the violation of the constraints is penalized too little (1.1) may end up generating points away from the feasible set, and when it is penalized too much the points generated by (1.1) may have tendency to jump around.

In a recent paper, Bertsekas [3] has succeeded in proving convergence and rate of convergence results for the multiplier method; actually, under the standard 2nd order sufficiency conditions for $\bar{x}$ to be an isolated local minimum (i.e. when - see Luenberger [7], p.226 - f and g are C^2 in a ball $B(\bar{x}, \epsilon)$, at $\bar{x}$ the differential dg of g has rank m and $df + \bar{y}^T dg = 0$ for some $\bar{y} \in \mathbb{R}^m$, at $(\bar{x}, \bar{y}, 0)$ the hessian of $L(\cdot, \bar{y}, 0)$ is positive definite on the kernel of $dg(\bar{x})$) and under a Lipschitz condition on $d^2f, d^2g_1, \dots, d^2g_m$ in $B(\bar{x}, \epsilon)$ he proves that given a bounded subset $V \subseteq \mathbb{R}^m \exists r_1 \geq 0$ and $M_1 \geq 0$ such that for all $r \geq r_1$ and $y \in V$

(i) $\exists$ one minimizer $x(y, r)$ of $L(\cdot, y, r)$ in $B(\bar{x}, \epsilon)$, $\epsilon > 0$

(ii) $|x(y, r) - \bar{x}|$ and $|y + rg(x(y, r)) - \bar{y}| \leq \frac{M_1 |y - \bar{y}|}{r}$.

In the proof, such conclusions are obtained by requiring - in particular - r to be so large as to make

$$(1.4) \quad d^2L(\bar{x}, \bar{y}, 0) + r dg^T(\bar{x}) dg(\bar{x}) \text{ positive definite.}$$

In section 2 it is given a computable estimate of r^* such that (1.4) holds for all $r > r^*$; actually, we feel that such result (local estimate theorem) is a special case of a global one.

The main difficulty for obtaining such more general result is that a certain set of matrices should possess uniformly bounded inverses. However, we shall not investigate the global problem here, except for setting it (in section 2), conjecturing its solution and extending (in section 3) a lemma in Polyak [8] about Hilbert space operators possessing uniformly bounded inverses, to a more general situation. Actually we feel that such more general lemma or some variant of it, besides having some interest in themselves, may be useful for establishing our conjecture about the global result.

Finally some numerical results, obtained by making use of r^* in a simple way within the standard multiplier method, are reported in section 4 and compared with analogous ones available in the letterature.

2. A LOCAL ESTIMATE OF r^*

We shall first state the general problem we are concerned with. Let the norm $\|\cdot\|$ be defined on the space of finite dimensional linear operators $L(R^n, R^m)$ by $\|A\| = \sup \{ (Ax, Ax)^{1/2} / (x, x)^{1/2} : x \in R^n \}$, for $n \geq m > 0$ consider a map $G : X \subseteq R^n \rightarrow L(R^n, R^m)$ and let $N(G(x))$ be the kernel of $G(x)$, for each $x \in X$; also, consider a map $Q : Z = X \times Y \subseteq R^n \times R^m \rightarrow L(R^n, R^n)$ and for r real let Q_r be defined by

$$Q_r(x, y) = Q(x, y) + r G^T(x) G(x) \quad , \quad (x, y) \in X \times Y$$

The problem we are concerned with is the following:

if it is known that for $(x, y) \in Z$, $Q(x, y)$ has a coerciveness property on the subspace $N(G(x))$; when and how it possible to estimate r^* such that for every $r > r^*$ $Q_r(x, y)$ has such coerciveness property on all of R^n ?

Here we shall give an answer to such problem in the simpler case where the sets X and Y contain just one point each.

Theorem 2.1 (Local estimate)

Let n, m be positive integers ($n \geq m$), and suppose that

- (i) $G \in L(R^n, R^m)$ has rank m ;
- (ii) $Q \in L(R^n, R^n)$ is symmetric and $(Qs, s) > 0$, all $s \in N(G)$, $s \neq 0$.

For r real consider $Q_r = Q + r G^T G$ and let r^* be defined by (2.3) below. Then

$$r > r^* \Rightarrow (Q_r s, s) > 0, \text{ all } s \in \mathbb{R}^n, s \neq 0.$$

Remark 2.2. Since Q_r is a matrix of numbers, to say that it is coercive it is the same as saying that it is positive definite.

Proof of theorem 2.1.

Let us observe that there are congruent transformations $B(\cdot)B^T$ that reduce $G^T G$ to a diagonal matrix; such transformations - by a classical result of Sylvester - leave invariant the number of positive and the number of zero eigenvalues. A natural choice for one such transformation is given by $B \in L(\mathbb{R}^n, \mathbb{R}^n)$ defined as

$$(2.1) \quad B = [G^T | e_{j_1}, \dots, e_{j_{n-m}}]^{-1}$$

where e_{j_p} is the j_p -th column of the identity matrix and j_p is the row index of the p -th row of G^T that is linearly dependent on the previous ones. Indeed it can be easily seen that such B acts as left inverse of G and

$$BG^T GB^T = \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix}, \quad I \quad m \times m.$$

Let us now partition $BQ_r B^T$ symmetrically

$$(2.2) \quad BQ_r B^T = BQB^T + \begin{bmatrix} rI & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} P_r & \Pi^T \\ \Pi & p \end{bmatrix}, \quad P_r \quad m \times m;$$

put

$$M(p; \Pi) = \begin{cases} 2\|p\|^{-1} \|\Pi\|^2 - 2\|\Pi\| + \|p\| & \text{if } m < n \\ 0 & \text{else} \end{cases}$$

and define r^* by

$$(2.3) \quad r^* = \max \{0, \max_{1 \leq i \leq m} [-(BQB^T)_{ii} + \sum_{j=1, j \neq i}^m |(BQB^T)_{ij}| + M]\}.$$

We claim that such r^* will do; indeed let $r > r^*$ be given. It has to be shown that

$$(2.4) \quad (s, Q_r s) > 0 \quad \text{all } s \in \mathbb{R}^n, s \neq 0.$$

Since $G^T G$ is positive semidefinite, from the definition of Q_r it is seen that (2.4) with $r \geq 0$ holds if s is restricted to the kernel of G i.e.

$$(2.5) \quad (s, Q_r s) > 0, \text{ all } s \in N(G), \quad s \neq 0;$$

for $s \in R^n$ let $w \in R^n$ be defined by $s = B^T w$, call w' (respectively w'') the vector formed by the first m (last $n-m$) components of w and observe that

$$(s, Q_r s) = (w, B Q_r B^T w) = (w', P_r w') + 2(w'', \Pi w') + (w'', p w'')$$

implies in view of the consequence of (2.5)

$$s \in N(G) \Leftrightarrow w' = 0,$$

that the submatrix p is positive definite; thus M is a well defined real number.

We now claim that for $w \in R^n$ with $(w, w) = 1 = \|w'\|^2 + \|w''\|^2$ one has

$$(2.6) \quad (w, B Q_r B^T w) \geq (\lambda(P_r) - \|p\| + 2\|\Pi\|) \|w'\|^2 - 2\sqrt{2} \|\Pi\| \|w'\|^2 + \|p^{-1}\|^{-1}$$

where $\lambda(\cdot)$ denotes the least eigenvalue of the argument; indeed using the Cauchy-Schwarz inequality, the fact that $(1 - \sqrt{2}t)^2 \geq 0$ for t real and that $(w'', p w'') \geq \lambda(p) (1 - \|w'\|^2)$ one obtains

$$(w, B Q_r B^T w) \geq (\lambda(P_r) - \lambda(p) + 2\|\Pi\|) \|w'\|^2 - 2\sqrt{2} \|\Pi\| \|w'\| + \lambda(p)$$

and hence (2.6), because $\|p^{-1}\|^{-1} = \lambda(p) \leq \|p\|$.

From (2.6) one has

$$(2.7) \quad 2\|\Pi\|^2 - (\lambda(P_r) - \|p\| + 2\|\Pi\|) \|p^{-1}\|^{-1} < 0 \Rightarrow (w, B Q_r B^T w) > 0, \text{ all } w, \|w\|=1$$

from which it is seen that the proof of the theorem is completed as soon as it is shown that

$$(2.8) \quad \lambda(P_r) > M(p; \Pi);$$

this inequality does indeed hold because it follows from Gershgorin's theorem and the choice of r^* : in fact, there is an integer i ($1 \leq i \leq m$) such that

$$-\lambda(P_r) + [(B Q B^T)_{ii} + r] \leq \sum_{j=1, j \neq i}^m |(B Q B^T)_{ij}|,$$

and this inequality implies (2.8) because $r > r^*$.

If the condition " X and Y contain just one point each" is dropped and the hypotheses of theorem 2.1 are somewhat strengthened so as to make the matrices $B(x)$ (i.e. the left inverses of $G^T(x)$, $x \in X$) uniformly bounded in x , we conjecture that a result similar to the one above should hold. In the following section we shall examine some conditions insuring that linear operators depending on parameters possess uniformly bounded inverses.

3. UNIFORMLY BOUNDED INVERSES

In this section we generalize a result of Polyak, [8] lemma 1, about Hilbert space operators with uniformly bounded inverses and then give a variant of it.

Let H, h be real Hilbert spaces; their inner product as well as the induced norms will be denoted by the same symbols $(\cdot, \cdot)$ and $\|\cdot\|$, when there is no risk of misunderstanding; let the product space $H \times h$ be normed by $\|\cdot\|^2 = \|\cdot\|_H^2 + \|\cdot\|_h^2$, the space $L(H, h)$ of bounded linear operators from H to h be normed by $\|C\| = \sup\{\|Cs\|_h / \|s\|_H : s \neq 0\}$, and let X, Z, R be subsets of the Euclidean spaces R^V, R^U and R^O respectively. Suppose

(3.1) $\{C_x : x \in X\}$ is a bounded family of operators $C_x \in L(H, h)$ satisfying $\|C_x^* s\| \geq \sqrt{k_1} \|s\|$, all $s \in h$ and $x \in X$, some $k_1 > 0$, where C_x^* denotes the adjoint of C_x ;

(3.2) $\{U_z : z \in Z\}$ is a bounded family of symmetric operators $U_z \in L(H, H)$ satisfying $(U_z s, s) \geq k_2 \|s\|^2$, all $s \in H, z \in Z$, some $k_2 > 0$; let k_3 be a positive number such that $\|U_z\| \leq k_3$, all $z \in Z$;

(3.3) $\{u_r : r \in R\}$ is a family of operators $u_r \in L(h, h)$ satisfying $(u_r s, s) \leq k_4 \|s\|^2$, all $s \in h$ and $r \in R$, with k_4 real.

For each $(x, z, r) \in P = X \times Z \times R$, consider the operator $A = A_{x, z, r} : H \times h \rightarrow H \times h$ defined by

$$A = \begin{bmatrix} U_z & C_x^* \\ C_x & u_r \end{bmatrix}$$

If the families (3.1), (3.2) reduce to a single operator and if the u_r 's are the specific negative operators $-I/r$ where I is the identity and $r \in]0, \infty[$, Polyak proves in [8] that the inverse of A exists and is

bounded uniformly in r . Here we obtain a similar result for the more general situation (3.1), (3.2) and (3.3), where in particular (3.3) does not exclude u_r to be the null operator, a case important for some nonlinear programming methods, and indeed allows the operators u_r to be somewhat positive; the simplicity of the proof is enhanced by the explicit exhibition of the inverse of A .

Lemma 3.1 (Formula for the inverse operator)

Suppose that (3.1), (3.2), and (3.3) with $k_4 < k_1/k_3$ hold; for each $(x, z, r) \in P$ define the operator $F = F_{x,z,r}$ by

$$(3.4) \quad F_{x,z,r} = \begin{bmatrix} U_z^{-1} + S v^{-1} S^* & - S v^{-1} \\ - v^{-1} S^* & v^{-1} \end{bmatrix}$$

where $U_z^{-1} = U_z^{-1}$ and the operators $S = S_{x,z}$, $v = v_{x,z,r}$ are defined by $S = U_z^{-1} C_x^*$, $v = u_r - C_x U_z^{-1} C_x^*$. Then

- (i) $\exists K$ real: $\|F_{x,z,r}\| \leq K \quad \forall (x, z, r) \in P$;
- (ii) $\forall (x, z, r) \in P$: $F_{x,z,r}$ is the inverse of $A_{x,z,r}$.

Proof

In view of (3.1), (3.3) and the choice of k_4 , the family of operators on $h \{-v_{x,z,r}\}$ is uniformly coercive because

$$(3.5) \quad (-v_{x,z,r} s, s) = - (u_r s, s) + (C_x U_z^{-1} C_x^* s, s) \geq -k_4 \|s\|^2 + \frac{1}{\|U_z\|} \|C_x^* s\|^2 \\ \geq \left(\frac{k_1}{k_3} - k_4\right) \|s\|^2, \quad \text{all } s \in h \text{ and } (x, z, r) \in P$$

and hence $\{v_{x,z,r}^{-1}\}$ is a bounded family because (3.5) implies, in addition to the existence of the inverses, that

$$(3.6) \quad \|v_{x,z,r}^{-1}\| \leq \frac{1}{(k_1/k_3) - k_4}, \quad \text{all } (x, z, r) \in P;$$

thus the operator $F_{x,z,r}$ is well defined. A direct check of its product with $A_{x,z,r}$ shows that indeed (ii) holds. Also $\{U_z^{-1}\}$ is a bounded subset of $L(H, H)$ because the uniform coerciveness condition in (3.2) implies

$$(3.8) \quad \|U_z^{-1}\| \leq \frac{1}{k_2}, \quad \text{all } z \in Z;$$

in view of (3.6), (3.8) and (3.1), all the operators appearing in the definition (3.4) belong to bounded families, hence the uniform boundedness conclusion (i) is a simple consequence of the fact that

$$\|F_{x,z,r}\| \leq (\|U_z^{-1} + S v^{-1} S^*\| + \|S v^{-1}\| + \|v^{-1} S^*\| + \|v^{-1}\|)_{x,z,r}.$$

The lemma is thus established.

We shall now give a variant of lemma 1: roughly speaking, if the set $\{u_r\}$ is sufficiently coercive, the hypotheses on $\{C_x\}$ can be weakened. More precisely, let us assume that

(3.1)' $\{C_x : x \in X\}$ is a bounded subset of $L(H, h)$; let k_5 be a number such that $\|C_x^*\| \leq k_5$, all $x \in X$;

(3.3)' $\{u_r : r \in R\}$ is a subset of $L(h, h)$ satisfying: $(u_r s, s) \geq k_6 \|s\|^2$, all $s \in h$ and $r \in R$, $k_6 > 0$.

Lemma 3.2.

Under the hypotheses (3.1)', (3.2), and (3.3)' with $k_6 > k_5^2/k_2$ the conclusions of lemma 3.1 hold.

Proof

Observe that

$$\begin{aligned} (v_{x,z,r} s, s) &= (u_r s, s) - (C_x U_z^{-1} C_x^* s, s) \geq k_6 \|s\|^2 - \|U_z^{-1}\| \|C_x^*\|^2 \|s\|^2 \\ &\geq (k_6 - k_5^2/k_2) \|s\|^2 \end{aligned}$$

holds for all $s \in h$ and $(x, z, r) \in P$; hence

$$\|v_{x,z,r}^{-1}\| \leq (k_6 - k_5^2/k_2)^{-1}.$$

From now on the proof continues as in lemma 3.1 and is omitted.

Let us now return to the general problem formulated at the beginning of section 2 and make more precise our conjecture about its solution.

In view of the lemmas above, the rank condition as well as the positive definiteness condition of theorem 2.1 should be strengthened so as to hold uniformly in the parameters. More precisely we conjecture that under the following type of hypotheses

- the map G is bounded and satisfies: $\|G^T(x) s\| \geq k_1 \|s\|$, for all $s \in R^m$, all $x \in X$, and some $k_1 > 0$, where $G^T(x)$ denotes the transpose of $G(x)$;
- the map Q is bounded, its values are symmetric (matrices), and satisfies $(Q(x; y) s, s) \geq k_2 \|s\|^2$, for all $s \in N(G(x))$, all $(x; y) \in X \times Y$, and some $k_2 > 0$;

it is likely to hold a global estimate of the following type

$$r^* = \max\{0, \sup_{x,y} \max_{1 \leq i \leq m} [-(BQB^T)_{|x,y\rangle ii} + \sum_{j=1, j \neq i}^m |(BQB^T)_{|x,y\rangle ij}| + \hat{M}]\}.$$

4. SOME NUMERICAL RESULTS

The estimate of the penalty parameter arrived at here has been used - in a simple way - within the standard multiplier method scheme, and implemented in a computer program. This latter has been tested in particular on the following example of constrained minimization problem (P):

(4.1) minimize $f(x) = x_1 x_2 x_3 x_4 x_5$, subject to

$$g_1(x) = x_1^2 + x_2^2 + x_3^2 + x_4^2 + x_5^2 - 10 = 0$$

$$g_2(x) = x_2 x_3 - 5x_4 x_1 = 0$$

$$g_3(x) = x_5^3 + x_2^3 + 1 = 0$$

whose solution (to 4 figures) is

$$f(\bar{x}) = -2.920, \quad \bar{x} = (-0.7636, 1.596, 1.827, -0.7636, -1.717).$$

Numerical results for (4.1) were given originally by Powell [9] and subsequently by Asaadi [2] where, incidentally, there is a misprint in $g_2(x_2 - x_3)$ instead of $x_2 x_3$. Actually in [2], the initial point x_0 used in solving (4.1) is declared

$$x_0 = (-1, 1.5, 2, -1, -2),$$

and (4.1) is among a number of examples solved by means of a few different nonlinear programs with the purpose of comparing their efficiency. Let N_i stand for number of iterations (outer iterations if there are as in the multiplier method inner iterations i.e. within each unconstrained minimization) and N_e stand for number of function evaluations. From [2] one sees that to solve (4.1) to 4 figures (N_i ; N_e) is

- (7,148) with the method of Kowalik, Osborne and Ryan
- (4.2)
- (6, 90) with Powell's version of the multiplier method
 - (6,126) with Fiacco and McCormick penalty method (SUMT).

By using the minimizations (1.2) with the iterations (1.3) initiated at $y_0 = 0$ and with the penalty parameter r_k equal for all k to the constant r^* computed in correspondance of the initial point, we have obtained the solution with $(N_i; N_e) = (3; 78)$. The usefulness of inserting the estimate r^* , is not however in such possible slight improvement of the efficiency in terms of "number of function evaluations" as used in [2]; actually in [2] it is not said how has been fixed r_k nor are tabulated initial guesses if any - i.e. minimizations (1.2) and iteration (1.3) - that might have been necessary in order to find "good values" of the penalty parameter, for instance in the computations (4.2); such possible

guess work, that need not indeed be negligible, is reduced or avoided by using an estimate such as r^* .

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