IMPLEMENTABLE POLICIES FOR IMPROVING THE
BIOMASS YIELD OF A FISHERY

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Chapter 1. Introduction

The Beverton-Holt Model of a fishery was first presented in [1] and is still the main tool in planning policies where the age structure of the fish is considered important (see [2] - [4]), even though Leslie Matrix techniques have been applied. The model assumes that no stock-recruitment relationship exists and that fishing mortality depends on the net choice and the applied fishing intensity. This paper deals with choosing mesh sizes and planning fishing strategies to improve yields.

[2], [3], [5] have given various analyses of the optimal way to harvest a single year class under various restriction. Chapter 2 solves the general problem formulated in [2] subject to an implementable class of controls and presents a simple numerical way of computing the optimal solution. A method for choosing optimal mesh sizes for maximum sustainable yields is then presented, based on a technique given in [6].

[4], [7], [8] considered the effect of all the year classes being fished together, since no net is completely selective. Chapter 3 defines periodicity in fisheries control policies and analyses the structure of periodic solutions, allowing one to obtain better long term yields than the maximum sustainable yield. [2], [3], [4], [7], [8], [9] have discussed the concept of "pulse fishing", whereby one fishes with maximum possible intensity at certain times. "Bang-bang" solutions to the periodic problem of this type are then discussed.

Assuming that one has chosen a long-term policy, necessary conditions on the age structure are presented in Chapter 4 that insure that future average yields over periods of fixed duration are at least as good as the long term average yield. This allows one to formulate strategies for recovering overexploited fisheries in a control theoretic framework. A numerical example provides some interesting insights.

CHAPTER 2

THE SINGLE YEAR-CLASS MODEL

The Beverton-Holt Model of a fishery deals with a stock of a given species contained within certain geographical limits that does not interact with other fish of the same species. Whereas most of the fishery economics literature deals with aggregated numbers of fish, or their total weight, the Beverton-Holt Model also considers their age structure. Each year-class is regarded as having dynamics such as mortality and weight growth independent of the others.

At the beginning of each successive year i a number of fish, called the recruitment R_i , enter the fishery. These are considered to be independent of previous populations, an assumption that is not as rash as it initially sounds when one investigates the available stock-recruitment models. Considering that a single female cod can produce millions of eggs, that of the order of 99.998% of the eggs fail to survive, and that survivorship depends largely on local environmental conditions, it is not surprising that recruitments can vary by factors of more than a hundred and are considered random, uncorrelated with previous populations.

Suppose that at time $t = 0$, a number of fish R arrive at the fishery. At any future time t the number of fish in this year-class $N(t)$ suffers natural mortality M and fishing mortality $F_M(t)$, both proportional to $N(t)$. An upper limit T on the fishes' lives is also decided upon, above which no fish can survive:

$$\begin{aligned}\frac{dN(t)}{dt} &= -[M + F_M(t)] N(t), \quad 0 \leq t < T, \quad M > 0, \quad F_M(t) \geq 0 \\ N(0) &= R > 0 \\ N(t) &= 0, \quad t \geq T\end{aligned}\tag{2.1}$$

Associated with the year-class is a weight function $w(t)$ which describes the average weight of a fish of age t . We require $w(t)$ to have continuous first derivative, increasing and bounded for $t \in [0, T]$, and $\frac{\dot{w}}{w}(t)$ decreasing, a standard property of natural growth curves. (The von Bertalanffy curve $w(t) = w_\infty [1 - e^{-r(t-t_0)}]$ is often a good fit).

We assume too that the fishing mortality $F_M(t)$ is linearly dependent on fishing intensity, a function of the number of vessels used, their tonnage, and the time spent fishing. (see [1], [3], [8]).

We shall henceforth use $F(t)$ instead of $F_M(t)$ in (2.1) and call it fishing intensity. It is piecewise continuous and physically bounded as follows:

$$0 \leq F(t) \leq \bar{F}\tag{2.2}$$

Given a constant fixed cost per unit fishing effort c , a fixed unit price for fish, and having decided on a discount rate r , Clark et al. [2] showed that the present

value of the stock PV is given by the following expression:

$$PV = \int_0^{\infty} e^{-rt} F(t) \{N(t)w(t) - c\} dt \quad r \geq 0, \quad c \geq 0 \quad (2.3)'$$

We are concerned with finding strategies $F(t)$, $t \in [0, T]$ which maximize this quantity.

If $F(t) = 0$, $t \in [0, T]$, then the natural biomass curve $B(t) = N(t)w(t)$ has shape as in Fig. 1. We assume that c is such that either $c > B(0)$ in which case we define $t^* = 0$, or else $\exists t^* \in [0, T]$ such that $B(t^*) = c$ and $B'(t^*) > 0$. Otherwise the value of the stock never exceeds the cost of extracting it. Either $B(T) > c$ in which case we define $t^{**} = T$ or else $\exists t^{**} \in [t^*, T]$ where $B(t^{**}) = 0$ and $B'(t^{**}) < 0$. We also define the value curve $V(t)$ for any $F(t)$:

$$V(t) = e^{-rt} \{N(t) w(t) - c\} \quad (2.4)$$

Fig. 1. Dynamics of Single year Class.

These curves are plotted in Fig. 1, together with $e^{-rt}B(t)$, and defining the ages at which they achieve their respective maxima t_o , t_c , and t_r , it is easily seen that these can be computed from the following relationships:

$$t_o \in (t^*, \infty) \quad \text{where} \quad \frac{\dot{w}}{w}(t_o) = M \quad (2.5a)$$

$$t_c \in (t^*, \infty) \quad \text{where} \quad B(t_c) \{r + M - \frac{\dot{w}}{w}(t_c)\} = cr \quad (2.5b)$$

$$t_r \in [0, \infty) \quad \text{where} \quad \frac{\dot{w}}{w}(t_r) = M + r \quad (2.5c)$$

Because $\frac{\dot{w}}{w}(t)$ is decreasing and since $r, M, c \geq 0$, it can also be seen

from (2.5a) - (2.5c) that:

$$t_r \leq t_c \leq t_o \quad (2.6)$$

Note: It is possible that $T < t_r$. (2.6)

Implementable Optimal Solutions of the One-dimensional Problem:

If $\bar{F}$ is very large, the situation corresponds to our being able to harvest all the fish in one fell swoop. If we wish to maximize the biomass yield, it is obvious that we should do so at the point this curve reaches its peak, i.e. at t_o . Clark et al. [2] and Hannesson [3] used an investment technique first described by Fisher to show that the point of "instant fishing" for a given c and r is t_c or that the optimal policy for $\bar{F}$ large enough was one with controls not piecewise constant, and therefore not easily implementable. Thus bounds on the point of harvest are given by (2.6) for any cost c , given r .

In practice, however, $\bar{F}$ is relatively small (e.g. $\bar{F} < 1.5$ for $M \approx .15$). This means that we shall require more time within which to harvest the year-class. It will be shown that the optimal implementable policy for any $\bar{F}$ will be of the form:

$$\begin{aligned} F(t) &= 0 & 0 \leq t < t_1 \\ &= \bar{F} & t_1 \leq t < t_2 \\ &= 0 & t_2 \leq t \leq T \end{aligned} \quad (2.7)$$

where $t_1 \leq t_c \leq t_2$. That is, there is only one period of fishing (t_1, t_2) and it must include t_c . A simple numerical method for determining t_1 and t_2 will then be outlined.

Definition: $t_1 \in [0, T]$ is called a switching-on time if t_2 is called a switching-off time if

$$\begin{aligned} F(t_1 - \delta) &= 0 \quad \text{and} \quad F(t_1 + \delta) = \bar{F} \\ F(t_2 - \delta) &= \bar{F} \quad \text{and} \quad F(t_2 + \delta) = 0 \end{aligned} \quad \forall \delta \in (0, \epsilon), \text{ some } \epsilon > 0. \quad (2.8)$$

Using standard optimization techniques (see [10]) we form the Hamiltonian of the system described by (2.1) and (2.3):

$$\begin{aligned} H(t) &= -F(t)e^{-rt} \{N(t)w(t) - c\} - \lambda(t) \{M+F(t)\} N(t), \quad t \in [0, T] \\ &= -F(t) \{V(t) + \lambda(t)N(t)\} - \lambda(t)MN(t) \end{aligned} \quad (2.9)$$

where $V(t)$ is as in (2.4) and $\lambda(t)$, the adjoint variable, is defined by:

$$\begin{aligned}
 -\dot{\lambda}(t) &= \frac{\partial H}{\partial N} \\
 &= -\{M+F(t)\} \lambda(t) - F(t) w(t) e^{-rt}
 \end{aligned} \tag{2.10a}$$

$$\text{and satisfies the transversality condition } \lambda(T) = 0 \tag{2.10b}$$

Using a strong variation of the Maximum Principle we know that, except for singular subarcs, the optimal $F(t)$ will minimize $H(t) \forall t \in [0, T]$ subject to (2.2). Along a singular subarc over I , a finite subset of $[0, T]$, we have:

$$V(t) + \lambda(t) N(t) = 0 \quad \forall t \in I$$

$$\text{i.e.} \quad \dot{V}(t) + \dot{\lambda}(t) N(t) + \lambda(t) \dot{N}(t) = 0 \quad \forall t \in I$$

$$\text{or} \quad \left\{ \frac{\dot{w}}{w}(t) - M - r \right\} N(t) w(t) + r c = 0 \quad \forall t \in I$$

This is the singular arc that Clark et al [2] and Hannonsson [3] proposed as the optimal policy for $c, r \neq 0$, and $\bar{F}$ large enough such that the path can be followed. Further differentiation yields the corresponding control:

$$F(t) = \frac{\dot{w}(t) - (M+r) \frac{\dot{w}(t)}{w(t)}}{\frac{\dot{w}(t)}{w(t)} - (M+r) \frac{\dot{w}(t)}{w(t)}} - M \quad \text{which can be satisfied only for } t \in [t_c, t_o].$$

However, as they pointed out, such a policy is not feasible practically, and we shall therefore search for extremal solutions that yield piecewise constant controls (and exclude sequences of these that may tend weakly to the singular solution). Henceforth we shall regard this as our class of admissible controls. For $\bar{F}$ not large enough to follow singular arcs, our solutions are the same as those from the class of piecewise continuous controls. [Numerical examples show the differences between the singular solutions and those proposed below are $< 1\%$]. Our implementable solution must therefore minimize $H(t)$, which yields:

$$\begin{aligned}
 F(t) &= \bar{F} & V(t) + \lambda(t) N(t) &> 0 \\
 &= 0 & &< 0
 \end{aligned} \tag{2.11}$$

and since the switching curve is continuous we have that at the switching times t_i :

$$V(t_i) + \lambda(t_i) N(t_i) = 0 \tag{2.12}$$

Lemma 1: If t_1 is a switching-on time, then $\dot{V}(t_1^-) > 0$.

Similarly $\dot{V}(t_2^+) < 0$ for t_2 switching-off time. (2.13)

Proof: We first show that:

$$F(t) = 0 \quad \forall t \in [a, b] \Rightarrow \lambda(t) N(t) = \text{constant on } [a, b].$$

$$F(t) = 0 \Rightarrow \dot{N}(t) = -MN(t), \quad \dot{\lambda}(t) = M\lambda(t)$$

$$\frac{d}{dt} [\lambda(t) N(t)] = \dot{N}(t) \lambda(t) + N(t) \dot{\lambda}(t) = 0$$

$$\Rightarrow \lambda(t) N(t) = \text{constant}$$

$$(2.8), (2.11), \Rightarrow \{V(t) + \lambda(t) N(t)\}_{t_1 - \delta} < 0 \quad \text{some } \delta > 0$$

$$(2.12) \Rightarrow \{V(t) + \lambda(t) N(t)\}_{t_1} = 0$$

$$\text{Therefore: } \frac{V(t_1 - \delta) - V(t_1) + \lambda(t_1 - \delta) N(t_1 - \delta) - \lambda(t_1) N(t_1)}{\delta} > 0$$

$$(2.14) \Rightarrow \lim_{\delta \rightarrow 0^+} \frac{V(t_1 - \delta) - V(t_1)}{\delta} > 0$$

i.e. $\dot{V}(t_1^-) > 0$ since derivative obviously exists where $F(t)$ is constant.

Similarly $\dot{V}(t_2^+) < 0$.

Theorem 1. If t_1 is an optimal switching-on time and t_2 is an optimal switching-off time, then:

- i) There is at most one fishing period.
- ii) $t_1 \in [0, t_c]$
- iii) $t_2 \in [t_c, T]$

Proof. i) Suppose that there is more than one switching-on time in $[0, T]$. Then let t_2^a correspond to the switching-off time of a period followed by a switching-on time $t_1^b < T$.

Then by Lemma 1:

$$\dot{V}(t_2^a) < 0 \Rightarrow e^{-rt_2^a} \left\{ \left(\frac{\dot{w}}{w}(t_2^a) - M - r \right) B(t_2^a) + rc \right\} < 0$$

If $t_2^a < t_c$, then $t \in [t_2^a, t_c]$ imply $B(t)$ increasing, $\frac{\dot{w}}{w}(t)$ decreasing; while $V(t)$ is decreasing $\forall t \geq t_2^a$ (by (2.6)). Together these imply that $\dot{V}(t) < 0 \forall t \in (t_2^a, T)$, which implies $\dot{V}(t_1^b) < 0$, contradicting Lemma 1.

So there is at most one switching-on time $t_1 \in [0, T]$.

ii) A switching-on time t_1 must exist; then from the shape of $V(t)$ in Fig. 1., (2.5b) and (2.13), it is obvious that $t_1 \leq t_c$.

iii) Let t_2 be the subsequent switching-off time and suppose that

$$t_2 < \min \{t_c, T\}$$

Together with Lemma 1 this implies that $\exists \delta t > 0$ such that $\dot{V}(t + \delta t) < 0$ so putting $t_3 = t_2 + \delta t$ we get $e^{-rt_3} \left\{ \left(\frac{\dot{w}}{w}(t_3) - M - r \right) w(t_3) N(t_2) e^{-M(t_3 - t_2)} + rc \right\} < 0$

$$\Rightarrow N(t_2) > \frac{rc}{w(t_3) e^{-M(t_3 - t_2)} \left\{ M + r - \frac{\dot{w}}{w}(t_3) \right\}} \triangleq k$$

But from (2.5b) and the shape of natural $V(t)$ we have that $e^{-rt_3} \left\{ \left(\frac{\dot{w}}{w}(t_3) - M - r \right) w(t_3) \bar{N} e^{-M(t_3 - t_2)} + rc \right\} > 0$ where $\bar{N} = \text{Re}^{-Mt_2} \Rightarrow \bar{N} < k < N(t_2)$

But $N(t_2) = \text{Re}^{-Mt_1 - (M+F)(t_2 - t_1)} = \bar{N} e^{-\bar{F}(t_2 - t_1)} < \bar{N}$ contradiction.

$$\text{Therefore } t_2 \geq \min \{t_c, T\}$$

This theorem implies that the optimal fishing period must include time t_c . Being the maximum of the natural "value curve" it can be thought of as the time of "most efficient fishing", i.e. the time at which one gets most for one's effort.

Note: It is readily seen that $t_1 \geq t^*$ and $t_2 \leq t^{**}$, for otherwise one is unnecessarily adding negative parts to the objective function.

The shape of the biomass curve under fishing is next discussed.

a) If $\{N(T)w(T) - c\} \geq 0$, such as when $c=0$, then by (2.11) $F(T) = \bar{F}$, i.e.

$$t_2 = T \text{ and } \lambda(t_2) = 0.$$

b) If $\{N(T)w(T) - c\} > 0$, then $F(T) = 0$ and $t_2 < T$.

(2.10) $\Rightarrow \lambda(t_2) = 0$, and (2.12) gives $N(t_2) w(t_2) = c$.

These imply that

$$t_2 = \min \{T, \tau\} \text{ where } N(\tau)w(\tau) = c \quad (2.15)$$

Since $\lambda(t_2) = 0$ (2.10a) gives

$$\lambda(t_1) = \bar{F} \int_{t_2}^{t_1} e^{-rs} w(s) e^{(M+\bar{F})(t_1-s)} ds \text{ and substituting into (2.12)}$$

we get:

$$e^{-rt_1} \{N(t_1)w(t_1) - c\} = \bar{F} \int_{t_1}^{t_2} e^{-rs} N(s) w(s) ds \quad (2.16)$$

where the left hand side is the natural value curve, independent of $\bar{F}$ for given c . As $\bar{F} \rightarrow 0+$, right hand side $\rightarrow 0$; so by choosing $\bar{F}$ small enough one can make t_1 as close to t^* as one wishes. Also as $\bar{F} \rightarrow 0+$, $N(t_2) \rightarrow N(\min \{T, t^{**}\})$ i.e. $\bar{F} \rightarrow 0 \Rightarrow$ the optimum fishing period $(t_1, t_2) \rightarrow (t^*, \min \{T, t^{**}\})$. Fig. 2 illustrates the varying trajectories for different $\bar{F}$'s using data for plaice from [1]. with $r = .1$ and $c = .05$.

Fig.2. Optimal Implementable Trajectories.

From (2.16) can also be derived the following expression:

$$e^{-rt_1} \{B(t_1) - c\} = \bar{F} \int_{t_1}^{t_2} e^{-rs} \{B(s) - c\} ds + \bar{F} c \int_{t_1}^{t_2} e^{-rs} ds$$

i.e. The instantaneous value of the fishery at starting time

= expected revenue + discounted cost.

This is a generalization of the result Goh [5] proved for $c, r = 0$.

Implementing Numerical Algorithms to solve the 1-D Problem

The problem is to find $(t_1, t_2) \in [t^*, T]$ which maximizes

$$J = \bar{F} \int_{t_1}^{t_2} e^{-rs} \{N(s)w(s) - c\} ds$$

Now given t_1 , (2.15) allows us to calculate a corresponding t_2 which will be optimal if t_1 is. So the problem reduces to a one-dimensional minimization problem, with a convenient starting interval for the search, $[t^*, t_c]$.

$$\min_{t_1 \in [t^*, t_c]} V(t_1) = -\bar{F} \int_{t_1}^{t_2(t_1)} e^{-rs} \{N(s)w(s) - c\} ds \quad (2.17)$$

$$\text{where } t_2(t_1) = T \text{ if } N(t_1)e^{-(M+\bar{F})(T-t_1)} w(T) > c$$

$$= \text{solution of } N(t_1)e^{-(M+\bar{F})(T-t_1)} w(\tau) = 0 \text{ otherwise.}$$

$$t_1 \in [t_1, T]$$

A Numerical Example and some Observations

The above algorithm was used to compute the optimal fishing periods for various costs, discount rates, and $\bar{F}$'s. The parameters used are those for North Sea Plaice as above, although the below observations apply also to other data used (North Sea Cod). It must be remembered that we assume that we are exploiting each year-class optimally, i.e. that we do not expend any fishing effort on the year-class after t_2 , a condition not easy to implement.

Fig.3. Optimal Fishing Periods.

i) Fig. 3 shows the optimal fishing periods for different costs and discount rates. It is interesting to note that as costs increase to $\max \{B(t)\}$, the period becomes a pulse at t_0 . This is because at high costs $t \in [0, T]$ the profit margin is low and $[t^*, t^{**}]$ gets smaller while $t_0 \in [t^*, t^{**}]$. Thus high costs eliminate the advantage of considering discount rates.

ii) It is instructive to consider the situation from the viewpoint of a private company, interested in maximizing its profits from a single year stock. If r represents the rate of interest in other forms of investment, then the profit from the stock at time T is:

$$P = \int_0^T F(t) e^{r(T-t)} \{N(t)w(t) - c\} dt \quad (2.18)$$

which has the same optimal solution (t_1, t_2) as (2.3)

$$P_{\text{opt}} = e^{rT} \bar{F} \int_{t_1}^{t_2} V(t) dt \quad (2.19)$$

The profits that can be gained using differing interest rates, and the corresponding drop in the actual mass of fish-meat extracted are illustrated in Fig.4.

Fig.4. Profits due to interest.

Fig.5. Yields for varying $\bar{F}$'s.

Two features are apparent:

1) A high interest rate makes possible such increased profits that it acts as a large incentive for the private firm to adopt policies which drastically reduce the biomass yield.

2) High fishing costs tend to nullify this effect somewhat, a fact that should be considered when planning taxation.

iii) Fig. 5 illustrates how much improvement in profits is possible by increasing fishing capacity. It shows that for $\bar{F} > .4$, very little extra revenue is gained, implying that a rather conservative policy of using low intensity effort for a long time is not only more stable, but also almost as profitable. Increasing costs accentuate this trend.

Application to choice of mesh size

In practice the fishing effort $F(t)$ does not affect the fishing mortality $F_M(t)$ of all the age groups equivalently. Depending on the mesh size of the net, the same fishing effort has more effect on old fish than on young ones. This can be represented by means of a selectivity curve $s(t)$, a monotonic increasing function of the age of the fish t .

$$\text{i.e. } F_M(t) = s(t) F(t) \quad 0 \leq s(t) \leq 1, \quad t \in [0, T]$$

So $s(t)$ represents the probability of a t -year old fish being affected by the net.

A class of nets can be parameterized by a parameter such as the mean cut-off age. Denote $s_p(t)$ the curve corresponding to parameter $p \in \bar{P}$, a closed subset of R^1 .

(2.20)

Knife-edge Selection and Sustainable Yields

Since the number of fish of any year-class after T years is too low to significantly affect results, the precise choice of T is not very important (see [1]). We can therefore choose T to be integral which implies that at any time there are T distinct year classes in the fishery.

Beverton and Holt [1] observed that if one fishes constantly with $F(t) = F_s$ the catch in one year due to all the T year classes, is equal to the total yield of one year class over T years (2.21).

A useful simplification that we shall consider first is to regard a net as exploiting all fish above a cut-off age t_K equally, and not affecting younger fish.

$$\begin{aligned} s_{t_K}(t) &= 0 & 0 \leq t \leq t_K \\ &= 1 & t_K < t \leq T \end{aligned} \quad (2.22)$$

Assuming that $c = 0$ and that one can select a knife-edge mesh with parameter t_K as in (2.22), choosing $t_K = t_1$ for the given $\bar{F}, r$ will obviously give the maximum possible sustained yield since, as Goh [5] and Clarke et al [2] pointed out,

one is then best exploiting each year-class individually.

If $c \neq 0$, then one can use a technique very similar to that described by Turvey [6] to compute the optimal choice of t_K and $\bar{F}$ for a long term yield. The method is described here both for completeness and because it presents a simply computable explicit solution to a well defined problem, as opposed to Turvey's theoretical formulation.

Using the observation (2.21), the problem is to choose $\bar{F}$, t_K so as to find the maximum sustainable yield Y_s .

$$\begin{aligned} Y_s &= \max_{t_K \in [0, t_0], \bar{F}} \int_0^T \bar{F} e^{-rt} \{s_{t_K}(t) N(t) w(t) - c\} dt \\ &= \max_{\bar{F}} \max_{t_K} \left\{ \int_0^T \bar{F} e^{-rt} N(t) w(t) dt - c \bar{F} \int_0^T e^{-rt} dt \right\} \end{aligned} \quad (2.23)$$

since s_{t_K} is knife-edge. The solution of the eumetric curve

$$E(\bar{F}) = \max_{t_1} \int_{t_1}^T \bar{F} e^{-rt} N(t) w(t) dt \quad (2.24)$$

was derived above and one can plot this function and the corresponding optimal cut-off ages $t_K(\bar{F}) = t_1(\bar{F})$ for varying $\bar{F}$'s as in Fig. 6.

$E(\bar{F}) \rightarrow N(t_r)w(t_r)$, and $t_K(\bar{F}) \rightarrow t_r$ as $\bar{F} \rightarrow \infty$.

Fig.6. Max. Sust. Yield Policy ($c \neq 0$).

The second term in (2.23) is linear in $\bar{F}$ and can easily be drawn for a given c . Solving (2.23) now requires choosing $\bar{F}_{opt}$ such that the distance between the two curves is greatest. The solution to this is of the Golden Rule type, where the slope of the eumetric curve is $c \int_0^T e^{-rt} dt$. t_{Kopt} is the corresponding best cut-off point.

Extensions for Different Nets

The above method can easily be extended to more general descriptions of nets. All that is different is that the eumetric curve is derived from minimizing with respect to a parameter p instead of t_K . Nets are never actually knife-edge, although most treatments to date have searched for the optimal mean cut-off age t_M by identifying it with the optimal cut-off age t_K .

By considering the solutions of the knife-edge problems, it is obvious that its yields must be greater than those from any other selectivity curve. The curves due to different $\bar{F}$'s for a mean cut-off class of meshes with selectivity curve slope equal to .5 is shown in Fig. 6. They are very similar to those for knife-edge selection (<.1%) and show that when considering sustainable yields the approximation mentioned above is fairly accurate.

This technique can be easily used for any class of meshes parameterized by a parameter p . The use of computers to numerically integrate functions makes it feasible to parameterize as accurately as possible sets of selectivity curves, giving more accurate choices of mesh. However, the general crudeness of the model probably gives such exercises little significance.

Optimum Solution of Conglomerate Problem for large $\bar{F}$ and c

It can be seen from Fig. 3 that for large values of $\bar{F}$ and c , the optimal harvesting of a single year-class may require fishing for less than one year. In this case it is possible by a suitable choice of net to obtain a periodic solution which makes full use of the available $\bar{F}$, rather than using $\bar{F}_{opt}$ as calculated above.

Given $\bar{F}$, r and c we can compute the optimal fishing period for a single year class $(t_1, t_2) < 1$. Suppose that we fish each year class when its age corresponds to this period.

Then provided the following conditions hold:

$$\begin{aligned} s_p(t) &= 0 & t &< t_2 - 1 \\ s_p(t) &= 1 & t &> t_1 \end{aligned} \tag{2.25}$$

each year class is optimally exploited, without interfering significantly in the dynamics of the other year classes.

Since fishing is traditionally periodic in a sense, (2.25) could provide a convenient method for establishing mesh sizes.

CHAPTER 3

THE CONGLOMERATE YEAR-CLASS PROBLEM

In considering strategies for maximizing the long-term yield of a fishery, it is worth considering what forms of control are likely not only to be effective, but also implementable. Limiting the size of the harvested fish or the extent of catches is not easily enforced. Planning $\bar{F}$ by means of controlling the building and power of fleets, and changing mesh sizes, are long-term measures, difficult to implement legislatively because of the "momentum" of the present fishing capital. In what follows we shall assume that we are dealing with a fixed sub-optimal net, and a given $\bar{F}$. The implementable measures we shall consider are:

- a) closing certain areas
- b) declaring closed times
- c) controlling the extent of fishing.

These can be summarized as follows

$$\text{Find } F(t), t \geq 0, \quad 0 \leq F(t) \leq \bar{F}$$

We shall assume $c = 0$ for simplicity, but the analysis that follows is easily adapted to include it. Similarly discount rates are not considered; these could in fact be regarded as being included in a modified weight function $e^{-rt} w(t)$ since such stringent conditions on the shape of $w(t)$ are no longer needed. The T-dimensional model can now be described:

Let each component $N_i(t)$ of $N \in \mathbb{R}^T$ be the number of fish in the age group $[i-1, i)$. Then

$$\frac{dN(t)}{dt} = -\{M + F(t)S(t)\} N(t) \text{ for almost all } t \geq 0 \quad (3.1)$$

$$N(0) = N_0 \geq 0$$

where M = unit matrix $\times$ natural mortality

$F(t) \in \mathbb{R}, 0 \leq F(t) \leq \bar{F}, t \geq 0$ represents fishing intensity

$$S(t) = \begin{bmatrix} s_p(\tau) & & & & \\ & s_p(\tau+1) & & & \\ & & \ddots & & \\ & & & 0 & \\ 0 & & & & \ddots \\ & & & & & s_p(\tau+T-1) \end{bmatrix}$$

where τ is the fractional part of t and $s_p(t)$ is the selectivity curve as described above for some parameter p .

The objective function is now:

$$\max J = \int_0^{\infty} F(t) W^T(t) S(t) N(t) dt \quad (3.2)$$

where $W(t) = (w(\tau), w(\tau+1), \dots$

$, w(\tau+T-1))^T$, τ is the fractional part of t and $w(t)$ is the weight function.

Note that the definition of $N(t)$, $S(t)$, $W(t)$ implies that at times $t = 1, 2, 3$ there are discontinuous boundary conditions similar to those used by Walters [7]. Specifically for $j \in I^+ = \{0, 1, 2, 3, \dots\}$

$$\begin{array}{lll}
 N_1(j) = R_j & W_1(j) = w(0) & S_1(j) = S_p(0) \\
 N_2(j) = N_1(j-) & W_2(j) = W_1(j-) & S_2(j) = S_1(j-) \\
 N_3(j) = N_2(j-) & W_3(j) = W_2(j-) & S_3(j) = S_2(j-) \\
 \vdots & \vdots & \vdots \\
 N_T(j) = N_{T-1}(j-) & W_T(j) = W_{T-1}(j-) & S_T(j) = S_{T-1}(j-)
 \end{array} \quad (3.3)$$

These relationships are illustrated in Fig. 7.

The system is a T -dimensional Bilinear system as discussed in [11].

We shall also assume that recruitments are equal, stationary, and independent.

$$R_j = R \quad j \in I^+.$$

Fig.7. Boundary Conditions.

Periodic Solutions and their Structure

Since the system must be optimized over infinite time, it is reasonable both theoretically and practically to search for periodic solutions.

Definition. The system described by (3.1) and (3.3) is said to be periodic with period P if $F(t) = F(t+P) \quad \forall t \geq 0$ and $\bar{t} > 0$ such that $N(\bar{t}) = N(\bar{t}+P) = N(\bar{t}+2P) = \dots$ (3.4)

The following theorem follows immediately:

Theorem. For any control, if the system described by (3.1) and (3.3) is periodic, then the period time P is an integer.

Proof. Suppose P , the period time is not an integer and let $\bar{t}$ be as in (3.4).

Then $N_1(\bar{t}) = N_1(\bar{t}+P)$ (3.5)

Now choose $t^* = \min \{ \{\bar{t}\}, \{\bar{t}+P\} \}$

$t^* + \Delta t = \max \{ \{\bar{t}\}, \{\bar{t}+P\} \}$ where $\{x\}$ is the fractional part of $x \in \mathbb{R}$.

Since $N_1(i) = R \quad \forall i \in \mathbb{I}^+$, and $F(t)$ is periodic

$$(3.5) \Rightarrow \operatorname{Re}^{-Mt^*} - \int_0^{t^*} F(t+\Delta t)s(t)dt = \operatorname{Re}^{-M(t^*+\Delta t)} - \int_0^{t^*+\Delta t} F(t)s(t)dt$$

$$\text{i.e.} \quad \int_0^{t^*} F(t+\Delta t)s(t)dt = M\Delta t + \int_0^{\Delta t} F(t)s(t)dt + \int_{\Delta t}^{t^*+\Delta t} F(t)s(t)dt$$

$$\text{or} \quad M\Delta t + \int_0^{\Delta t} F(t)s(t)dt + \int_0^{t^*} F(t+\Delta t) \{s(t+\Delta t) - s(t)\} dt = 0$$

But $M\Delta t > 0$, $s(t)$ is increasing so all right hand terms ≥ 0 . So we get a contradiction, which implies that P is an integer.

Note. Theorem 2 implies that $N(t) = N(t+P) \quad \forall t \geq 0$.

Next, the structure of a system with integral period P is considered. We define $N^S(0)$ to be the stable age structure at the beginning of the period, with $N^S(t)$ the age structure for $t \in [0, P)$.

$$\text{We also define:} \quad f_i(t_1, t_2) = e^{-M(t_2-t_1)} - \int_{t_1}^{t_2} F(t) s(i+t)dt \quad \text{for } (t_1, t_2) \subset [0, P]$$

This represents the fraction of fish left at time t_2 if we start fishing at t_1 and i indicates the age of the fish when we start fishing. Similarly define

$$g_i(t_1, t_2) = \int_{t_1}^{t_2} F(t)w(t)e^{-M(t-t_1)} - \int_{t_1}^t F(t)s(i+t)dt \quad \text{at the total catch from the same}$$

year-class over the same time with unit starting conditions.

Periodicity implies that $N^S(0-) = N^S(P-)$

This allows us to construct $N^S(0)$ and define $\alpha_i \in \mathbb{R}$, $i=1,2, \dots$ as follows:

$$\begin{aligned}
N_1^S(0) &= R \\
N_2^S(0) &= N_1^S(P-) = Rf_O(P-1, P) = R\alpha_1 \\
N_3^S(0) &= N_2^S(P-) = Rf_O(P-2, P) = R\alpha_2 \\
&\vdots \\
N_p^S(0) &= N_{p-1}^S(P-) = Rf_O(1, P) = R\alpha_{p-1} \quad (3.6) \\
N_{p+1}^S(0) &= N_p^S(P-) = N_1^S(0) f_O(0, P) \\
N_{p+2}^S(0) &= N_{p+1}^S(P-) = N_2^S(0) f_1(0, P) \\
&\vdots \\
N_T^S(0) &= N_{T-1}^S(P-) = N_{T-p}^S(0) f_{T-p}(0, P) = R\alpha_{T-1}
\end{aligned}$$

Fig. 8 clarifies what α_i , β_i represent. Note that constructing $N^S(0)$ from $F^S(t)$, $t \in [0, P]$ computationally merely requires "wrapping around" the $N_i^S(P-)$ for each i to $N_{i+1}^S(0)$ as illustrated.

It is also apparent that after T years of applying a periodic control to a system with constant recruitment, a stable age structure $N^S(0)$ will be constructed by (3.6)

Fig.8. Periodic Structure.

Fig. 8. α_i is the fraction of fish left after being fished "along the appropriate "arrow", β_i the corresponding catch ($T=S, P=3$).

We define β_i in a similar way:

If $f_j(t_1, t_2) = \alpha_i$, then $\beta_i = g_j(t_1, t_2) \quad i = 1, 2, \dots, T-1$

and $\beta_T = g_{T-P}(0, P)$

$$\beta_{T+1} = g_{T-P+1}(0, P-1) \quad (3.7)$$

$$\beta_{T+P-1} = g_{T-1}(0, 1)$$

Associated with this control and period P is the cost functional derived from (3.2).

$$\text{average yield} = \frac{1}{P} \int_0^P F^S(t) W^T(t) S(t) N^S(t) dt \quad (3.8)$$

Applying a periodic policy for a long while implies that any recruitment R_j is subject to one of P fishing policies, depending on where within the cycle $[0, P]$ it appears. [e.g. The yield due to $R=1$ at $t=2$ in Fig. 8. is $\gamma_3 = \beta_1 + \alpha_1 \beta_4$.] Suppose that these different yields due to a recruitment $R=1$ are $(\gamma_1, \gamma_2, \dots, \gamma_P)$; then we can exploit the linearity of catch with recruitment as follows.

The long term expected average yield is given by:

$$\begin{aligned} E[Y] &= \sum_{j=0}^{\infty} (E[R_j] \gamma_1 + E[R_{j+1}] \gamma_2 + \dots + E[R_{j+P}] \gamma_P) \\ &= R \sum_{j=0}^{\infty} \sum_{i=1}^P \gamma_i \quad \text{because we assumed } R_i \text{'s independent and} \end{aligned}$$

$$E[R_j] = E[R_{j+1}] = \dots = R$$

So the expected long term yield is the same as that obtained by considering the long term deterministic case with constant recruitment R .

The problem is therefore to find $P, F^S(t)$ for $t \in [0, P]$ so as to maximize (3.8). P is an integer and since periods of more than fifteen years are beyond consideration practically, we need only solve a finite number of optimizations of (3.8) to find optimal $F^S(t)$ for $P \leq 15$. Note, however, that we do not claim to solve the long-term problem, only the periodic one.

Numerical Solutions of the Periodic Problem

The periodic formulation of the problem can be transformed into a standard P -dimensional problem as discussed in [11]. It can be solved using numerical algorithms for constrained control such as those in [11], [12], [13].

The Switching-Time-Variation Method (STVM) by Mohler [11] was used to find controls that satisfied the Strong Maximum Principle and could reasonably be assumed

to be global solutions. However, the implementation of the algorithm is difficult and computationally time consuming (especially evaluating the switching curve). Several other simply implementable techniques were also used to get suboptimal solutions. The three procedures are described below.

i) Following Hannesson [3] and to a certain extent Pope [4] we assume that we fish for one of every P years. A one-dimensional minimization is then carried out on the degree of fishing intensity during that year. However, above a certain low P , the required intensity rapidly exceeds physical bounds. Hannesson [3] achieved better results because he was also selecting the cut-off age of the mesh t_k . The results, however, were inferior to ii) below.

ii) In this method the full fishing capacity $\bar{F}$ was implemented and it was assumed that the single stretch of fishing during $[0, P]$ would end at P . The starting time $t_1 \in [0, P]$ was then chosen to maximize the yield. The results of this crude method were surprisingly good. A two-dimensional search on both the starting and finishing times $[t_1, t_2] \subset [0, P]$ improved the yield marginally ($<1\%$).

iii) The Switching-Time-Variation Method was used with much difficulty. Forming the Hamiltonian, one must define an adjoint variable as in the single-dimensional case and deduce boundary conditions at integral times similar to (3.3) (see [10]). A version of the Strong Maximum Principle indicates that the optimal strategy must be "bang-bang".

If the selectivity curve was very steep, then the best policy within a period usually consisted of about $P/2$ gradually lengthening pulses, the total duration of which was roughly the solution of ii) above. Shallow selectivity curves tended to reduce the number of pulses within each period. Again their total duration was that of ii).

- Notes
- 1) Two one-year periods are a local solution of the two-year single period problem and similarly for longer periods. This is an example of a Hamiltonian with several local minima in function space.
 - 2) Yields of up to 9.34% increase on the maximum sustainable yield were possible with current data on the North Sea 3N-0 cod stock with $\bar{F}=2.$, a reasonable figure. It was generally indicated that improvements in yield due to periodic policies are chiefly attained due to using the full available intensity for a certain length of time.
 - 3) The yields are relatively insensitive to the actual distribution of this length of time, the optimal yield using iii) never exceeding that of ii) by more than 1% in the examples used. This indicates that simple optimizations such as ii), possibly extended to two dimensions, would yield results as good practically as those by the sophisticated but

much more difficult techniques.

- 4) It is important to realize that if one is pulsing with large fishing intensity, the solution is more sensitive to small variations in the total length of fishing time, so this must be strictly determined and implemented.
- 5) When $c \neq 0$ or $r \neq 0$ larger improvements than the above are envisaged, partly due to policies like those discussed earlier (chapter 2), and also due to profitability arguments. (e.g. If the total biomass is less than the cost of extracting it, one wouldn't consider fishing. See also [2]).

Table 1 compares the yields obtained by methods i), ii), iii) to the maximum sustainable yield. The data used is that for the ICNAF 3N-0 North Sea Cod Fishery. In this case the solution by iii) is the same as the two-dimensional version of ii), i.e. there is only one period of fishing.

The figures refer to percentage improvements on the maximum sustainable yield obtained by F^S (optimal) = .285.

Period	1	3	6	10
i) $\bar{F} \text{ opt} \leq 2$	.285	.883	2.	2.
Increase	0	.59	4.67	decreasing
ii) $t_1 \text{ (opt)}$	.86	2.57	4.94	8.22
Increase	1.00	1.54	4.77	9.23
iii) $[t_1, t_2] \text{ (opt)}$	[.86, 1.00]	[3.36, 2.07]	[4.36, 5.35]	[7.86, 9.67]
Increase	1.00	1.62	5.05	9.25

Table 1. Percentage Improvements in Max. Sust. Yield.

CHAPTER 4

NECESSARY CONDITIONS FOR CONTINUED SUSTAINABLE YIELDS

Suppose that we have established a periodic control with period P . Then using (3.6), (3.7) we can construct the corresponding α_i , $i=1, T$.

$$\beta_i, i=1, 2, \dots, T+P-1.$$

Next we define:

$$\begin{aligned} \bar{b} \in R^P &= (\beta_1, \beta_2, \dots, \beta_P)^T \\ b \in R^T &= (\beta_{P+1}, \beta_{P+2}, \dots, \beta_{P+T-1}, 0)^T \\ a \in R^T &= (\alpha_1, \alpha_2, \dots, \alpha_P, 0, \dots, 0)^T \end{aligned} \quad (4.1)$$

$$A \in R^{TxT} = \begin{bmatrix} & & & 0 & & 0 & & \\ & & & \dots & & \dots & & \\ & & & \alpha_{P+1} & & & & \\ & & & \alpha_{P+2} & & 0 & & \\ & & & \vdots & & \vdots & & \\ & & & 0 & & \vdots & & 0 \\ & & & & & \vdots & & \\ & & & & & \alpha_T & & \end{bmatrix} \quad \begin{matrix} P \\ T-P \end{matrix}$$

Then just before the beginning of the period of control, we have the age structure $N(O-)$. At the end of the period the new age structure will be $N(P-)$ and they are related as follows:

$$N(P-) = AN(O-) + Ra \quad (4.2)$$

and the corresponding catch

$$\begin{aligned} C &= b^T N(O-) + (R, R, \dots, R) \bar{b} \\ &= b^T N(O-) + R\beta \quad \text{where } \beta = \sum_{i=1}^P b_i \end{aligned} \quad (4.3)$$

But the stable age structure N^S is invariant under (4.2)

$$\text{i.e. } N^S(P-) = N^S(O-) = AN^S(O-) + Ra \quad (4.4)$$

$$\text{or } N^S(O-) = (I-A)^{-1} Ra, \text{ a unique solution}$$

(Note $(I-A)^{-1} = I + A + A^2 + \dots$ converges in a finite number of steps).

$$C_s = R \{b^T (I-A)^{-1} a + \beta\} \quad (4.5)$$

(4.4) and (4.5) are useful expressions for the stable age structure and catch.

We seek conditions on $N(O-)$ that ascertain that the expected yield with variable recruitments R_j ($E[R_j] = R$, $\text{co}[R_i, R_j] = 0$, $i \neq j$) from time $t = 0$ is at

least as much as if we had the stable age structure $N^S(O-)$. Thus we can judge whether the fishery is in good shape with respect to its age structure or not. It also provides a convenient target set in planning strategies for the "recovery" of a fishery, understood to mean the above.

Suppose that the age structure just before we start applying the periodic control is $N(O-)$. Note that the two right hand terms in expression (4.3), refer respectively to the catch from the known age structure and future expected recruitments. Then in the first period of control the expected catch:

$$EC_1 = b^T N(O-) + E_1 \text{ where } E_1 \text{ is the expected yield due to new recruitments.}$$

To satisfy the above conditions we require:

$$EC_1 \geq EC_s$$

i.e. $b^T N(O-) + E_1 \geq b^T N^S(O-) + E_1$

or $b^T N(O-) \geq b^T (I-A)^{-1} Ra$ because recruitments are independent

Similarly we require that in the second period after $t = 0$

$$EC_2 = b^T AN(O-) + E_2 \geq b^T AN_s(O-) + E_2$$

or $b^T AN(O-) \geq b^T A(I-A)^{-1} Ra$

The year-classes in $N(O-)$ will only affect the yields over the next $(T-1)$ years or $n_p = (\text{integer above } \{(T-1)/p\})$ periods. (Note: $A^{n_p} = 0$).

Proceeding as above we obtain the conditions

$$b^T A^i N(O-) \geq b^T A^i (I-A)^{-1} Ra \quad i=0,1,2,\dots,(n_p-1) \quad (4.6)$$

As i increases $b^T A^i$ has fewer non-zero elements, so the conditions applies to fewer elements of $N(O-)$, namely, the younger age groups. (3.14) defines a closed non-empty set ($N^S(O-)$ belongs to it) within which we are assured of further yields at least as good as "sustainable" yields with our chosen control policy.

Note that we assume that the average yield over each period of P years is above a certain minimum. The smaller P is, the larger the number of constraint equations on $N(O-)$. This corresponds to controlling the variance of future yields more strictly. For instance, assuring a certain average yield over ten years is easier than assuring the same average yield over each of two five year periods. So we need also to decide how regular we require future yields to be.

A computable constraint equation of the form (4.6) allows us for the first time to formulate strategies for the recovery of a fishery in a control theoretic framework. We first decide on the long term policy we wish to apply (i.e. P and $F(t)$, $t \in [0, P]$). Suppose we decide that Y years is the time within which we

wish the fishery to recover if possible. Then, knowing the current age structure N_c , we can construct the constraint set (4.6) and solve the system described by (3.1) and (3.2) with:

$$\begin{aligned} N(0) &= N_c \\ \text{and } b_1^T N(Y-) &= a_i \quad i = 0, 1, 2, \dots, (n_{p-1}) \end{aligned} \quad (4.7)$$

A Numerical Example

Below are the constraint equations corresponding to being able to preserve a four-yearly average using the maximum-sustainable-yield long term policy for the North Sea ICNAF 3N-0 cod stock.

$$F_{\text{max.sust.yield}} = .285, \quad T = 12$$

If $N(0-) \in R^{12}$ is the age structure we require:

$$\begin{bmatrix} .70 & 1.04 & 1.61 & 2.16 & 2.76 & 3.41 & 4.16 & 4.98 & 5.01 & 4.47 & 2.99 & 0 \\ .53 & .53 & .60 & .72 & .72 & .64 & .43 & 0 & 0 & 0 & 0 & 0 \\ .14 & .10 & .06 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} N(0-) \geq \begin{bmatrix} 3.83 \\ 1.39 \\ .198 \end{bmatrix} \quad (4.8)$$

The following are examples of age structures that satisfy the condition (4.8).

- i) The stable age structure $N^S(0-)$ obviously satisfies (4.8) with equality:
(.819, .624, .413, .254, .156, .0967, .0595, .0366, .0221, .0137, .0087, .0053)
- ii) $N^N(0)$, the age structure for constant recruitments $R=1$ and no fishing, provides upper bounds on the components for constant recruitments.
(.819, .670, .549, .449, .368, .301, .247, .202, .165, .135, .111, .091).
- iii) In fact, a little less than the first five terms of ii) satisfy (4.8)
(.819, .670, .549, .449, .214, 0, 0, 0, 0, 0, 0, 0)
This implies that if the recruitments over the last five years were all $R = 1$ and we do not interfere with them, the fishery can be said to have recovered, even if there are very few fish of any other age group there.
- iv) (5.44, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0).
This shows that one recruitment of $5.44/.819 = 6.72$ and no fishing in the subsequent year will also recover the fishery.
- v) (2.73, 1.83, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0).
Similar to iv), but requires two recruitments of 3.34 each and two years of no fishing.

These arguments ignore stock-recruitment aspects which could well be affected in such a consideration. However, we have a simple method of determining how to recover a fishery as we learn what the current age structure is. (e.g. We do not fish until the current age structure satisfies 4.8).

The terminal time constraint also allows us to describe an elementary feedback policy in the sense that if we know the current age structure, we can maximize the expected yield over the next t years and still make sure that the expected age structure $N(t-)$ is such that the fishery age-structure is still viable.

CHAPTER 5

COMMENTS

Perhaps the most striking observation throughout this study is that increases in fishing intensity above the level of maximum sustainable yield suffer greatly from diminishing returns, and in fact the best increases over these were less than 10%. Increasing costs reduce the level of intensity even further, provided meshes are correctly selected.

Although the Maximum Principle implies that optimal policies will be "bang-bang", these interact much more strongly with the fishery if they are slightly misused. It may be wiser to sacrifice the slight increase in yield possible by such methods for greater long term stability by using more conservative, low intensity, sustainable yield policies. However, higher costs should yield more benefits from pulse fishing, and the analysis of Chapter 3 could be carried out for a more realistic cost function.

Perhaps the next important improvement to the economic model described in Chapter 2 should be a removal of the assumptions of constant price of fish and constant cost per unit fishing effort. These two variables are closely linked to the discount rate and costs are certainly not linear in fishing effort. A more realistic financial description would probably be solvable by methods such as those described above, but with greater significance.

CHAPTER 6

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