A NON COOPERATIVE GAME IN A DISTRIBUTED PARAMETER SYSTEM

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ABSTRACT :

This paper is devoted to study a class of differential games for distributed parameter systems. Essentially we study a Nash equilibrium point for a system governed by a parabolic equation. A method based on the SCARF-HANSEN [1] algorithm for solution of non-cooperative games is given.

I - INTRODUCTION

The general formulation of non-cooperative games is the following .

$$(1.1) \quad \left\{ \begin{array}{l} \text{Let } u_i \quad i=1,2,\dots,n \text{ be a family of Banach spaces} \\ \text{and } J_i \quad i=1,2,\dots,n \text{ a family of functionals, each of} \\ \text{them being defined on } u = \prod_{i=1}^n u_i. \text{ Then we have the following :} \end{array} \right.$$

Definition

Let u_{ad} be a subset of u , a point $u \in u_{ad}$ is a Nash equilibrium point if it satisfies :

$$(1.2) \quad \left\{ \begin{array}{l} J_i(u_1, \dots, u_{i-1}, u_i, u_{i+1}, \dots, u_n) \leq J_i(u_1, \dots, u_{i-1}, v_i, u_{i+1}, \dots, u_n) \\ \text{for all } v_i \text{ such that} \\ (u_1, \dots, v_i, \dots, u_n) \in u_{ad} \end{array} \right.$$

This definition follows the paper of Nash [1]. The main result is then,

Theorem 1.1 ROSEN [1]

We assume that

$$(1.3) \quad u_{ad} \text{ is a convex compact subset of } u$$

$$(1.4) \quad \left\{ \begin{array}{l} \text{the mapping } v \rightarrow J_i(v) \text{ is continuous on } u \text{ and the mapping} \\ v_i \rightarrow J_i(v_1, \dots, v_i, \dots, v_n) \\ \text{is convex and continuous on } u, \text{ for } i=1,2,\dots,n. \end{array} \right.$$

Then there exists at least an equilibrium point on u_{ad} .

In order to be able to treat the infinite dimensionnal case we have the following extension of the result of ROSEN.

Theorem 1.2

(1.5) If u_i is a reflexive Banach space for $i=1,2,\dots,n$ and u_{ad} is a closed bounded convex subset of u then Theorem 1.1 still holds with assumption (1.4) with continuity in the weak topology of u . ■

Remark 1.1

Uniqueness of the solution requires more specific (and unfortunately restrictive) hypothesis. The following assumption is an extension of monotonicity property :

If J_i is Gateaux-differentiable with respect to its i^{th} component and verifies the following property :

$$(1.6) \left\{ \begin{array}{l} \text{There exists a } \lambda_1, \dots, \lambda_n \quad \lambda_i \geq 0 \quad \text{such that} \\ \sum_{i=1}^n \lambda_i \left(\frac{\partial J_i}{\partial u_i}(u) - \frac{\partial J_i}{\partial u_i}(v), u_i - v_i \right)_i \geq 0 \end{array} \right.$$

where $(,)_i$ denote the duality pairing between V_i and V_i' , then there exists a unique equil.point. ■

II - A CLASS OF DIFFERENTIAL GAMES

Now we can state our game in the frame of optimal control theory for linear parabolic systems (cf. LIONS [1]).

Let V and H be two Hilbert spaces with V dense in H with continuous injection so that we have

$$V \subset H = H' \subset V'.$$

In addition we consider a family of operator

$$A(t) \in \mathcal{L}(V; V') \quad t \in [0, T]$$

and

$$\{B_i \in \mathcal{L}(u_i; L^2(0, T; V'))\}_{i=1,2,\dots,n}.$$

If f is given in $L^2(0, T; V')$ et $y_0 \in H$ we may consider the parabolic system

$$(2.1) \quad \left\{ \begin{array}{l} \frac{dy}{dt} + A(t)y = f + \sum_{i=1}^n B_i v_i \quad \text{sur }]0, T[\\ y(0) = y_0. \end{array} \right.$$

The equation admits a unique solution $y(v)$ depending continuously on v under assumption of "measurability" of $A(t)$ and coercivity of $A(t)$:

$$(A(t)\varphi, \varphi)_{V, V'} + \lambda \|\varphi\|_H^2 \geq \alpha \|\varphi\|_V^2 \quad \forall \varphi \in V.$$

Observation is given by a family of criteria

$$(2.2) \quad \left\{ \begin{array}{l} J_i(v) = \phi_i(y(v)) + \theta_i(v) \quad \text{where } \phi_i \text{ is convex continuous in } y \text{ and} \\ \theta_i \text{ is convex continuous with respect to } v_i, \text{ for } i=1,2,\dots,n \\ \text{continuous with aspect to } v. \end{array} \right.$$

If we assume that u_{ad} is convex compact we can apply directly Theorem 1.1. If u_{ad} is compact only for the weak topology of u (eg. u_{ad} is a bounded closed convex subset of u) then we need a further assumption :

$$(2.3) \quad \left\{ \begin{array}{l} v \rightarrow \phi(y(v)) + \theta_i(v) \\ \text{is continuous for the weak topology of } u ; \end{array} \right.$$

An example of this situation is the following :

The linear-quadratic problem with distributed observation

We know (cf. LIONS [1]) that the solution $y(v)$ of (2.1) belongs to $W(0,T)$ where

$$(2.4) \quad W(0,T) = \{ \varphi \in L^2(0,T;V) \mid \frac{d\varphi}{dt} \in L^2(0,T;V') \}$$

which is an Hilbert space with the norm

$$(2.5) \quad \|\varphi\|_W^2 = \int_0^T \{ \|\varphi(t)\|_V^2 + \|\frac{d\varphi}{dt}(t)\|_{V'}^2 \} dt .$$

$$(2.6) \quad \left\{ \begin{array}{l} \text{Furthermore if the injection of } V \text{ into } H \text{ is compact then "the injection of} \\ W \text{ in } H = L^2(0,T;H) \text{ is compact".} \end{array} \right.$$

Hence if we consider the family of functional

$$(2.7) \quad J_i(v) = \int_0^T \|y(v) - z_{d,i}\|_H^2 dt$$

The assumption (1.6) is verified ; then we have :

Theorem 2.1

The linear-quadratic game given by (2.1) and (2.2) admits at least an equilibrium point in the bounded closed convex subset u_{ad} of u , under assumption (2.6).

Remark 2.1

Another approach for existence of a Nash point is, in the case of uncoupled constraints

$$(2.8) \quad u_{ad} = \pi_{i=1}^n u_{ad,i} \quad u_{ad,i} \subset u_i$$

is the theory of variational inequalities (cf. BENSOUSSAN [1], BENSOUSSAN-LIONS-TEMAM [1]) and the compactness argument of Theorem 2.1 corresponds to a pseudo-monotonicity property (in terminology of LIONS [2]) of the operator $\mathcal{Q}$ defined by

$$(2.9) \quad (\mathcal{Q}(u), v) = \sum_i \left(\frac{\partial J_i}{\partial v_i}(u_i), v_i \right). \quad \blacksquare$$

III - AN ALGORITHM FOR SOLUTION OF THE GAME

In this section we give an example of a feasible algorithm (for solution of the game) based upon a method of SCARF [1].

If we consider the function

$$(3.1) \quad Z(u, v) = \sum_{i=1}^n J_i(u_1, \dots, u_{i-1}, v_i, u_{i+1}, \dots, u_n)$$

it is easy to show that the equilibrium point ⁽¹⁾ is a fixed point for the mapping

$$(3.2) \quad u \rightarrow \Gamma(u) = \{v \in U_{ad} \mid Z(u, v) \leq Z(u, w) \quad \forall w \in U_{ad}\}.$$

Let us note that the mapping (3.2) is a point-to-point mapping in the case of strict convexity of J_i for $i=1, 2, \dots, n$.

It follows from this remark that the search of a Nash point for the game is equivalent to the search of a fixed point for (3.2).

The original algorithm of SCARF consists in finding a fixed point for a continuous mapping Γ of a simplex of R^N in itself. Let S be the simplex of R^N :

$$S = \{v \in R^N \mid v_i \geq 0 \sum_{i=1}^N v_i = 1\}.$$

and let us consider a partition T_k of S in sub-simplices $S_1, \dots, S_k$. The principle of the method is to give a systematic way for walking over the vertices of the triangulation up to a vertex u "which is not far from its image $\Gamma(u)$ ". More precisely the result is

Theorem 3.1 (SCARF [1]).

Let $\epsilon > 0$ be given. Then there exists a partition T_k of S (in sub-simplices) such that the final vertex u of the algorithm satisfies.

$$\|\Gamma(u) - u\|_{R^N} \leq \epsilon. \quad \blacksquare$$

Remark 3.1

There is an other version of this algorithm which finds a fixed point for a continuous mapping of a convex polyedron in itself but the principle of the method is the same. $\blacksquare$

Remark 3.2

There are many others algorithm to solve a fixed point problem (to say nothing of contraction mappings !), but there are all based upon the SPERNER's lemma. Hence they possess a combinatorial aspect very sensitive to the dimensionality cf. EAVES [1], KUHN [1]. $\blacksquare$

An implementable version of the SCARF's algorithm is due to SCARF-HANSEN [2]. $\blacksquare$

IV - AN EXAMPLE

Let Ω be an open subset of R^p , Γ its boundary, $T > 0$ given and

$$Q = \Omega \times]0, T[, \quad \Sigma = \Gamma \times]0, T[.$$

We consider the following equation

(1) In the case of uncoupled constraints (2.8).

$$(4.1) \quad \begin{cases} \frac{dy}{dt} - \sum_{i=1}^p \frac{\partial}{\partial x_i} (a_i(x) \frac{\partial y}{\partial x_i}) = f + \sum_{j=1}^n v_j(t) \delta(x-c_j) \\ y|_{\Sigma} = 0 \\ y(x,0) = y_0(x) \end{cases}$$

where c_j are given points in Ω .

If we denote by $y(v)$ the unique solution of (4.1) with $v=(v_1, \dots, v_n)$ then we introduce the family of functionals :

$$(4.2) \quad J_i(v) = \int_0^T \int_{\Omega_i} |y(x,t;v) - z_{d_i}(x,t)|^2 dx dt + v_i \int_0^T |v_i(t)|^2 dt \quad (v_i \geq 0)$$

Ω_i being a subset of Ω with $\text{meas.}(\Omega_i) \neq 0$ for $i=1,2,\dots,n$. Then if we introduce

$$u_{ad,i} = \{v_i(t) | \alpha_i \leq v_i(t) \leq \beta_i\}.$$

We can define our game by :

$$(4.3) \quad \text{"To find a Nash point for the family of functionals (4.2)".} \quad \blacksquare$$

Remark 4.1

In spite of the (relative) simplicity of this problem, we cannot apply directly the theory of section 2 because of the Dirac's measures in (4.1). But this difficulty can be skipped easily by using an approximation of the δ -function.

Remark 4.2

This game is an idealization of an economic problem arising in allocation of water resources between regions.

The first equation of (4.1) represents the pressure of water in a phreatic. This phreatic supplies regions, each of them represented by its withdrawal :

$$v_j(t) \delta(x-c_j)$$

where c_j is the location of the well number j and $v_j(t)$ is the flow-rate of water.

The cost function given by (4.2) has no direct interpretation. Actually it is necessary to modelize the economical structure of the region to exhibit the dependance of the profit with respect to the supply of water.

For a general approach of this problem cf. BREDEHOEFT-YOUNG [1].

Remark 4.3

The SCARF's algorithm is not directly applicable to problem (4.3) but it is possible to use a method analogous to the relaxation method combined with SCARF's algorithm cf. YVON [1]. $\blacksquare$

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