ON CONSTRUCTING INVARIANT SETS IN LINEAR DIFFERENTIAL GAMES

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1. Introduction

In the theory of differential games considerable success was achieved for the cases of linear games of pursuit and evasion.

In this contribution we investigate another game problem: how to find /extremal/ invariant sets, or in other words, stable sets of preference. Before giving exact definitions let us consider two simple examples of the type of problems we are going to study. Suppose there are given two controllable objects, the crocodile, x , and the boy, y ,

$$\ddot{x} = u, \quad \|u\| \leq \zeta, \quad \dot{y} = v, \quad \|v\| \leq \sigma, \quad x, y \in \mathbb{R}^k \quad (1)$$

Do numbers $\mathcal{H}_1, \mathcal{H}_2$ exist such that the conflicting players can guarantee, each, by some behaviour, which do not anticipates future, $\mathcal{H}_2 \geq \|x(t) - y(t)\| \geq \mathcal{H}_1$, at least for all large enough t , where $\mathcal{H}_1$ is minimized by the crocodile, $\mathcal{H}_2$ maximized by the boy?

Here we propose some methods which might be useful in dealing with similar problems in linear games. The main idea is to periodize the control process: to return, after time intervals of length t_0 , to a suitable chosen /parametrized/ set K_0 . The chief tools of our construction are the tools of the theory of linear differential games of pursuit, introduced by L.S.Pontrjagin [1]. The formulation of our problem as well as our results are in close connection with and were inspired by the evasion problem in the treatment of L.S.Pontrjagin, [2].

2. Definition of the problem

A linear differential game is a control process, idealized by the equation

$$\dot{z} = Cz + u - v, \quad z \in R^n, \quad u \in P, \quad v \in Q, \quad z(0) = z_0, \quad (2)$$

where C is a quadratic constant matrix, z the phase vector, the control parameters u and v belong to compact, convex sets P and Q , are chosen respectively by the first and second player.

The main feature of the control problem in a game consist in that the first player /from the part of whom we shall investigate the control process/ does not know at the moments $t > 0$ the values $v(s)$, $s > t$, that is the control of his opponent in the future. He shall construct his control $u(t)$ by what we call a behaviour, using only the knowledge of the matrix C , sets P and Q and the values $z(s)$, /perhaps $v(s)$ /, for $s \leq t$. For each measurable function $v(t)$ the constructed $u(t)$ should also be measurable. Following the idea of L.S. Pontrjagin we use behaviours which are build up in discrete steps in time. We consider cases /Theorem 1/, in which it is sufficient to know only the values $z(t_0, n)$, $n=1, 2, \dots$, in the periods $[n t_0, (n+1) t_0]$.

Let us mention briefly that to search for optimal behaviour /without "memory"/ of the type $u(t) = U(z(t))$ with a continous /possibly multi valued/ function $U(z)$, would lead, even in the simple game (1), not only to a strongly nonlinear /and discontinuous!/ partial differential equation of first order, of Bellman-Isaacs type, but would not give a better result than the open loop control $u(t) = \bar{u}(t)$, see [3].

The methods we propose make use from linearity more fully.

Definition: A closed set $K \in R^n$ is said to be invariant /from the point of view of the first player/, iff for each $z_0 \in K$ there is an appropriate behaviour for the first player which guarantees $z(t) \in K$, for all $t > 0$.

A set K is said to be strongly stable, if for each $\varepsilon > 0$ there exist $\delta(\varepsilon) > 0$, $\delta(\varepsilon) \rightarrow 0$ for $\varepsilon \rightarrow 0$, such that for each z_0 $d(z_0, K) \leq \delta(\varepsilon)$, the first player can guarantee $z(\varepsilon) \in K$; here $d(\cdot, K)$ is the distance function from the set K .

Let us remark, that in a pursuit game the optimal pursuit time $T(z_0)$ is continous iff the terminal manifold is strongly stable.

The optimisation problem we deal with may be formulated as the problem of finding that invariant set K , for which a functional, $f_0(K)$, defined on sets in R^n assumes its minimal value.

As in the evasion problem, the above problem relates to the whole interval $[0, \infty)$, so might be called stationary. Let M be a linear subspace, called terminal manifold, π the operator of orthogonal

projection to the orthogonal complement of M , L , and take, with $\|\pi z\| = d(z, M)$, $\|z\|^2 = \langle z, z \rangle$,

$$f_0(K) = -\min \{ d(z, M) \mid z \in K \}. \quad (3)$$

So we have another formulation of the evasion problem.

In many engineering, economic and biological applications we have the following problem of "stabilisation": hold $\|\pi z(t)\|$ as small as possible:

$$f_0(K) = \max \{ d(z, M) \mid z \in K \}. \quad (4)$$

As an example let us consider the following game: $x, y \in \mathbb{R}^k$

$$x^{(n)} + a_{n-1}x^{(n-1)} + \dots + a_1\dot{x} = \varphi u, \quad y^{(m)} + b_{m-1}y^{(m-1)} + \dots + b_1\dot{y} = \sigma v, \quad (5)$$

$$\|u\| \leq 1, \quad \|v\| \leq 1, \quad M = \{z \mid z' = x - y = 0\}, \quad \varphi, \sigma > 0$$

where for $d_i = a_i$ or b_i the polynomials $\sum d_i p^{i-1}$ are stable; i.e. their roots have negative real parts, therefore $d_i > 0$. A further interesting example, for which we shall investigate the problems (3), (4), is

$$\dot{x}^1 = \lambda x^2 + u, \quad \dot{x}^2 = -\lambda x^1 + v, \quad |u| \leq \varphi, \quad |v| \leq \sigma, \quad z \in \mathbb{R}^2, \quad M = 0 \quad (6)$$

$$\lambda, \varphi, \sigma > 0.$$

3. The idea of periodisation. Sufficient conditions of existence of compact invariant sets

Here we are interested to give some simple sufficient conditions for the existence of an invariant set for which (4) is finite and /possibly/ small. The idea of periodisation in a simple form may be expressed so: find a set K_0 and a positive number t_0 such, that for each $z_0 \in K_0$ there is a behaviour of the first player /from a given class of behaviours/, which guarantees $z(t_0) \in K_0$. Then the set of all points $z(s)$, $0 \leq s \leq t_0$ which are realized when the second player selects arbitrary controls $v(\tau)$, is an invariant set $K(t_0, K_0)$.

In the paper [1] L.S. Pontrjagin has given a procedure to compute the set $W(\mathcal{V}, M)$ of all points $z(0)$ for which $z(\mathcal{V}) \in M$ can be guaranteed for a fixed time, $\mathcal{V}$, and closed, not necessarily convex, set M , by introducing the operations (+) and (x) for convex sets and notion of alternating integral. For brevity we do not repeat here this defi-

nitions, only $A \preceq B = \{w | A \preceq B + w, w \in \mathbb{R}^n\}$.

Taking $M=K_0$, $v=t_0$, the condition of periodizability is

$$-e^{t_0 C} K_0 \subseteq \int_{-K_0, 0}^{t_0} (e^{\tau C} P \pm e^{\tau C} Q) d\tau = -e^{t_0 C} W(t_0, K_0) \quad (7)$$

and $K(t_0, K_0) = \bigcup_{t \in [0, t_0]} W(t, K_0)$.

The computation of the alternating integral is relatively simple if we have, for all $0 \leq t \leq t_0$, exact "sweeping" /M.S.Nikolskii, B.N. Psenitsmii/, see [4]. For this reason we shall search for the solutions (K_0, t_0) of the form

$$K_0 = - \int_0^{t_0} e^{\tau C} Q d\tau - K^0(t_0). \quad (8)$$

This leads to the condition /for simplicity we write $K^0 = K^0(t_0)$

$$e^{t_0 C} (K^0 + Q(t_0)) \subseteq K^0 + P(t_0) \quad (9)$$

where $Q(t_0) = \int_0^{t_0} e^{\tau C} Q d\tau$, $P(t_0) = \int_0^{t_0} e^{\tau C} P d\tau$.

Theorem 1: Let us suppose that there exist a positive number t_0 , for which the inclusion (9) has a /compact/ solution K^0 , then there exist a /compact/ invariant set $K = K(t_0, K_0)$. To construct the corresponding behaviour of the first player it is enough for him to know /remember/ only the values $z(nt_0)$, $n=1, 2, \dots$, on the time intervals $[nt_0, (n+1)t_0]$.

The proof of this theorem uses only the well known integral representation of the solutions of (2) and the basic properties of the operation (+). /Using the theory of the alternating integral one can show that the same periodisation can be attained by another behaviour which uses only arbitrarily small "memory" on $z(t_1)$, without knowing or computing, the values of $v(s)$./ This is proved in showing that, then, in the construction of the alternating integral the order of the operations ($\pm$) and + can be reversed. A third way to prove the first part of the Theorem 1 is to introduce the variables x, y , $x=Cx+u, y=Cy+v$ $x(0)=z$, $y(0)=0$ and to show that if $[x(nt_0) - e^{t_0 C} \cdot y((n-1)t_0)] \in -K^0$, then it is possible to guarantee: $[x((n+1)t_0) - e^{t_0 C} y(nt_0)] \in -K^0$.

The set K^0 in (9) /or K_0 in (7)/ might be interpreted as a "generalized" fixed-point of a /not everywhere defined/ mapping in the space, Ω , of convex, compact sets: $K^0 \subseteq e^{-t_0 C} [K_0 + P(t_0) \pm Q(t_0)]$.

Remark: (9) is fulfilled with $K^0=0$, if for some t_0 :

$$e^{t_0 C} Q(t_0) \subseteq P(t_0), \quad \text{especially if } e^{t_0 C} Q \subseteq P. \quad (10)$$

In the case $\pi \neq E$: identity, the choice $K^0 = (E - \pi) Q(2t_0)$ is natural in some cases. Then (9) is satisfied if

$$e^{t_0 C} \pi Q(t_0) + \pi e^{t_0 C} (E - \pi) e^{t_0 C} Q(t_0) \subseteq P(t_0) \quad (11)$$

and

$$(E - \pi) e^{t_0 C} (E - \pi) e^{t_0 C} Q(t_0) \subseteq (E - \pi) Q(2t_0). \quad (12)$$

We used $Q(2t_0) = Q(t_0) + e^{t_0 C} Q(t_0)$, and the inclusion $A \subseteq (E - \pi)A + \pi A$, for an arbitrary set A .

The equation (12) is satisfied for all t_0 if the mapping $(E - \pi) e^{t_0 C}$ is a contraction on the subspace $(E - \pi)R^n = M$, see example (5).

Technically, the following lemma is often useful in proving (9) or (11).

Lemma 1: Let $A(s) = \left\{ \begin{pmatrix} a_1(s)u \\ a_2(s)u \end{pmatrix} \mid \|u\| \leq 1 \right\} \in \mathcal{R}$, $B(s) = \left\{ \begin{pmatrix} b(s)u \\ 0 \end{pmatrix} \mid \|u\| \leq 1 \right\} \in \mathcal{R}$

$$\text{then } \int_{t_1}^{t_2} A(s) ds \supseteq \int_{t_1}^{t_2} B(s) ds \quad \text{holds if the equations}$$

$$\int_{t_1}^{t_2} a_1(s) ds - \int_{t_1}^{t_2} a_1(s) ds \geq \int_{t_1}^{t_2} b(s) ds, \quad \int_{t_1}^{t_2} a_2(s) - \int_{t_1}^{t_2} a_2(s) = 0$$

have a solution for some $t_3 \in (t_1, t_2)$.

Applications. The condition (10) is fulfilled in the example (6) (4), if $\rho > \sigma$ and $t_0 = \pi \rho^{-1}$, $\pi = 3.14 \dots$. In the game (1), (4) it will be fulfilled with $t_0 = 4\sigma\rho^{-1}$. In the example (5) the inclusion (9) will be satisfied if we chose $K^0 = (E - \pi) Q(2t_0)$, where E : identity, if $\rho a_1^{-1} > \sigma b_1^{-1}$ and t_0 is large enough, see [5]. Another reasonable choice is: $K_0 = \{z \mid x=y, \dot{x}=\dot{y}\}$ which however, requires a stronger superiority, see [4].

A sufficient condition for the existence of a set $K^0 \in R^n$, satisfying (9), which is more special, is established in the following theorem.

Theorem 2: Let us suppose that

$$\tilde{K} = [\tilde{K}^0 + \int_0^\infty e^{-\tau C} P d\tau] \neq \int_0^\infty e^{-\tau C} Q d\tau \left\{ \tilde{K}^0 + \sum_{n=1}^\infty e^{-nt_0 C} P(t_0) \right\} \neq \sum_{n=1}^\infty e^{-nt_0 C} e^{t_0 C} Q(t_0) \neq \emptyset, \quad (13)$$

$e^{t_0 C} \tilde{K}^0 \subseteq \tilde{K}^0$, $\tilde{K}^0$ need not to be finite/, e.g. $e^{t_0 C} M = M = \tilde{K}^0$ then $K^0 = \tilde{K}$ satisfies (9), so it is possible to periodize the process with the set:

$$K_0^* = - \left[\left[\tilde{K}^0 + \int_0^\infty e^{-\tau C} P d\tau \right] \neq \int_0^\infty e^{-\tau C} Q d\tau \right].$$

This condition is necessary for the existence of a nontrivial set K^0 , for which (9) is satisfied, if $\int_0^T e^{\tau C} P d\tau$ is finite, which is true if the matrix C has roots only which positive real parts. If, on the contrary, $P(\infty)$, see (9), is finite, e.g. C is stable, then a necessary and sufficient condition for the existence of an invariant set, or for the existence of a solution of (9) is equally: $P(\infty) \neq Q(\infty) \neq \emptyset$.

Taking in (13) not difference of sums, but the "alternating" sum with an initial set $\bar{K}^0$, for example $\bar{K}^0=0$, and going to the limit $t_0 \rightarrow 0$, we get that if the alternating integral:

$$\bar{K}^0 \leq K^* = -\lim_{T \rightarrow \infty} e^{-TC} \bar{K}^0 + \int_0^T (e^{\tau C} P - e^{\tau C} Q) d\tau \quad (14)$$

exists in the limit, then K^* is an invariant set. If C has roots only with positive real parts, then taking $\bar{K}^0=0$, we get in (14) an invariant set which is extremal in the sense, that is it is not contained in any other invariant set, if the sets K^* and K_0^* are equal.

In order to diminish $f_0(K(t_0, K_0))$, the behavioral prescription for the first player might be changed to: remain in K_0 as long as possible, return to K_0 as soon as possible /this is because the smallest, fixible time of return is not necessarily t for all boundary points of K_0 /. If we have other, for example integral type, restrictions on the functions $u(\tau)$, $v(\tau)$: $\int_0^{t_0} u(\tau) d\tau \leq p_0$, $\int_0^{t_0} v^2(\tau) d\tau \leq q_0$, then only the sets $P(t_0)$, $Q(t_0)$ should be changed.

The set of all solutions K^0 , in $\mathfrak{R}$, of the inclusion (9), or (7), is convex. Therefore the set of all so constructed invariant sets $K=K(t_0, K_0)$, is also convex, and closed /in the topology, generated by Hausdorff metric $d(K_1, K_2)$ yet an individual $K(t_0, K_0)$ is, generally, not convex!

The functional $f_0(K)$ is convex, continuous. Therefore our optimisation problem fits in the framework of convex optimisation theory. Let us remind that $\mathfrak{R}$ can be embedded in a linear space. $f_0(\cdot)$ is Gateaux-differentiable. For approximation the Galerkin method might be used treating the problem as one, leading to a variational inequality, for the monotone operator $f'_0(K)$.

To represent the elements of $\mathfrak{R}$, a very useful device is introducing support functions of the sets K^0 , $Q(t_0)$, $P(t_0)$, ...,

$$m(K, \varphi) = \max\{\langle \varphi, z \rangle \mid z \in K\}, \|\varphi\| = 1 \quad \text{then} \quad f_0(K) = \max\{m(\varphi, K) \mid \varphi \in L\}.$$

(9) means, that: $m(\varphi, K^0) - \mu m(\varphi e^{t_0 C}, K^0) \geq m(\varphi, e^{t_0 C} Q(t_0)) - m(\varphi, P(t_0))$, for all φ , $\|\varphi\| = 1$, $\mu = \|e^{t_0 C}\|^{-1}$.

4. Application of a special representation of convex sets

In analyzing the inclusions (7), (9) another representation of the sets $S=K^0$, $P(t_0)$, $Q(t_0)$, ..., in the form

$$\left\{ S = \int_0^{t_0} s(\tau) e^{\tau C} W d\tau, K^0 = \int_0^{t_0} k^0(\tau) e^{\tau C} W d\tau, \dots \right\} = \mathcal{R}(C, W), \quad (15)$$

is useful, where W is a fixed compact, convex set, s a natural number, $s(\tau)$, $(k^0(\tau), \dots)$, a measurable and integrable scalar function /or more generally taken from a suitable class of "generalized" functions, distributions, containing the $\delta(\tau - \tau_i)$ functions, which we do not specify here/. The reason to introduce such representations lies in the simplicity of operations expressing, in terms of the scalar functions $s(\tau)$, the operations $S \rightarrow e^{\tau C} S$, $S_1 + S_2$, $\int S(\tau) d\tau$. If we ask for solutions, K^0 , of the inclusion (9), which admit a representation (15), then $k^0(\tau)$ should satisfy the inequality, /supposing the uniqueness of (15)

$$k^0(\tau) + p(\tau) \geq k^0(\tau - t_0) + q(\tau - t_0), \quad \text{for all } \tau > 0, \text{ if } W = -W \quad (16)$$

For an arbitrary initial, positive function $\underline{k}(\tau)$, $0 \leq \tau \leq t_0$ it is possible to compute from (16) the prolongation $K^0(\tau)$, $\underline{k} = k^0$ for $0 \leq \tau \leq t_0$, to the intervals $[t_0, 2t_0]$, $[2t_0, 3t_0]$, ..., and so on, as the smallest positive solution of (11). A sufficient condition for the existence of $k^0(\tau) \geq 0$, with compact support, is given in the following

Lemma 2: Let $r(t) = q(t - t_0) - p(t)$, $t > 0$, ($q(s) \equiv 0$ for $s < 0$), where the functions $p(\cdot)$, $q(\cdot)$ are taken from (15), /see (9)/. If there exist an integrable function $\underline{k}(\tau)$, $0 \leq \tau \leq t$, for which $h(\tau, n) \equiv 0$ for all large enough $n \geq N(\tau)$, where

$$h(\tau, 0) = \underline{k}(\tau), \quad h(\tau, n) = \max \{ h(\tau, n-1) + r(\tau + nt_0), 0 \}, \quad \text{pos } y = \begin{cases} y & \text{if } y > 0 \\ 0 & \text{if } y < 0 \end{cases}$$

then the inequality has a measurable solution $k(\tau) \geq 0$, which has finite support, if $N(\tau) \leq N$ for all $0 \leq \tau \leq t$.

Corollary: if for each τ , $0 \leq \tau \leq t$ the function $\ell(\tau) = \sum_{n=1}^{\infty} r(\tau + nt_0) < \infty$ and is integrable, then there exist a compact set K^0 satisfying (9). The special case $P \ni \exp(t_0 C)$. Q was mentioned before in (10).

In general, the above representation (15) do not exist for all compact convex sets S , neither is he unique, here one can apply the theory of G.Choquet. It will be unique if we require $\max_{\tau} |S(\tau)|$, or $\int S^2 d\tau$ to be minimal /for a fixed interval/. A natural choice,

for which (15) is unique,:

$$W = \{s w_0 \mid w_0 \in \mathbb{R}^n, s \in [-1, 1]\}, \lambda e^{\tau C} w_0 \neq w_0 \text{ for } \lambda > 0, \tau \in [0, t_0] ,$$

then from $S = S_1 + S_2$ and $S \in \mathcal{R}(C, W)$ follows, that $S_1, S_2 \in \mathcal{R}(C, W)$. For greater range of $\mathcal{R}(C, W)$ representable sets, one wishes the control system $\dot{z} = Cz + w_0$ to be controllable that is: $\text{rank}(w_0, Cw_0, C^{n-1}w_0) = n$. In the game (6), then, every centrsymmetrical convex compact set can be represented by (15), for arbitrary $\lambda, W = Q$ and $\bar{s}t_0 = \pi\lambda^{-1}$. In dimensions greater than 2 we get a special class of representable sets, which might be useful in other control problems too.

5. Construction of invariant sets in the evasion problem

First let us remark that it is possible to apply the same method, as in theorem 1, for treating the "evasion" problem (2), (3). However then one should be sure that the transient states $z(s)$, $0 \leq s \leq t$, do not belong to M , and one wishes that $z(t) \in K_0$ can be guaranteed for all $z_0 \in \mathbb{R}^n$ for a finite time $T = T(z_0)$.

Theorem 3: Suppose that there exist a positive number t_0 such that $\text{int}[\exp(t_0 C)Q(t_0)] \supset P(t_0)$, then with the choice of the periodizing set: $K_0 = \mathbb{R}^n \setminus Q(t_0)$, one can construct an invariant set which has positive distance from the point $z=0$.

Application: in the game (6), (3) if $\sigma > \rho$, taking $t_0 = \pi\lambda^{-1}$.

As another application of the idea of periodisation we are able to get a slight improvement of the construction of the evading strategy given in [2].

Theorem 4: Assume that the conditions of evasion in [2] are fulfilled. Then, there exist positive numbers ξ , θ such that for each initial position z_0 , the evading player can guarantee $\|\pi z(t)\| \geq \xi$ for all $t \geq T(z_0)$, where $T(z_0) \leq \theta$. The numbers depend only on the game, that is on (C, P, Q, M) .

To prove it we need to modify the behaviour, described in [2], briefly saying, in that we define the condition of switching to the evading manoeuvre, at moment $\underline{t}$, being the violation of the condition $z(\underline{t}) \in \text{int } K_0$, where

$$K_0 = \{z \mid \|\pi e^{\tau C} z\| \geq \xi_0, \text{ for all } 0 \leq \tau \leq \theta_0\} = K_0(\xi_0, \theta_0) \quad (17)$$

We use a different evading manoeuvre only for great "drifts" that is, if $\|\pi C e^{\tau C} z(\underline{t})\| \geq \mu_0$, here μ_0, ξ_0, θ_0 are constants depending only

on the game. In the latter case the curve $\pi e^{\tau C} z(t)$ is almost a straight line /near the origine, $\pi z=0$ /, therefore it is easy to Construct, using the "two dimensional superiority" a simple evading manoeuvre which shall guarantee $z(t+\theta_0) \in K(\xi_0, \theta_0)$, see the game (1) for example.

To "connect" the two behaviours we need the

Lemma 3: Suppose we have $z(t) \notin K_0(\xi_0, 0)$ for $t=t_1+k\theta_0$, $k=1,2,\dots,N$, then $\|\pi C z(s)\| \leq \mu_0 = \mu_0(\xi_0)$ for all $t_1 \leq s \leq t_1+N\theta_0$ if θ_0, N^{-1} are small enough, and depending only on the game. In the evading game (1) it is possible to construct the evading manoeuvre, which guarantees $\|x(t)-y(t)\| \geq \varepsilon$ for all $t > \theta_0$ using only the values $z(n\theta_0) \in K_0$, $n=1,2,\dots$, in the intervalls $[n\theta_0, (n+1)\theta_0]$, for some sufficiently small θ_0 .

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