AN INFORMATIONAL GAME PROBLEM

Yu.S. Osipov

Institute of Mathematics and Mechanics, Ural
Scientific Center of the Academy of Sciences
of the USSR

S. Kovalevskaja st. 16, Sverdlovsk, USSR

The purpose of this report is to give account of one of the possible methods of approaching to the control problems with incomplete information and to demonstrate some results which this method gives. For a certain class of the problems this method seems to be perspective in that it permits one not only to formulate the criteria of solvability of the discussed problems but also to describe a construction of the controls solving the problem, namely to formulate a generalization of Krasovskii's extremal aiming rule in the theory of positional differential games with complete information. The essence of the method is a substitution of the control problem with incomplete information to more general one in suitable functional space but with complete information. The pay for this substitution is an infinite dimension of a new state space. However the advantage obtained namely the completeness of information and suitable controlled process description permits overcoming of some serious obstacles. Naturally, such substitution must be invariant in the sense of the initial control problem.

Let us consider the following game problems of convergence.

I. Given is a controlled system

$$(1) \quad \dot{x} = B(t)u - C(t)v + w(t), \quad u \in P, \quad v \in Q, \quad t_0 \leq t \leq \theta.$$

Here x is a state vector; u (v) is controlled by the first (second) player; P and Q are compacts. Coefficients in (1) are integrable in Lebesgue sense. The absence of $A(t)x$ from (1) does not diminish the generality of the system [1]. In the space $\{x\}$ given is a convex compact M . The purpose of the first player is to bring $x(t_1)$ to the set M at the moment $t = \theta$. The purpose of the second player is antagonistic. Choosing a control $v(t_1)$ at the moment t the first player possesses only the

information about the convex compact R_t which contains the realized point $x(t)$ of (1). The opponents' behavior is restricted only by the constraint $v(t) \in Q$ and the cases, where the choice of $v(t)$ is based also on the information concerning the control being realized, are not excluded. Let us describe the character of R_t changing. Assume the diameter of R_t is not greater than $\varphi(t)$ where $\varphi(t)$, $t_0 \leq t \leq \vartheta$ is not an increasing function. Further we will distinguish cases (1°) and (2°): for any t_* , $t^* \geq t_*$, R_{t^*} consists only of the points to which it is possible to bring (1) at the moment t^* starting the motion at the moment t_* from the points of R_{t_*} (1°) under the controls being realized in the game on $[t_*, t^*]$, or (2°) under the realized control $u(t)$ and some open-loop measurable control $v(t)$.

Mathematical formalization of this problem is based on one-to-one correspondence between convex compacts and their supporting functionals and is achieved by initial problem absorption in more general control problem. Let $\mathcal{H}_1$ be Lebesgue space of integrable on the ball $E = \{l \mid \|l\| \leq 1\} \subset \{\infty\}$ with the square $(\|h\|, = (\int_E h^2(l) dl)^{1/2})$ functions; $\mu(l)$, $h_0(l)$ - supporting functionals of M, R_{t_0} ; $\mathcal{L} = \{h \in \mathcal{H}_1 \mid h(l) \leq \mu(l) \text{ for a.a. } l\}$; $d(q) = \max_l (q(l) + q(-l))$; $\mathcal{L}^\varepsilon$ - a closed ε -neighbourhood of $\mathcal{L}$ in $\mathcal{H}_1$. A set of elements $g_t \in \mathcal{H}_1$, $t_* \leq t \leq t^*$, will be called a precision of h_* on $[t_*, t^*]$ if $g_{t_*}(l) \leq h_*(l)$ for a.a. l , $d(g_t) \leq \varphi(t)$ and for any $t_* \leq t_1 \leq t_2 \leq t^*$ $g_{t_2}(l) \leq g_{t_1}(l) + \int_{t_1}^{t_2} [f_\xi(l) + l w(\xi)] d\xi$ for a.a. l .

Here $f_\xi = -l C(\xi) v[\xi]$ - in the case (1°) ($v[\xi]$ being realization of v on $[t_*, t^*]$) and $f_\xi = \min_{v \in Q} l C(\xi) v$, $v \in Q$, in the case (2°). A pair $p = \{t, h\}$, $t \in [t_0, \vartheta]$, $h \in \mathcal{H}_1$, will be called a position. A law which puts in correspondence to every p a set $U(p) \subset P$ will be called a strategy U . Let $p_* = \{t_*, h_*\}$ and Δ - finite separation of $[t_*, \vartheta]$ by the points τ_i ($\tau_0 = t_*$, $\tau_i < \tau_{i+1}$), $\delta(\Delta) = \max_i (\tau_{i+1} - \tau_i)$ be fixed. Then an aggregate of elements $h_t(l)_\Delta \in \mathcal{H}_1$, $t_* \leq t \leq \vartheta$, will be called the motion from p_* generated by U if $\forall t$ for almost all l $h_t(l) = g_t(l) + \int_{t_*}^t B(\xi) u[\xi] d\xi$ where $u(t) = u[\tau_i] \in U(\tau_i, h_{\tau_i}(l)_\Delta)$, $\tau_i \leq t \leq \tau_{i+1}$; g_t , $t_* \leq t \leq \vartheta$, is a certain precision of h_* . An element g_t describes the deformation and the displacement of R_t under the forces unsubordi-

nated to the first player, moreover a way of g_t forming can make use of any possible information about the system and the control being realized by the first player.

Problem I. Given is a system (1) and a position $p_0 = \{t_0, h^0\}$. It is required to find a strategy U with the property: $\forall \varepsilon > 0 \exists \delta_0 > 0 \forall h_t(\Delta) = h_t(l; p_0, U)$ with $\delta(\Delta) \leq \delta_0$ $h_g(l)_\Delta \in \mathcal{L}^\varepsilon$.

Remark I. The strategy U resolving the problem 1 guarantees bringing of every compact R_g realized in the game process into a sufficiently small Euclidean neighbourhood of M (while $\delta(\Delta)$ is small). Therefore the problem I can be considered as mathematical formalization of the initial control problem.

Let us describe the criteria of a solvability of the problem I and the method of the solving strategy construction. Let $\mathcal{H}_1 \supset \supset \mathcal{N}_t \neq \emptyset, t_0 \leq t \leq \vartheta$, and $r(t, h) = \inf_{y \in \mathcal{N}_t} \|h - y\|_1$,

$y \in \mathcal{N}_t$. Let $D(h, \mathcal{N}_t)$ be a union of all possible minimizing $r(t, h)$ sequences $\{y_k\}$ of the aggregates of all weak in $\mathcal{H}_1$ partial limits sequences $\{y_k\}$, where for almost all l $y_k(l) = y_k(l) - h(l)$ if $y_k(l) < h(l)$ and $y_k(l) = 0$ if $y_k(l) \geq h(l)$.

A strategy $U^e = U^e(t, h) = \{u^e \mid sB(t)u^e = \max sB(t)u\}$,

$$u \in P, \quad s = \int_E l g(l) dl, \quad g \in D(h, \mathcal{N}_t)\}$$

will be called an extremal with respect of the sets $\mathcal{N}_t$. A rule G which puts in correspondence to any vector $\{t', t'', h, u(\cdot)\}$, $t'' > t'$, a certain set $G(t', t'', h, u(\cdot)) = \{g_t\}$ of h precision on $[t', t'']$, moreover in the case (1°) , while g_t is being determined, we ought to put $f_\xi = lC(\xi)v$, where $v \in Q$, will be called a counterstrategy of precision. A set of elements $y_t(l)_\Delta \in \mathcal{H}_1$, $t_* \leq t \leq \vartheta$, will be called the motion from p_* generated by G if $\forall t$ for

$$\text{a.a. } l \quad y_t(l)_\Delta = g_t(l) + l \int_{t_*}^t B(\xi) u(\xi) d\xi$$

where $\{g_t, \tau_i \leq t < \tau_{i+1}\} \subset G(\tau_i, \tau_{i+1}, y_{\tau_i}(l)_\Delta, u(\cdot))$, $y_{\tau_0} = h_*$ and $u(t)$ is any open-loop control. Let Γ_{t_*} be a set of all $h_* \in \mathcal{H}_1$ with the property: $\forall G \quad \forall \Delta \quad \exists y_t(l)_\Delta$ such that $y_g(l)_\Delta \in \mathcal{L}^\varepsilon$. Γ_t is analog of the sets of positional absorption [1-3].

Theorem I. The problem I is solvable if and only if $h_0 \in \Gamma_{t_0}$. The solving strategy is a strategy extremal with respect of the sets

$\Gamma_t, t_0 \leq t \leq \vartheta$

As in the theory of differential games [1,3], there are regular cases [4,5] when Γ_t coincides with the sets of program absorption which can be described effectively.

2. Let us consider now a nonlinear case. Given is a system

$$(2) \quad \dot{x} = f(t, x, u), \quad u \in P, \quad t_0 \leq t \leq \vartheta,$$

where f is continuous and Lipschitzian in x . It is required to bring $x(\vartheta)$ into the compact M by choosing u . Initial state of (2) belongs to a certain compact S_{t_0} . While controlling, S_t becomes more precise and the results of the precision $S_t \subset S_{t_0}$ at the moment t and the function $h_t(l)$, $l \in S_t$, (3) are known to the first player. The value of this function is a state vector of (2) initiated from $x(t_0) = l$ under realized control $u(t)$, $t_0 \leq t \leq \vartheta$. A way of S_t forming can be arbitrary.

Obviously, the constructions from the point I can't be used here. Let us describe one of the possible ways to investigate this problem. Let $\mathcal{X}_n$ be Lebesgue space of n -dimensional functions integrable with the square on S_{t_0} , $\|h\|_n = (\int_{S_{t_0}} \|h(l)\|^2 dl)^{1/2}$.

A system of compacts of positive measure $S_t \subset S_{t_0}$, $t_* \leq t \leq t^*$, will be called a precision of S on $[t_*, t^*]$ if $S_{t_*} = S$, $S_{t_2} \subset S_{t_1}$, $t_2 > t_1$ and diameter of S_t is less than $\varphi(t)$.

A triple $p = \{t, S, h\}$ where $S \subset S_{t_0}$ and h is a contraction on S of an element from $\mathcal{X}_n$ will be called a position.

A rule putting in correspondence to any p a set $U(p) \subset P$ will be called a strategy U . A function $x_t(l)_\Delta$, $l \in S_t$, $t_* \leq t \leq \vartheta$, will be called the motion from $p_* = \{t_*, S_*, h_*\}$ generated by U if for any t $x_t(l)_\Delta$ is a contraction on S_t of a certain function from $\mathcal{X}_n$, $x_{t_*}(l) = h_*(l)$ for a.a. $l \in S_{t_*}$ and $\forall t \in (t_*, \vartheta]$ for a.a. $l \in S_{t_*}$, $t \in [t_*, t^*]$ $\dot{x}_t(l) = f(t, x_t(l), u(t))$

where $u(t) = u(\tau_i) \in U(\tau_i, S_{\tau_i}, x_{\tau_i}(l)_\Delta)$, $t \in [\tau_i, \tau_{i+1}) \cap [t_*, t^*]$ and S_t , $t_* \leq t \leq \vartheta$ is a certain precision of S_* . Let $\{m\} = \{m \in \mathcal{X}_n | m(l) \in M \text{ a.a. } l\}$ and $\{m\}^\varepsilon$ be a closed ε -neighbourhood of $\{m\}$ in $\mathcal{X}_n$.

Problem 2. Given is a system (2) and a position $p_0 = \{t_0, S_{t_0}, l\}$. It is required to find a strategy U with the property: $\forall \varepsilon > 0 \exists \delta_0 > 0 \forall x_t(l)_\Delta = x_t(l, p_0, U)_\Delta$, $\delta(\Delta) \leq \delta_0$, $\exists m \in \{m\}^\varepsilon$ such that $x_\vartheta(l) = m(l)$ for a.a. $l \in S_\vartheta$.

All told in remark I remains true.

Let G be a rule putting in correspondence to every position $\{t', S, h\}$ and to a number $t'' > t'$ an aggregate $G(t', t'', S, h)$ of precisions of S on $[t', t'']$. G will be called a stra-

tegy of precision. A function $y_t(l)_\Delta$, $l \in S_t$, $t_* \leq t \leq \vartheta$, will be called the motion from $p_* = \{t_*, S_*, h_*\}$ generated by G if $\forall t$

$y_t(l)_\Delta$ is a contraction on S_t of a certain function from $\mathcal{H}_n$, $y_{t_*}(l) = h_*(l)$ for a.a. $l \in S_{t_*}$ and $\forall t \in (t_*, \vartheta]$ for a.a. $l \in S_{t_*}$, $t \in [t_*, t^*]$ $y_t(l) \in \text{co}\{f(t, y_t(l)_\Delta, u) \mid u \in P\}$ where $\{S_t, \tau_i \leq t < \tau_{i+1}\} \subset G(\tau_i, \tau_{i+1}, S_{\tau_i}, y_{\tau_i}(l)_\Delta)$.

Let $\mathcal{H}_{t_*}$ be an aggregate of $h_* \in \mathcal{H}_n$ with the property:

$$\forall G \quad \forall \Delta \quad \forall S_* \quad \exists y_t(l)_\Delta \quad \exists m \in \{m\}$$

such that $y_t(l)_\Delta = m(l)$ for almost all $l \in S_{t_*}$. Let

$\mathcal{N}_t \subset \mathcal{H}_n$, $t_* \leq t \leq \vartheta$ be non-empty, $r_n(t, q) = \inf \|q - y\|_n$, $y \in \mathcal{N}_t$, and

$D(q, \mathcal{N}_t)$ be an aggregate of all weak in $\mathcal{H}_n$ limits of possible minimizing $r_n(t, q)$ sequences. A strategy $U^e = U^e(t, S, h) = \{u^e \mid \int_{S_t} (z(l) - h(l)) f(t, h(l), u^e) dl = \max_{u \in P} \int_{S_t} (z(l) - h(l)) f(t, h(l), u) dl, z \in D(\tilde{h}, \mathcal{N}_t)\}$ will be called an extremal strategy of the sets $\mathcal{N}_t$. Here $\tilde{h}$ is a continuation of h on S_{t_*} .

Theorem 2. The problem 2 is solvable if and only if $l \in \mathcal{N}_{t_*}$. Under this relation an extremal to the sets $\mathcal{N}_t$, $t_* \leq t \leq \vartheta$, strategy solves the problem.

$\mathcal{N}_t$ is analog of the sets of positional absorption from [1 - 3]. In the case of linear system following to [1, 2] one can distinguish regular cases when $\mathcal{N}_t$'s coincide with the sets of program absorption which have an effective description. The presence of the second player control in (2) does not change the formalization suggested. The changes in the definitions of the motions can be done by analogy with point 1.

References

1. N.N. Krasovskii. Game problems on encounter of motions. "Nauka", Moscow, 1970. (Russian).
2. N.N. Krasovskii. Differential games of convergence-evasion. Izv. AN SSSR. Tehn. kibernet., № 2, 3, 1973. (Russian).
3. Ju.S. Osipov. Differential games with delay. Dokl. AN SSSR, 196, № 4, 1971. (Russian).
4. N.N. Krasovskii, Ju.S. Osipov. Control problem with incomplete information. Izv. AN SSSR. Meh. Tv. Tel., № 4, 1973. (Russian).
5. N.N. Krasovskii, Ju.S. Osipov. On the theory of differential games with incomplete information. Dokl. AN SSSR, 215, № 4, 1974. (Russian).