SOME PROPERTIES OF NONZERO-SUM MULTISTAGE GAMES

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1. INTRODUCTION

This contribution deals with so called multistage games, that is games whose dynamics are governed by vector difference equation. This type of games is studied from the point of view of equilibrium, minimax and noninferior solutions on the open-loop and closed-loop strategy classes.

Previous research in this field was devoted mostly to the case of two-player, zero-sum, multistage games, e.g. see [1] - [4]. Only very few results are known for general, N-player, nonzero-sum, multistage games in comparison with the existing theory of differential games [5] - [7]. General multistage games were extensively studied in the author's thesis [10]. Some results obtained in [10] have been already published, see [8], [9].

Results presented in the following sections are based on [10]. We consider certain type of multistage games in which controls of each player satisfy given state-dependent constraints. For this multistage games we obtain necessary conditions for equilibrium, minimax and noninferior solutions by direct application of the discrete maximum principle proved in [10]. Applying these necessary conditions to the case of linear multistage games with quadratic cost functionals it was possible to obtain explicit form of all considered solution types.

2. FORMULATION OF MULTISTAGE GAME

Consider the dynamical system described by vector difference equation

$$x_{k+1} = f_k(x_k, u_k^1, \dots, u_k^N), \quad k = 0, 1, \dots, K-1, \quad (2.1)$$

where K is a given positive integer, $k = 0, 1, \dots, K$, denotes stage of the system and $f_k: E^n \times E^{m_1} \times \dots \times E^{m_N} \rightarrow E^n$, $k = 0, 1, \dots, K-1$. Let the cost functional, which is minimized by player i , $i = 1, \dots, N$, be given as

$$J_i = g^i(x_K) + \sum_{k=0}^{K-1} h_k^i(x_k, u_k^1, \dots, u_k^N), \quad i = 1, \dots, N. \quad (2.2)$$

Here $g^i: E^n \rightarrow E^1$; $h_k^i: E^n \times E^{m_1} \times \dots \times E^{m_N} \rightarrow E^1$, $i = 1, \dots, N$; $k = 0, 1, \dots, K-1$.

The admissible control vector u_k^i of player i at the stage k , $i = 1, \dots, N$; $k = 0, 1, \dots, K-1$, satisfies constraint

$$u_k^i \in U_k^i(x_k),$$

where

$$U_k^i(x) = \{u^i \mid Q_k^i(x, u^i) = 0, \quad q_k^i(x, u^i) \leq 0\}, \quad i = 1, \dots, N; \quad k = 0, 1, \dots, K-1, \quad (2.3)$$

and $Q_k^i: E^n \times E^{m_i} \rightarrow E^{r_i^i}$; $q_k^i: E^n \times E^{m_i} \rightarrow E^{s_i^i}$, $i = 1, \dots, N$; $k = 0, 1, \dots, K-1$.

The inequality sign for vectors in (2.3) we use in the following sense: Let $a \in E^p$; then $a \leq 0 \Leftrightarrow a_j \leq 0$, $j = 1, \dots, p$.

In this paper we shall consider two types of strategies defined further.

Definition 1. Any sequence

$$\{u^i\} = \{u_0^i, u_1^i, \dots, u_{K-1}^i \mid u_k^i \in U_k^i(x), k=0, 1, \dots, K-1\}$$

we shall denote as an admissible open-loop strategy of the i -th player, $i=1, \dots, N$.

Definition 2. In analogous way, any sequence

$$\{\varphi^i(x)\} = \{\varphi_0^i(x), \varphi_1^i(x), \dots, \varphi_{K-1}^i(x) \mid \varphi_k^i(x) \in U_k^i(x), k=0, 1, \dots, K-1\},$$

where $\varphi_k^i: E^n \rightarrow E^{m_i}$, $k=0, 1, \dots, K-1$, we shall denote as an admissible closed-loop strategy of the i -th player, $i=1, \dots, N$.

To have the problem just stated well-posed, it is also necessary to specify for which solution type necessary conditions should be derived. The following solution types will be considered: equilibrium, minimax and noninferior. For the exact definitions the reader should consult [5].

In general, we make the following assumptions.

Assumption 1. All functions appearing in (2.1) - (2.3) are continuously differentiable in all of their arguments.

Now denote

$$\tilde{h}_k^i(x, u^1, \dots, u^N) = h_k^i(x, u^1, \dots, u^N), \quad i=1, \dots, N; \quad k=0, 1, \dots, K-2;$$

$$\tilde{h}_{K-1}^i(x, u^1, \dots, u^N) = h_{K-1}^i(x, u^1, \dots, u^N) + g^i(f_{K-1}(x, u^1, \dots, u^N)), \quad i=1, \dots, N.$$

In E^{n+1} consider the sets

$$V_k^i(x, u^1, \dots, u^{i-1}, u^{i+1}, \dots, u^N) = \{(a, v) \mid a \in E^1, v \in E^n; a = \tilde{h}_k^i(x, u^1, \dots, u^N), \\ v = f_k(x, u^1, \dots, u^N), \forall u^i \in U_k^i(x)\}, \quad i=1, \dots, N; \quad k=0, 1, \dots, K-1. \quad (2.4)$$

Further define in E^{n+1} the convex cone

$$R = \{r \mid r \in E^{n+1}, r = (\rho, 0, \dots, 0), \rho \leq 0\}.$$

Assumption 2. For each $i=1, \dots, N$, the sets $V_k^i(\cdot)$, $k=0, 1, \dots, K-1$, in (2.4) are R -directional convex for any $x \in E^n$, $u_j^i \in E^{m_j}$, $j=1, \dots, N$, $j \neq i$. For the definition of so called directional convexity see [11].

For an admissible open-loop strategy N -tuple $(\{u^1\}, \dots, \{u^N\})$ and corresponding trajectory $x_0, x_1, \dots, x_K$, define as $I_k^i(x_k, u_k^i)$, $i=1, \dots, N$; $k=0, 1, \dots, K-1$, the set of numbers from the set $\{1, 2, 3, \dots, s_k^i\}$ denoting those components of $g_k^i(x_k, u_k^i)$, for which equality sign holds. The set $I_k^i(x_k, u_k^i)$ is then called the active set for player i , $i=1, \dots, N$, at the stage k , $k=0, 1, \dots, K-1$. In a similar way we also define the active set for a closed-loop strategy N -tuple.

As usual, we introduce the Hamiltonian of player i , $i=1, \dots, N$, at the stage k , $k=0, 1, \dots, K-1$, by formula

$$H_{k+1}^i(x, u^1, \dots, u^N) = -\tilde{h}_{k+1}^i(x, u^1, \dots, u^N) + \lambda_{k+1}^i f_k(x, u^1, \dots, u^N),$$

where row-vectors $\lambda_{k+1}^i \in E^n$ will be defined later.

3. NECESSARY CONDITIONS FOR EQUILIBRIUM SOLUTION

First let us consider equilibrium solution on the class of open-loop strategies. From its definition we can conclude that each player solves discrete optimization problem supposing, that other players use corresponding equilibrium strategies. Then we can immediately apply general results from [10] to this problem. Formulation from [12] can be also used, because it remains valid under the weaker assumption of R -directional convexity of reachable sets.

Theorem 1. Let the multistage game satisfy Assumptions 1 and 2.

Suppose that the admissible open-loop strategy N-tuple $(\{\bar{u}^1\}, \dots, \{\bar{u}^N\})$ is the equilibrium solution of this game on the class of open-loop strategies. Assume that for $i = 1, \dots, N$; $k = 0, 1, \dots, K-1$, the vectors (indices ℓ, m denote components of the vector constraints in (2.3))

$$\frac{\partial}{\partial u^i} Q_{k\ell}^i(x_k^*, \bar{u}_k^i), \quad \ell = 1, \dots, r_k^i; \quad \frac{\partial}{\partial u^i} q_{km}^i(x_k^*, \bar{u}_k^i), \quad m \in I_k^i(x_k^*, \bar{u}_k^i),$$

are linearly independent. Here $x_0^*, x_1^*, \dots, x_K^*$ denotes the equilibrium trajectory.

Then for each $i = 1, \dots, N$, there exist row-vectors

$$\lambda_{k+1}^i \in E^n, \quad \xi_k^i \in E^{r_k^i}, \quad \xi_k^i \in E^{s_k^i}, \quad k = 0, 1, \dots, K-1,$$

such that the conditions 1) - 3) are satisfied.

1)

$$\lambda_k^i = \frac{\partial}{\partial x} H_{k+1}^i(x_k^*, \bar{u}_k^i, \dots, \bar{u}_k^N) + \xi_k^i \frac{\partial}{\partial x} Q_k^i(x_k^*, \bar{u}_k^i) + \xi_k^i \frac{\partial}{\partial x} q_k^i(x_k^*, \bar{u}_k^i), \quad k = 0, 1, \dots, K-1,$$

where we defined $\lambda_0^i = 0$ and $\lambda_K^i = -\frac{\partial}{\partial x} g^i(x_K^*)$.

2)

$$\frac{\partial}{\partial u^i} H_{k+1}^i(x_k^*, \bar{u}_k^i, \dots, \bar{u}_k^N) + \xi_k^i \frac{\partial}{\partial u^i} Q_k^i(x_k^*, \bar{u}_k^i) + \xi_k^i \frac{\partial}{\partial u^i} q_k^i(x_k^*, \bar{u}_k^i) = 0, \quad k = 0, 1, \dots, K-1.$$

3)

$$\xi_k^i \leq 0, \quad \xi_k^i q_k^i(x_k^*, \bar{u}_k^i) = 0, \quad k = 0, 1, \dots, K-1.$$

To formulate analogous conditions for the class of closed-loop strategies, we must take into account function dependence given by Def. 2. Thus we obtain

Theorem 2. Consider again the multistage game as in the previous theorem and let the admissible closed-loop strategy N-tuple

$(\{\varphi^1(x)\}, \dots, \{\varphi^N(x)\})$ be the equilibrium solution on the class of closed-loop strategies. Suppose that all assumptions made in Theorem 1 are satisfied with $\bar{u}_k^i = \varphi_k^i(x_k^*)$, and moreover, let the functions $\varphi_k^i(x)$, $i = 1, \dots, N$; $k = 0, 1, \dots, K-1$, be continuously differentiable in the neighbourhood of the trajectory $x_0^*, x_1^*, \dots, x_K^*$.

Then, for $i = 1, \dots, N$, there exist row-vectors

$$\lambda_{k+1}^i \in E^n, \quad \xi_k^i \in E^{r_k^i}, \quad \xi_k^i \in E^{s_k^i}, \quad k = 0, 1, \dots, K-1,$$

such that the conditions 1) - 3) are satisfied.

1)

$$\begin{aligned} \lambda_k^i = & \frac{\partial}{\partial x} H_{k+1}^i(x_k^*, \varphi_k^1(x_k^*), \dots, \varphi_k^N(x_k^*)) + \xi_k^i \frac{\partial}{\partial x} Q_k^i(x_k^*, \varphi_k^i(x_k^*)) + \xi_k^i \frac{\partial}{\partial x} q_k^i(x_k^*, \varphi_k^i(x_k^*)) + \\ & + \sum_{j=1}^N \left[\frac{\partial}{\partial u^j} H_{k+1}^i(x_k^*, \varphi_k^j(x_k^*), \dots, \varphi_k^N(x_k^*)) \right] \left[\frac{\partial}{\partial x} \varphi_k^j(x_k^*) \right], \quad k = 0, 1, \dots, K-1, \end{aligned}$$

where we define $\lambda_0^i = 0$ and $\lambda_K^i = -\frac{\partial}{\partial x} g^i(x_K^*)$.

2)

$$\frac{\partial}{\partial u^i} H_{k+1}^i(x_k^*, \varphi_k^1(x_k^*), \dots, \varphi_k^N(x_k^*)) + \xi_k^i \frac{\partial}{\partial u^i} Q_k^i(x_k^*, \varphi_k^i(x_k^*)) + \xi_k^i \frac{\partial}{\partial u^i} q_k^i(x_k^*, \varphi_k^i(x_k^*)) = 0, \quad k = 0, 1, \dots, K-1.$$

3)

$$\xi_k^i \leq 0, \quad \xi_k^i q_k^i(x_k^*, \varphi_k^i(x_k^*)) = 0, \quad k = 0, 1, \dots, K-1.$$

On comparing Theorems 1 and 2 we see that the equilibrium costs differ, in general, as a result of summation term in condition 1) of the last theorem. Because this equation was treated more detailly in [8], we shall not discuss it here further.

4. NECESSARY CONDITIONS FOR OTHER SOLUTIONS

From the definition of the minimax solution we see that, in fact, only two person, zero-sum, multistage game must be solved in which opponent of the i -th player are all remaining players. For such a game both theorems stated in the previous section are identical, i.e. closed-loop solution is synthesis of open-loop problems. Having this in mind, that is $u_k^i = \varphi_k^i(x)$, $i = 1, \dots, N$; $k = 0, 1, \dots, K-1$, the following theorem is stated only for the class of the open-loop strategies to make notation simpler.

Theorem 3. Let two-person, zero-sum, multistage game in which results minimax problem of player i , satisfy Assmps. 1 and 2. Suppose that the admissible strategy N -tuple $(\{u^i\}, \dots, \{\bar{u}^i\}, \dots, \{u^N\})$ is the saddle-point of this game (minimax solution). Corresponding trajectory denote $\bar{x}_0, \bar{x}_1, \dots, \bar{x}_K$.

Assume that for $k = 0, 1, \dots, K-1$, the vectors

$$\frac{\partial}{\partial u^i} Q_{kl}^i(\bar{x}_k, \bar{u}_k^i), \quad l = 1, \dots, r^i; \quad \frac{\partial}{\partial u^i} q_{km}^i(\bar{x}_k, \bar{u}_k^i), \quad m \in I_k^i(\bar{x}_k, \bar{u}_k^i),$$

and the vectors

$$\frac{\partial}{\partial u^i} Q_{kl}^j(\bar{x}_k, u_k^j), \quad l = 1, \dots, r^j; \quad \frac{\partial}{\partial u^i} q_{km}^j(\bar{x}_k, u_k^j), \quad m \in I_k^j(\bar{x}_k, u_k^j),$$

are linearly independent ($j = 1, \dots, N, j \neq i$).

Then there exist vectors

$$\bar{\lambda}_{k+1}^i \in E^n, \quad \bar{\xi}_k^j \in E^{r^j}, \quad \bar{\xi}_k^j \in E^{s_k^j}, \quad j = 1, \dots, N; \quad k = 0, 1, \dots, K-1,$$

such that conditions 1) - 5) are satisfied.

1)

$$\begin{aligned} \bar{\lambda}_k^i = & \frac{\partial}{\partial x} \bar{H}_{k+1}^i(\bar{x}_k, u_k^1, \dots, \bar{u}_k^i, \dots, u_k^N) + \bar{\xi}_k^i \frac{\partial}{\partial x} Q_k^i(\bar{x}_k, \bar{u}_k^i) + \bar{\xi}_k^i \frac{\partial}{\partial x} q_k^i(\bar{x}_k, \bar{u}_k^i) \\ & - \sum_{\substack{j=1 \\ j \neq i}}^N [\bar{\xi}_k^j \frac{\partial}{\partial x} Q_k^j(\bar{x}_k, u_k^j) + \bar{\xi}_k^j \frac{\partial}{\partial x} q_k^j(\bar{x}_k, u_k^j)], \quad k = 0, 1, \dots, K-1, \end{aligned}$$

where we set $\bar{\lambda}_0^i = 0$ and $\bar{\lambda}_K^i = -\frac{\partial}{\partial x} g^i(\bar{x}_K)$.

Used notation

$$\bar{H}_{k+1}^i(x, u^1, \dots, u^N) = -h_k^i(x, u^1, \dots, u^N) + \bar{\lambda}_{k+1}^i f(x, u^1, \dots, u^N), \quad k = 0, 1, \dots, K-1.$$

2)

$$\frac{\partial}{\partial u^i} \bar{H}_{k+1}^i(\bar{x}_k, u_k^1, \dots, \bar{u}_k^i, \dots, u_k^N) + \bar{\xi}_k^j \frac{\partial}{\partial u^i} Q_k^j(\bar{x}_k, u_k^j) + \bar{\xi}_k^j \frac{\partial}{\partial u^i} q_k^j(\bar{x}_k, u_k^j) = 0, \\ j = 1, \dots, N, \quad j \neq i; \quad k = 0, 1, \dots, K-1.$$

3)

$$\frac{\partial}{\partial u^i} \bar{H}_{k+1}^i(\bar{x}_k, u_k^1, \dots, \bar{u}_k^i, \dots, u_k^N) + \bar{\xi}_k^i \frac{\partial}{\partial u^i} Q_k^i(\bar{x}_k, \bar{u}_k^i) + \bar{\xi}_k^i \frac{\partial}{\partial u^i} q_k^i(\bar{x}_k, \bar{u}_k^i) = 0,$$

4)

$$\bar{\xi}_k^j \leq 0, \quad \bar{\xi}_k^j q_k^j(\bar{x}_k, u_k^j) = 0, \quad k = 0, 1, \dots, K-1; \quad j = 1, \dots, N, \quad j \neq i.$$

5)

$$\bar{\xi}_k^i \leq 0, \quad \bar{\xi}_k^i q_k^i(\bar{x}_k, \bar{u}_k^i) = 0, \quad k = 0, 1, \dots, K-1.$$

The minimax solution is extremely pessimistic, and thus little probable. Moreover, in some well-posed realistic cases we have

$J_i = +\infty$. So we are sometimes forced to construct strategy from other reasons.

Now we shall consider noninferior solution of multistage games. From the definition of noninferior set (see [5]) it follows, that in this case we have only a discrete optimization problem with vector-valued performance index. As controls we can choose any admissible strategy N-tuple. Such optimization problems we treated in [10], [11]. It is evident, that on both, open-loop and closed-loop strategy classes we get same necessary conditions for optimal solution.

First we reformulate Assumption 2. Instead of (2.4) we now define in E^{n+1} the sets

$$\hat{V}_k(x) = \{ (a, v) \mid a \in E^N, v \in E^n; a_j = \tilde{h}_k^j(x, u^1, \dots, u^N), j=1, \dots, N, \\ v = f_k(x, u^1, \dots, u^N); u^i \in U_k^i(x), i=1, \dots, N \}, k=0, 1, \dots, K-1,$$

and the convex cone

$$\hat{R} = \{ r \mid r \in E^{N+n}, r = (\rho_1, \dots, \rho_N, 0, \dots, 0); \rho_i \leq 0, i=1, \dots, N \}.$$

Assumption 2a. The sets for any

are R-directional convex

Theorem 4. Consider discrete optimization problem with vector valued cost functional $J = (J_1, \dots, J_N)$, in which results the problem of finding noninferior solution set for multistage game. Let the Assumps. 1, and 2a be satisfied for this problem. Further suppose that the admissible N-tuple belongs to the noninferior set (denoted $\{\hat{u}^1, \dots, \hat{u}^N\}$), and denote by $\hat{x}_0, \hat{x}_1, \dots, \hat{x}_K$ corresponding trajectory. Assume that for $i=1, \dots, N; k=0, 1, \dots, K-1$, the vectors

$$\frac{\partial}{\partial u^i} Q_{kl}^i(\hat{x}_k, \hat{u}_k^i), l=1, \dots, r_k^i; \frac{\partial}{\partial u^i} q_{km}^i(\hat{x}_k, \hat{u}_k^i), m \in I_k^i(\hat{x}_k, \hat{u}_k^i),$$

are linearly independent.

Then there exists a vector $\mu \in E^N, \mu^i \geq 0, i=1, \dots, N, \sum_{i=1}^N \mu^i = 1$, and vectors

$$\hat{\lambda}_{k+1} \in E^N, \hat{\xi}_k^i \in E^{r_k^i}, \hat{\xi}_k^i \in E^{s_k^i}, k=0, 1, \dots, K-1; i=1, \dots, N,$$

such that conditions 1) - 3) are satisfied.

1)

$$\hat{\lambda}_k = \frac{\partial}{\partial x} \hat{H}_{k+1}(\hat{x}_k, \hat{u}_k^1, \dots, \hat{u}_k^N) + \sum_{i=1}^N [\hat{\xi}_k^i \frac{\partial}{\partial x} Q_k^i(\hat{x}_k, \hat{u}_k^i) + \hat{\xi}_k^i \frac{\partial}{\partial x} q_k^i(\hat{x}_k, \hat{u}_k^i)], k=0, 1, \dots, K-1,$$

where we define $\hat{\lambda}_0 = 0$ and $\hat{\lambda}_K = - \sum_{i=1}^N \mu^i \frac{\partial}{\partial x} g^i(\hat{x}_K)$.

We used notation

$$\hat{H}_{k+1}(x, u^1, \dots, u^N) = - \sum_{i=1}^N \mu^i h_k^i(x, u^1, \dots, u^N) + \hat{\lambda}_{k+1} f_k(x, u^1, \dots, u^N), k=0, 1, \dots, K-1.$$

2)

$$\left. \begin{aligned} \frac{\partial}{\partial u^i} \hat{H}_{k+1}(\hat{x}_k, \hat{u}_k^1, \dots, \hat{u}_k^N) + \hat{\xi}_k^i \frac{\partial}{\partial u^i} Q_k^i(\hat{x}_k, \hat{u}_k^i) + \hat{\xi}_k^i \frac{\partial}{\partial u^i} q_k^i(\hat{x}_k, \hat{u}_k^i) \end{aligned} \right\} \begin{aligned} i=1, \dots, N \\ k=0, 1, \dots, K-1. \end{aligned}$$

$$3) \hat{\xi}_k^i \leq 0, \hat{\xi}_k^i q_k^i(\hat{x}_k, \hat{u}_k^i) = 0,$$

It is easy to prove that, taking

$$J = \sum_{i=1}^N \mu^i J_i, \mu^i > 0, \sum_{i=1}^N \mu^i = 1,$$

as a scalar cost functional, and solving appropriate optimal control problem, the corresponding strategy N-tuple will be in the non-

inferior set. Thus solving $(N-1)$ -parameter family of scalar optimal control problems we get noninferior set for the game in question except, maybe, for points which are obtained for some $u^i = 0$. In general, such points are not necessarily noninferior. Also certain convexity assumptions must be satisfied for such conclusion - see [11]. In a concrete game, just mentioned points must be treated separately in order to obtain the whole noninferior set.

From practical point of view we can such solutions (with some $u^i = 0$) simply neglect, because they totally ignore the cost functional of some players. So it is hardly probable that these players will take part in cooperation in such case.

Remark 1. Let $U_k^i \neq U_k^i(x)$, $i = 1, \dots, N$; $k = 0, 1, \dots, K-1$. Except of Theorem 2, all other ones are valid, if in Assump. 1, with respect to u^i , $i = 1, \dots, N$, only continuity of f_k, h_k^i , $i = 1, \dots, N$; $k = 0, 1, \dots, K-1$, is required. Of course, then only maximum condition can be always used - [10].

Remark 2. Now suppose that the initial state x_0 is given. All stated theorems remain valid under the following change. In each theorem in condition 1) delete relation $x_0 = 0$, which was sooner used to compute x_0 and which is now meaningless.

Remark 3. Maybe, it would be more appropriate to assume $U_k^i \neq U_k^i(x)$ as in Remark 1, when dealing with the open-loop strategy class. Otherwise could happen that if, e.g. one player deviates from his equilibrium strategy, this may result in the inadmissible equilibrium strategies of some other players.

5. LINEAR MULTISTAGE GAMES WITH QUADRATIC COSTS

In this section we apply Theorems 1 - 4 to the special class of multistage game in order to obtain the explicit form for each solution. To avoid rather complicated notation, we suppose that the system is autonomous, i.e. its parameters does not change with k . The same technique is applicable to more complicated cases of these games which were solved in [10]. We shall consider following linear multistage game with quadratic cost functionals (x_0 given)

$$x_{k+1} = Ax_k + \sum_{j=1}^N B_j u_k^j, \quad k = 0, 1, \dots, K-1; \quad (5.1)$$

$$J_i = \frac{1}{2} x_K^T S_i x_K + \frac{1}{2} \sum_{k=0}^{K-1} [x_k^T Q_i x_k + \sum_{j=1}^N (u_k^j)^T R_{ij} u_k^j], \quad i = 1, \dots, N. \quad (5.2)$$

The dimensions of all variables are the same as in Sec.2. Dimensions of all matrices are then also determined. Without any loss of generality it is assumed, that S_i, Q_i, R_{ij} , $i, j = 1, \dots, N$, are symmetric. In order to apply Theorems 1 - 4, we make always certain assumptions on mentioned matrices.

Case 1. First let us consider closed-loop equilibrium solution. Assume that the matrices R_{ii} , $i = 1, \dots, N$, are positive definite and the matrices Q_i, S_i , $i = 1, \dots, N$, positive semidefinite. Applying Theorem 2 we get

$$\phi_k^i(x_k) = \tilde{R}_{ii}^T B_{k+1}^T P_{k+1}^i x_k, \quad i = 1, \dots, N; \quad k = 0, 1, \dots, K-1, \quad (5.3)$$

where symmetric matrices P_k^i , $i = 1, \dots, N$; $k = 1, \dots, K$, are solutions of N coupled discrete matrix Riccati-type equations

$$P_k^i = -Q_i + W_i^T [P_{k+1}^i - \sum_{j=1}^N P_{k+1}^j B_j \tilde{R}_{jj}^T B_j^T P_{k+1}^j] W_k, \quad i = 1, \dots, N; \quad k = 1, \dots, K-1, \quad (5.4)$$

with end conditions

$$P_K^i = -S_i, \quad i = 1, \dots, N. \quad (5.5)$$

Here we denoted

$$W_k = [1 - \sum_{j=1}^N B_j \tilde{R}_{jj}^T B_j^T P_{k+1}^j]^{-1} A, \quad k = 0, 1, \dots, K-1. \quad (5.6)$$

If we formally consider eq.(5.4) now also for $k = 0$, we get equilibrium costs for each player

$$J_i^* = -\frac{1}{2} x_0^T P_o^i x_0, \quad i = 1, \dots, N. \quad (5.7)$$

If the solution can be obtained in just described way, then under made assumptions this solution will be unique.

Case 2. Now let us find open-loop equilibrium solution. The assumptions on various matrices are the same as in Case 1. From (5.2) we conclude the existence of unique equilibrium solution. There exist two equivalent ways to obtain open-loop strategies.

A) Suppose that $\{\bar{u}^1, \dots, \bar{u}^N\}$ is desired equilibrium solution. From the point of view of player i he only solves optimal control problem with parameters $\{\bar{u}^j\}$, $j=1, \dots, N$, $j \neq i$. From Theorem 1 we obtain for player i , $i = 1, \dots, N$,

$$u_k^i = R_{ii}^{-1} B_i^T [\mathcal{P}_{k+1}^i (W_k^i x_k + (p_k^i)^T)], \quad k = 0, 1, \dots, K-1, \quad (5.8)$$

where symmetric matrices $\mathcal{P}_k^i$ and row-vectors p_k^i , $k = 1, \dots, K$, are given by equations

$$\mathcal{P}_k^i = -Q_i + A^T \mathcal{P}_{k+1}^i [1 - B_i R_{ii}^{-1} B_i^T \mathcal{P}_{k+1}^i]^{-1} A, \quad k = 1, \dots, K-1; \quad \mathcal{P}_K^i = -S_i, \quad (5.9)$$

$$p_k^i = [(w_k^i)^T \mathcal{P}_{k+1}^i + p_{k+1}^i] A, \quad k = 1, \dots, K-1; \quad p_K^i = 0. \quad (5.10)$$

We used notation

$$W_k^i = [1 - B_i R_{ii}^{-1} B_i^T \mathcal{P}_{k+1}^i]^{-1} A, \quad \left. w_k^i = [1 - B_i R_{ii}^{-1} B_i^T \mathcal{P}_{k+1}^i]^{-1} \left[\sum_{j=1, j \neq i}^N B_j \bar{u}_k^j + B_i R_{ii}^{-1} B_i^T (p_k^i)^T \right] \right\} \quad k = 0, 1, \dots, K-1. \quad (5.11)$$

In fact, we constructed immediately synthesis of open-loop strategies for the i -th player. Eliminating now x_k in (5.8) by successive substitution via (5.1) and doing the same for all $i = 1, \dots, N$, we see, that we are given by the system of linear algebraic equations for computing u_k^i , $i = 1, \dots, N$; $k = 0, 1, \dots, K-1$.

If we consider (5.9) - (5.10) also for $k = 0$, we can write costs of the i -th player

$$J_i^* = -\left[\frac{1}{2} x_0^T \mathcal{P}_0^i x_0 + p_0^i x_0 + q_0^i \right], \quad i = 1, \dots, N, \quad (5.12)$$

where

$$q_k^i = q_{k+1}^i + \frac{1}{2} (w_k^i)^T \mathcal{P}_{k+1}^i w_k^i + p_{k+1}^i w_k^i - \frac{1}{2} \sum_{j=1, j \neq i}^N (\bar{u}_k^j)^T R_{ij} \bar{u}_k^j - \frac{1}{2} [(w_k^i)^T \mathcal{P}_{k+1}^i + p_{k+1}^i] \cdot B_i R_{ii}^{-1} B_i^T [\mathcal{P}_{k+1}^i w_k^i + (p_{k+1}^i)^T], \quad k = 0, 1, \dots, K-1; \quad q_K^i = 0. \quad (5.13)$$

B) Suppose that each player uses open-loop synthesis (5.8). Then using the same procedure as in Case 1 we obtain

$$\bar{u}_k^i = R_{ii}^{-1} B_i^T (\Pi_{k+1}^i)^T \Omega_k x_k, \quad i = 1, \dots, N; \quad k = 0, 1, \dots, K-1, \quad (5.14)$$

where, in general, non-symmetric matrices Π_k^i we get from the discrete Riccati-type equations ($i = 1, \dots, N$; $k = 1, \dots, K$):

$$\Pi_k^i = -Q_i + \Omega_k^T \Pi_{k+1}^i A, \quad k = 1, \dots, K-1, \quad \left. \Pi_K^i = -S_i \right\} \quad i = 1, \dots, N, \quad (5.15)$$

where

$$\Omega_k = [1 - \sum_{j=1}^N B_j R_{jj} B_j^T (\Pi_{k+1}^j)^T]^{-1} A, \quad k = 0, 1, \dots, K-1. \quad (5.16)$$

Now we do not find an analogy of (5.12).

Comparing Eqs.(5.3) and (5.14) we see that these strategy N-tuples are not necessarily identical.

Case 3. Now let us try to find the minimax strategy for the i -th player. It is convenient to seek directly closed-loop strategy. Matrices $Q_i, S_i, R_{ij}, -R_{ij}, j=1, \dots, N, j \neq i$, are assumed positive definite so as to simplify next calculations. From Theorem 3 we have for minimax strategy $\{\bar{u}^i\}$ of player i and corresponding strategies of opponents $\{u^j\}, j=1, \dots, N, j \neq i$, the relations

$$\begin{aligned} \bar{u}_k^i &= \bar{q}_k^i(x_k) = R_{ii}^{-1} B_i^T \bar{P}_{k+1}^i \bar{W}_k^i x_k \\ u_k^{j \neq i} &= q_k^{j \neq i}(x_k) = -\bar{R}_{ij}^{-1} B_j^T \bar{P}_{k+1}^i \bar{W}_k^i x_k \end{aligned} \quad \left. \begin{array}{l} k=0, 1, \dots, K-1, \\ j=1, \dots, N, j \neq i, \end{array} \right\} \quad (5.17)$$

where symmetric matrices $\bar{P}_k^i$ are given by relation

$$\bar{P}_k^i = -Q_i + A^T [(\bar{P}_{k+1}^i)^{-1} - \sum_{j=1}^N B_j \bar{R}_{ij}^{-1} B_j^T]^{-1} A, \quad k=0, 1, \dots, K-1; \quad \bar{P}_K^i = -S_i, \quad (5.18)$$

and

$$\bar{W}_k^i = [1 - \sum_{j=1}^N B_j \bar{R}_{ij}^{-1} B_j^T \bar{P}_{k+1}^i]^{-1} A, \quad k=0, 1, \dots, K-1. \quad (5.19)$$

Minimax costs for player i are given by formula ($k=0$ in (5.18))

$$\bar{J}_i = -\frac{1}{2} x_0^T \bar{P}_0^i x_0. \quad (5.20)$$

Case 4. To find noninferior strategy set we assume matrices $Q_i, S_i, R_{ij}, i, j=1, \dots, N$, positive definite. Then it is possible to obtain noninferior set (with exceptions discussed in Section 4) by solving the optimal control problem with functional

$$J(\mu) = \sum_{i=1}^N \mu^i J_i, \quad \mu^i > 0, \quad i=1, \dots, N, \quad \sum_{i=1}^N \mu^i = 1. \quad (5.21)$$

We have denoted $\mu = (\mu^1, \dots, \mu^N)$. By rather straightforward manipulations we obtain for noninferior strategies the relations

$$\hat{u}_k^i = \hat{q}_k^i(x_k) = [\sum_{j=1}^N \mu^j R_{ji}]^{-1} B_i^T \hat{P}_{k+1}(\mu) \hat{W}_k(\mu) x_k, \quad i=1, \dots, N; \quad k=0, 1, \dots, K-1, \quad (5.22)$$

where

$$\hat{P}_k(\mu) = -\sum_{j=1}^N \mu^j Q_j + A^T [(\hat{P}_{k+1}(\mu))^{-1} - \sum_{i=1}^N B_i [\sum_{j=1}^N \mu^j R_{ji}]^{-1} B_i^T]^{-1} A, \quad k=0, 1, \dots, K-1; \quad \hat{P}_K(\mu) = -\sum_{j=1}^N \mu^j S_j, \quad (5.23)$$

and

$$\hat{W}_k(\mu) = [1 - \sum_{j=1}^N B_j [\sum_{i=1}^N \mu^i R_{ji}]^{-1} B_i^T \hat{P}_{k+1}(\mu)]^{-1} A, \quad k=0, 1, \dots, K-1. \quad (5.24)$$

If we consider Eqs. (5.23) again also for $k=0$ we can get total costs of all players

$$J(\mu) = -\frac{1}{2} x_0^T \hat{P}_0(\mu) x_0. \quad (5.25)$$

Each player can compute his own costs using Eq.(5.22).

Remark. From computational point of view it is necessary to assume existence of the all inverses on (5.6), (5.11) and (5.16), in order to apply the scheme described in Cases 1 and 2. On the other hand, the assumptions made in Cases 3 and 4 are sufficient for the existence of inverses in (5.19), (5.24).

6. CONCLUSIONS

For rather general class of N-player, nonzero-sum, multistage games we obtained necessary conditions for equilibrium, minimax and noninferior solution types. We used general version of the discrete maximum principle proved in [10].

As we could expected (by analogy with differential games) also here there exist two different types of equilibrium points depending on the considered strategy class. As a rule, open-loop and closed-loop equilibrium costs are not necessarily identical.

For the class of linear multistage games with quadratic cost functionals we also derived the explicit form of each solution.

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