CLOSED - LOOP DIFFERENTIAL GAMES

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The object of this report is closed-loop differential games which are a formalization of problems in dynamics related to control in presence of uncertainty or conflict. Under the game-theoretic approach to the problem we assume that the uncontrollable factors may appear to be most unfavourable from the point of view of one or the other criterion, important for the evaluation of the process. Thus a game is formed in between our ally, who selects the control in pursuing the one or the other aim and the antagonist who governs the disturbances. The given report contains some conclusions accumulated and summarized by some research members of our Institute [1]. Naturally, our results are closely related to those of other authors. However the framework of this report does not enable us to produce a bibliography.

The model of the differential games under consideration is as follows. Given is a differential equation

$$\dot{x} = f(t, x, u, v), \quad u \in \mathcal{P}, \quad v \in \mathcal{Q}, \quad t \geq t_0. \quad (1)$$

where x is the state space vector; u and v are vectors of control forces for the first and the second player; $\mathcal{P}$ and $\mathcal{Q}$ are compacts. Given is a functional

$$J = J(x[t]; t_0 \leq t) \quad (2)$$

which evaluates the result of the game.

Here we deal with closed-loop differential games. In assigning his control at current moment of time t the ally may be guided by complete knowledge of realized position $\{t, x\}$ or more generally by the knowledge of the whole history $\{t, x[\tau];$

$$t_0 \leq \tau \leq t\} = \{t, x_t[\cdot]\} \quad (t \geq t_0) \quad \text{of the motion.}$$

The idea of feedback control is simply formalized in the following

manner.

The pure strategy $\bar{u}$ for the first player-ally we shall identify with any fixed function $u(t, x) \in \mathcal{P}$. The equation (1), the strategy $\bar{u}$ and the initial position $\{t_*, x_*\}$ generate a bundle of motions $x[t] = x[t, t_*, x_*, \bar{u}]$ ($x[t_*] = x_*$) each of which is the limit for a certain converging sequence of Euler splines $x_{\Delta}^{(k)}[t] = x_{\Delta}[t, t_*, x_*^{(k)}, \bar{u}, v^{(k)}[\cdot]]$ ($k = 1, 2, \dots$) with $\delta^{(k)} \rightarrow 0$. Here the

Euler splines $x_{\Delta}[t] = x_{\Delta}[t, t_*, x_*, \bar{u}, v[\cdot]]$ are determined as the solutions of the equation

$$\dot{x}_{\Delta}[t] = f(t, x_{\Delta}[t], u(\tau_i, x_{\Delta}[\tau_i]), v[t]), \quad (\tau_i \leq t < \tau_{i+1}) \quad (3)$$

where $t_* = \tau_0$, $\tau_i \rightarrow \infty$ with $i \rightarrow \infty$, $v[t]$ is any measurable function, $v[t] \in Q$ and $\delta = \max_i (\tau_{i+1} - \tau_i)$. The function f in equation (1) is assumed to be continuous, locally Lipschitz in x and to satisfy the conditions for a uniform prolongation of solutions (3). Then the nonempty bundle of motions $\{x[t, t_*, x_*, \bar{u}]\}$ does always exist with no reservations on the functional properties of $u(t, x)$. At the same time one may always return from ideal limit motions $x[t]$ to actually realized Euler splines $x_{\Delta}[t]$ which yield a good approximation for the ideal motions. The pure strategy reflects the idea of forming the control on the basis of information on the current position $\{t, x[t]\}$ under the condition that the antagonist in forming his control at time t may make use of any conceivable information whatever. In particular, prior to the assignment of his own control force, he may be assumed to know the control force for the ally at same moment of time or even in advance for moments of time in the nearest future.

The mixed strategy $\tilde{u}$ as introduced in the sequel reflects the idea that the ally again forms his control using the information on the current position $\{t, x[t]\}$ but now with condition that precise knowledge of current control force values for the ally is inaccessible for the antagonist. The latter may now but make a guess for a convenient response reaction on his part with the aid of a certain probabilistic technique. Thus a mixed strategy of the first player is identified with the function $\mu(du | t, x)$ which

assigns for every possible position $\{t, \alpha\}$ a probabilistic measure $\mu(du)$ on $\mathcal{P}$. The motions $\alpha[t] = \alpha[t, t_*, \alpha_*]$, $\tilde{U}$ (understood to be ideal and deterministic) are determined again by a limit transition but now from Euler broken lines $\alpha_\Delta[t]$

$$\dot{\alpha}_\Delta[t] = \iint_{P \times Q} f(t, \alpha_\Delta[t], u, v) \mu(du | \tau_i, \alpha_\Delta[\tau_i]) \nu_t(dv) \quad (4)$$

$$(\tau_i \leq t < \tau_{i+1})$$

where $\nu_t(dv)$ is a probabilistic measure on Q weakly measurable in t . The transition from deterministic ideal motions $\alpha[t]$ to actually realized stochastic motions $\tilde{\alpha}_\Delta[t]$ is now achieved by means of stochastic control procedures (see as follows at p. 000).

The counterstrategy $U^{(v)}$ is identified with function $u(t, \alpha, v) \in \mathcal{P}$ which is Borel measurable in v . It reflects the idea of forming the control by the ally at every moment of time t on the basis of the realized position $\{t, \alpha[t]\}$ and of the realized at same moment of time t control v for the antagonist. This leads to the so-called informational discrimination of the antagonist. In the case of counterstrategies the motions $\alpha[t, t_*, \alpha_*]$, $U^{(v)}$ are determined again by a limit transition, but now from Euler splines $\alpha_\Delta[t]$:

$$\dot{\tilde{\alpha}}_\Delta[t] = f(t, \alpha_\Delta[t], u(\tau_i, \alpha_\Delta[\tau_i], v[t]), v[t]) \quad (5)$$

$$(\tau_i \leq t < \tau_{i+1})$$

where $v[t]$ is measurable in t and $v[t] \in Q$. The transition from ideal motions $\alpha[t]$ to those actually realizable $\alpha_\Delta[t]$ is here again achieved by direct return to Euler splines $\alpha_\Delta[t]$ which now again gives a good approximation to the ideal motions. It is noteworthy to say that the concept of counterstrategy is perhaps in its essence the very construction of nearest to control laws in differential games for the ally which are essential in the approach of L.S. Pontriagin and his followers [2]. On the contrary, the main attention in this report will be given to control laws formed in terms of formalized pure or mixed strategies-gi-

ving thus a preference to informational discrimination of the ally. It is useful to underline however that with certain conditions presumed (see e.g. as follows at p. 000) the transition from pure to counterstrategies gives no variation in guaranteed result for the game of the ally.

The definitions for strategies V for the second player and of the corresponding motions $x[t]$ and $x_a[t]$ are formed obviously from given definitions by interchange of letters u and v . In passing from information space of positions $\{t, x\}$ to information space of histories $\{t, x_t[\cdot]\}$ all the definitions of above are obviously transformed by direct substitution of history in place of position. In the sequel the terms U and V will denote either pure or mixed strategies or counterstrategies.

The game is determined as a pair of problems each of which is formulated for the player - ally. The problem for the first player is formalized as the one of assigning an optimal minmax strategy U° which satisfies the condition

$$\sup_{x_t[\cdot]} y(x[\cdot; t_0, x_0, U]) = \min_U \sup_{x_t[\cdot]} y(x[\cdot; t_0, x_0, U]), \quad (6)$$

the problem for the second player is to specify an optimal maxmin strategy V° which satisfies the condition

$$\inf_{x_t[\cdot]} y(x[\cdot; t_0, x_0, V]) = \max_V \inf_{x_t[\cdot]} y(x[\cdot; t_0, x_0, V]) \quad (7)$$

Equality

$$\inf_U \sup_{x_t[\cdot]} y(x[\cdot]) = \sup_V \inf_{x_t[\cdot]} y(x[\cdot]) = y^\circ \quad (8)$$

being fulfilled, the value y° will be called the value of the game. If there exists a pair of strategies $\{U^\circ, V^\circ\}$ with equality (3) fulfilled, the given pair will be called the saddle-point or point of equilibrium for the game. A minimaximizing sequen-

ce $\{U^{(k)}, V^{(k)}\}$ ($k=1,2,\dots$) for (8) will be called the ε -equilibrium sequence etc.

The quality of the formalization proposed here may be evaluated at first by verifying the existence of equilibrium and ε -equilibrium situations for more or less simple type of games. The foundation for solving the given question is given by the following theorem on the alternative for a standard approach-evasion game. The game is formed of two problems. Given are closed sets M_c and N_c in space $\{t, x\}$, the position $\{t_0, x_0\}$ and the point $\vartheta^* > t_0$. Problem of approach for the first player-ally: Specify a strategy U^* which ensures the encounter

$$\{\vartheta, x[\vartheta]\} \in M_c, \quad \{t, x[t]\} \in N_c, \quad t_0 \leq t \leq \vartheta \leq \vartheta^* \quad (9)$$

for all motions $x[t] = x[t, t_0, x_0, U^*]$. Problem of evasion for the second player-ally: specify a strategy V^* , which excludes the encounter (9) for all motions $x[t] = x[t, t_0, x_0, V^*]$.

The theorem on the alternative. In the approach-evasion game there either exists a mixed strategy $\tilde{U}^*$ which solves the problem of approach or in contrary there exists a mixed strategy $\tilde{V}^*$ which solves the problem of evasion related even to ε -neighbourhoods $M_c^{(\varepsilon)}$ and $N_c^{(\varepsilon)}$ of sets M_c and N_c . If for any selection of vector s , of t and of x the minor game

$$\min_{u \in P} \max_{v \in Q} s' \{t, x, u, v\} = s' \{t, x, u^*, v^*\} \quad (10)$$

does have a saddle-point $\{u^*, v^*\}$, then the alternative is valid in pure strategies $\bar{U}$ and $\bar{V}$. (In our notations the vectors always stand as columns. The upper prime denotes transposition). In general the alternative is always valid in the pairs of classes $\{\{\bar{U}\} \{V^{(u)}\}\}$, $\{\{U^{(v)}\} \{\bar{V}\}\}$.

A similar theorem is true for the information space of histo-

ries $\{t, x_t[\cdot]\}$ with functional sets $\mathcal{M}_c$ and $\mathcal{N}_c$ closed in $\{C_{[t_0, t]}, t_0 \leq t < \infty\}$.

The approach-evasion game constitutes an element in the investigation of the general differential game (6), (7). Consider as an example the game on the minmax-maxmin for the functional

$$y = \min_{t_0 \leq t \leq \vartheta} g(x[t], \mathcal{L}) \quad (11)$$

where $g(x, \mathcal{L})$ is the euclidean distance of point x from set $\mathcal{L}$ in space $\{x\}$, the moment ϑ being given. Select sets

$$\begin{aligned} \mathcal{N}_c &= [\{t, x\} : t \geq t_0, -\infty < x < \infty] \\ \mathcal{M}_c &= [\{t, x\} : t_0 \leq t \leq \vartheta, g(x, \mathcal{L}) \leq c] \end{aligned} \quad (12)$$

for $c \geq 0$. Assume c_0 is the lower bound over all c for which the first assertion of the theorem on the alternative is true in the class of mixed strategies with $\{t_0, x_0\}$ and $\vartheta^* = \vartheta$ given. Then it is easy to deduce from the alternative that c_0 is the value of the game and that there exists a pair of optimal mixed strategies $\{\tilde{U}^\circ, \tilde{V}^\circ\}$ which constitute the saddle-point. Further assume the saddle-point condition for the minor game (10) is fulfilled. Then for the game under consideration there exists a saddle-point in the class of pure strategies $\{\bar{U}^\circ, \bar{V}^\circ\}$. The optimal strategies $\bar{U}^\circ$ and $\bar{V}^\circ$ ensure the inequalities

$$\begin{aligned} \min_{t_0 \leq t \leq \vartheta} g(x[t, t_0, x_0, \bar{U}^\circ]) &\leq c_0 \\ g(x[t, t_0, x_0, \bar{V}^\circ]) &\geq c_0, \quad t_0 \leq t \leq \vartheta \end{aligned} \quad (13)$$

This being true for any $\varepsilon > 0$ there exists a $\delta(\varepsilon) > 0$ such that the same strategies when being applied to Euler splines $x_\Delta[t]$ with increment $\delta < \delta(\varepsilon)$ ensure the inequalities

$$\min_{t_0 \leq t \leq \vartheta} \varphi(x_\Delta[t, t_0, x_0, \bar{U}^\circ, v[\cdot]]) \leq c_0 + \varepsilon \quad (14)$$

$$\varphi(x_\Delta[t, t_0, x_0, \bar{V}^\circ, u[\cdot]]) \geq c_0 - \varepsilon, \quad t_0 \leq t \leq \vartheta$$

A rather more complicated example is given by the game formed again for the minmax-maxmin of functional (11), but now under condition that moment ϑ is not given but is assigned separately for each motion $x[t]$ with one of the following conditions fulfilled:

(1) $\vartheta = \vartheta^*$, (2) when point $x[t]$ runs into the closed domain G , (3) when point $x[t]$ runs into closed set H prior to the motion having attained the open domain F . Making use of a version of the theorem on the alternative for the functional space of histories $x_t[\cdot]$ we are again convinced that there exists a value c_0 of the game. There exists a mixed strategy $\bar{V}^\circ \div \div v^\circ(dv | x_t[\cdot])$ here which ensures the inequality $\varphi \geq c_0$ and for any $c^* > c_0$ there exists a mixed strategy $\bar{U} \div \div \mu^*(du | x_t[\cdot])$ which ensures the inequality $\varphi < c^*$.

A similar utilization of the principal element in the form of an appropriate approach-evasion game admits the investigation also of other differential games. Here the theorem on the alternative yields a conclusion that with saddle-point condition for the minor game (10) being fulfilled, the absence of informational discrimination of the antagonist does not worsen the result of the game for the ally.

When describing actual schemes of control one ought to admit the existence of errors (perhaps of minor value) in measurements of the realizations of $x[t]$. Generally speaking the game-theoretic problems of dynamics in adopted formalization are actually improperly posed. Indeed it is possible to demonstrate examples of games with functional (11) when there exists a pure strategy $\bar{V}^\circ$ which guarantees inequalities (13) and (14) with $c_0 > 0$, but which leads to the following phenomena. If when forming the Euler splines

$x_\Delta[t]$ one substitutes controls $v^\circ(\tau_i, x_\Delta^*[\tau_i])$ instead of $v^\circ(\tau_i, x_0[\tau_i])$, where measured values $x_\Delta^*[\tau_i]$ deflect from actual values $x_\Delta[\tau_i]$ by minor value $\|x_\Delta^* - x_\Delta\| \leq \tau$

then with $\delta \rightarrow 0$, the strategy $\bar{V}^0$ already does not ensure even the inequality $\rho(\alpha_\Delta[t], l) > 0$. It is of interest to note by the way that in the examples described, the strategy $\bar{V}^0$ is formed by means of discontinuous functions $v^0(t, x)$ and there exist no approximations by continuous strategies $\bar{V}^* + v^*(t, x)$ which could ensure the inequality $\rho(\alpha[t, t_0, x_0, \bar{V}^*], l) > c_0 - \varepsilon$, taken even for the ideal motions $x[t, t_0, x_0, \bar{V}^*]$.

Hence a problem of regularizing improperly posed solutions of differential games does arise. The problem is soluble for any approach-evasion game and hence for any game which may be reduced to its elements i.e. to appropriate approach-evasion games. Briefly speaking, the proposed regularized solution, which is said to be the control with leader, is constructed as follows. A computational model system

$$\dot{w} = f^*(t, w, u^*, v^*), \quad u^* \in P^*, \quad v^* \in Q^* \quad (15)$$

is adjoined to the actual system (1) and a game is formed in between u^* and v^* . Suppose for the given initial position the problem of encounter with set M_c till moment ϑ^* is solvable for the first player. Then one may specify the three controls $\{u, u^*, v^*\}$ so as to lead the motion $w[t]$ to desired state

$$\{\vartheta, w[\vartheta]\} \in M_c, \quad \{t, w[t]\} \in N_c, \quad t_0 \leq t \leq \vartheta \leq \vartheta^* \quad (16)$$

and at the same time so as to force the actual motion $\alpha_\Delta[t]$ and the motion for the model to track each other. Thus the motion

$\alpha_\Delta[t]$ is led to a minor ε -neighbourhood of state (16) if only the value $\alpha = \alpha_\Delta^*[t]$ introduced into the control law

$$\bar{U} \div \{u(t, x, w), u^*(t, x, w), v^*(t, x, w)\} \quad (\text{or into}$$

$U^{(v)} \div \{u(t, x, w, v), u^*(t, x, w), v^*(t, x, w)\})$ does differ from actual value $\alpha_\Delta[t]$ by sufficiently small number and the increment δ for Euler splines $\alpha_\Delta[t]$ is sufficiently small. The proposition stated here is an accurate mathematical theorem. A similar proposition is also true for the second player's

evasion problem with obvious transformation of the procedure and interchange of letters u and v . The scheme of control with a leader results in an interesting interaction of game-type control problems with problems on stability of motion. Here it is also worthwhile to explain the stochastic procedure of control with a leader, which approximates the ideal motion for mixed strategies. Assume each of the players receives his information in the form of a signal

$y^{(j)}[t]$ ($j = 1, 2$) statistically dependent on the actual motion $x[t]$. Assume the conditional expectations for the deviation of the measured values $\tilde{x}_{\Delta}^{*}[t]^{(j)}$ generated by signals $\tilde{y}_{\Delta}^{(j)}[t]$ with respect to actual values $\tilde{x}_{\Delta}[t]$ satisfy the inequalities

$$E(\|\tilde{x}_{\Delta}^{*}[t]^{(j)} - \tilde{x}_{\Delta}[t]\| | \tilde{x}_{\Delta}[t]) \leq \tau^{(j)} \quad (j = 1, 2) \quad (17)$$

Assume each player forms his control on the basis of his own strategy $\tilde{U} \div \mu(du | t, y_t^{(1)}[t])$ and $\tilde{V} \div \nu(dv | t, y_t^{(2)}[t])$ ($y_t^{(j)}[t] = \{y_t^{(j)}[\tau], t_0 \leq \tau \leq t\}$) within a scheme discrete in time t with his own subdivision $\Delta^{(j)}$ of the time axis t . In forming the realization $\tilde{x}_{\Delta}[t]$ of the stochastic motion (1) the j -th player being in situation $\{\tau_i^{(j)}, \tilde{y}_{\tau_i^{(j)}}^{(j)}[t]\}$ selects his control realization for the coming time interval as a result of a stochastic test with probabilistic distributions

$$\mu(du | \tau_i^{(1)}, y_{\tau_i^{(1)}}^{(1)}[t]) \quad \text{or} \quad \nu(dv | \tau_i^{(2)}, y_{\tau_i^{(2)}}^{(2)}[t])$$

respectively. In order to found this type of control scheme properly, it is necessary to assume the abilities of the controllers to be in some sense so "flexible" that they would indeed prevail over the informational abilities. These conditions may be expressed in exact form, but the framework of this report does not enable us to do it here. We can say only that the last conditions hold in particular if the information lag exceeds the value of the increment for Euler splines.

Under given conditions the following proposition is true. Assume with $\{t_0, x_0\}$, $\vartheta^* > t_0$, $\mathcal{M}_c$ and $\mathcal{N}_c$ given, the problem of approach is soluble in the class of mixed strategies

$\bar{U} \div \mu(du | t, x)$ within the ideal scheme. Then the procedure of control with leader makes possible the construction of a mixed strategy $\tilde{U} \div \mu(du | t, y_t^{(1)}[t])$ which for any $\varepsilon > 0$ and $p^* < 1$ selected in advance ensures, for the stochastic Euler splines $\tilde{x}_{\Delta}[t]$,

the encounter

$$\{\vartheta, \tilde{x}_\Delta[\vartheta]\} \in \mathcal{M}_c^{(\varepsilon)}, \quad \{t, \tilde{x}_\Delta[t]\} \in \mathcal{N}_c^{(\varepsilon)}, \quad t_0 \leq t \leq \vartheta \leq \vartheta^*, \quad (18)$$

with probability $p \geq p^*$ if only the value $\tilde{\gamma}^{(1)} > 0$ in (17) and the increment $\delta^{(1)} = \sup_i (\tau_{i+1}^{(1)} - \tau_i^{(1)})$ for the splines are sufficiently small. On the contrary, assume the problem of evasion is soluble in the class of mixed strategies $\tilde{V} \div v(dv | t, x)$ within the ideal scheme. Then there exists an $\varepsilon > 0$ such that the procedure of control with leader makes it possible to construct a mixed strategy $\tilde{V} \div v(dv | y_t^{(2)}[.])$ which for any $p^* < 1$ selected in advance excludes the encounter (18) with probability $p \geq p^*$, if only the values $\tilde{\gamma}^{(2)} > 0$ in (17) and the step $\delta^{(2)}$ are sufficiently small.

As an example of effective construction of solution strategies $\tilde{U}$ and $\tilde{V}$ we here discuss at first the method of extremal aiming of the latter being applied to a game with functional (11) considered for the sake of determinancy in the class of mixed strategies. Assume that at moment t in the course of the game the vector $x[t] = x_*$ is realized. The method of extremal aiming gives a recommendation which is to freeze position $\{t, x_*\}$ and to consider an auxiliary open loop control problem in a fictitious time scale $t \leq \tau \leq \vartheta$: this is to specify over the set of measures $\mu_\tau(du)$ and $\nu_\tau(dv)$ an optimal pair $\{\mu_\tau^\circ, \nu_\tau^\circ\}$ which ensures for motions

$$\dot{x} = \iint_{\mathcal{P}Q} f(\tau, x, u, v) \mu_\tau(du) \nu_\tau(dv), \quad x(t) = x_* \quad (19)$$

the condition

$$\varepsilon^\circ(t, x_*) = g(x^\circ(\tau_0), l) = \min_{\mu_\tau} \min_{\nu_\tau} g(x(\tau), l) \quad (20)$$

The solution $\{x^\circ(\tau), \mu_\tau^\circ, \nu_\tau^\circ\}$ of the problem satisfies under certain regularity assumptions the minimax condition

$$\begin{aligned}
& \iint_{PQ} s'(\tau) f(\tau, x^\circ(\tau), u, v) \mu_\tau^\circ(du) \nu_\tau^\circ(dv) = \\
& = \min_{\mu} \max_{\nu} \iint_{PQ} s'(\tau) f(\tau, x^\circ(\tau), u, v) \mu(du) \nu(dv)
\end{aligned} \tag{21}$$

which holds almost everywhere and which is an analogy of the maximum principle. Here $s(\tau) (t_* \leq t \leq \tau)$ is a solution of the equation

$$\dot{s} = - \left[\iint_{PQ} \left\{ \frac{\partial f}{\partial x} \right\} \mu_\tau^\circ(du) \nu_\tau^\circ(dv) \right] s, \quad s(\tau_0) = \left[\frac{\partial \rho}{\partial x} \right]_{\{\tau_0, x^\circ(\tau_0)\}} \tag{22}$$

Assume $S(t, x_*, \tau_0)$ is the set of all vectors $s(t)$ in (22) which correspond to all possible solutions of problem (20) for given position $\{t, x_*\}$ with one or the other value of τ_0 from (20). Then under certain regularity conditions formulated in terms of the properties of $S(t, x, \tau_0)$ one may construct on the basis of sets S the optimal mixed strategies $\tilde{U}^\circ \div \mu(du|t, x)$ and $\tilde{V}^\circ \div \nu(dv|t, x)$ which ensure (12) with precisely the equality $c_0 = \varepsilon^\circ(t_0, x_0)$ of (20) being true. Thus, the regularity conditions being fulfilled, the control force law when formed by extremal aiming rule in the actual time scale is determined with the aid of a superspeed solution of an auxiliary open loop control problem (20), achieved in a superspeed fictitious time scale $t \leq \tau \leq \bar{\tau}$. The solutions of the auxiliary problems thus act as a sort of a location system, which escorts the actual motion. For the special case of linear equation (1), i.e. with

$$\dot{x} = A(t)x + f(t, u, v) \tag{23}$$

and with convex set L the computations for the extremal aiming rule are visibly realizable and it is possible to form the control law for the game in the actual time scale, provided a modern computational device of good abilities is available.

We gave a description of an example on the additional open-loop construction for the closed-loop differential game. Such open-loop constructions of extremal aiming may vary greatly depending

upon the selected classes of strategies, upon the value to be attained and so on. We now refer to auxiliary constructions of another type, stopping for the sake of determinancy but only on the discussion of a representation of such a class for the case of mixed strategies.

Consider the sets

$$H(t, \alpha) = \bigcap_{\mu} \left(\bigcup_{\nu} \int \int \{f(t, \alpha, u, v)\} \mu(du) \nu(dv) \right). \quad (24)$$

Assume that in the domain $\{t, \alpha\}$ required, the sets $H(t, \alpha)$ are nonempty. Then any solution $\alpha = w(t)$ of the contingent equation

$$\dot{\alpha} \in H(t, \alpha) \quad (25)$$

yields for the first player a so-called stable path. If there exists at least one path connecting initial position $\{t_0, \alpha_0\}$ with target $\mathcal{M}_c$, then a mixed strategy $\tilde{U} \div \mu(du | t, \alpha)$ may be constructed on the basis of this path which leads all the motions $\alpha[t]$ (the latter being ideal) to set $\mathcal{M}_c$ along the path described.

Further consider the function

$$\alpha(\ell, t, \alpha) = - \min_{\mu} \max_{\nu} \ell' \int \int \{f(t, \alpha, u, v)\} \mu(du) \nu(dv) \quad (26)$$

With sets $H(t, \alpha)$ (24) nonempty in the domain $\{t, \alpha\}$ of interest and function $\alpha(\ell, t, \alpha)$ convex in ℓ , the bundle of solutions $\alpha(t, t_0, \alpha_0)$ (25) forms a stable set W for the second player. Therefore it is possible to construct a mixed strategy $\tilde{V} \div \nu(dv | t, \alpha)$ for the second player which would hold all the ideal motions $\alpha[t] = \alpha[t; t_0, \alpha_0, \tilde{V}]$ within W . If for $t_0 \leq t \leq \bar{t}^*$ the bundle W gives no intersection with $\mathcal{M}_c$ the problem of evasion is then solved.

Let us compare the given conditions with the regularity assumptions of the method of extremal aiming. The general nonlinear case gives a complicated picture. However for the special equation (23) linear in α with $\mathcal{L}$ convex, the answer is simple enough. With $H(t, \alpha)$ (24) nonempty and $\alpha(\ell, t, \alpha)$ (26) convex,

the regularity assumptions for the method of extremal aiming are always fulfilled. A stable path for the first player $\alpha = w^\circ(t) = w^\circ(t, t_0, \alpha_0)$ ($t_0 \leq t \leq \bar{t}$) which is the solution of (25) nearest to $\mathcal{L}$, lies at a distance of

$$\varepsilon = \min_{t_0 \leq t \leq \bar{t}} \rho(w^\circ(t), \mathcal{L}) = \varepsilon^\circ(t_0, \alpha_0) \quad (27)$$

from $\mathcal{L}$.

A stable integral manifold $W = \{ \alpha = w(t, t_0, \alpha_0) \mid t_0 \leq t \leq \bar{t} \}$ for the second player obtained for contingent equation (25) lies at a distance of $\varepsilon = \varepsilon^\circ(t_0, \alpha_0)$ from $\mathcal{L}$. An opposite assertion is false: the regularity assumptions for the method of extremal aiming generally do not yield the nonemptiness of set $H(t, \alpha)$ or the convexity of function $\varphi(\ell, t, \alpha)$ (26) in ℓ . Finally we note that in the particular case of an absolutely linear system (23) the conditions on $H(t, \alpha)$ and $\varphi(\ell, t, \alpha)$ under discussion correspond to the so-called condition of complete sweeping over for the direct method of L.S. Pontriagin [2] which in turn are a generalization of the similarity conditions for linear plants in the pursuit-evasion game [1].

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