A NEW HEURISTIC METHOD FOR GENERAL

MIXED INTEGER LINEAR PROGRAMS: A

REPORT ON WORK IN PROGRESS

(ABSTRACT)

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For linear integer programs, the method to be described starts with the continuous optimum and maintains primal feasibility. Instead of modifying the constraints of the original problem, the method modifies the objective function. A new, non-linear, objective function is set up, based on the original objective function and on two integrality measures. A lattice is constructed in the interior of the constraint polyhedron to enable us to move around inside this region. The simplex method is used to search for the maximum of the derived objective function which is so constructed that a maximum is likely to coincide with an integer solution to the problem. If a maximum is found which is also an integer solution, the parameters of the objective function are changed, and a search for a better integer solution is begun. If a maximum is found which is not an integer solution, the parameters are changed and the search continues. The search stops when no change of parameters will open up new areas of search. The effect of varying the parameters is to stretch or squeeze the derived objective function so that we can escape from local optima that are not integer solutions, and so that we can escape from integer solutions that are not global optima.

Experience with small test problems of proven difficulty (many of them pathological cases) has shown that an appropriate initial choice of parameters, based on the structure of the problem, and automatic adjustment of the parameters during the search, leads quickly to the optimum integer solution. This optimum is often the first integer solution found. The method has not yet been tried on large-scale problems.

It is established that the method generates feasible extreme points amongst which is the optimum integer solution, if there is one. It is also established that there is a path of feasible extreme points between any feasible integer solution and the continuous optimum. The method is guaranteed not to return to an already visited extreme point and to be finite. However, the rules used in setting and varying the parameters have not been proved to lead to the optimum integer solution if one exists. Work is proceeding on this point.

The following account will make the foregoing remarks clearer:

The general mixed integer linear programming problem may be stated as follows:

$$\text{Maximize } F_1 = \sum_{j=1}^p \gamma_j \delta_j + \sum_{j=p+1}^{m+n} c_j x_j \quad (1)$$

$$\text{subject to:} \quad \sum_{j=1}^p a_{ij} \delta_j + \sum_{j=p+1}^{m+n} a_{ij} x_j = b_i \quad i=1, \dots, m \quad (2)$$

where the δ_j and the x_j are non-negative variables, the δ_j are to be integers (not necessarily zero or one) and the γ_j, c_j and a_{ij} are constants. There are p integer variables.

Suppose that the maximum of F_1 has been found without the integrality condition for the δ_j and this maximum value of F_1 is Ω . Let us also write

$$F_2 = \sum_{j=1}^p f_j \quad (3)$$

where, using $\{\delta_j\}$ to denote the fractional part of δ_j ,

$$f_j = -5.0 \text{ for each integral } \delta_j \text{ (that is for } \{\delta_j\} \geq .9995 \text{ or } \{\delta_j\} \leq .0005), \quad (4a)$$

$$f_j = -5.0 - \{\delta_j\} \text{ for } 0.0005 < \{\delta_j\} \leq 0.5, \quad (4b)$$

$$f_j = -6.0 + \{\delta_j\} \text{ for } 0.5 < \{\delta_j\} \leq 0.9995 \quad (4c)$$

$$F_3 = \text{Number of integer valued } \delta_j \quad (5)$$

$$\text{and} \quad F_4 = (S + F_2 + ZF_3) / (Q - F_1) \quad (6)$$

where S, Q and Z are parameters which are calculated and varied as described in the algorithm.

At the s 'th iteration a constraint of the form

$$F_1 \geq \Omega - \xi_1 - \xi_2 - \dots - \xi_{s-1} - \xi_s \quad (7)$$

is added, where ξ_s is chosen so that

- (a) each constraint is feasible (i.e. does not violate any of the constraints of the original problem),
- (b) at least one of the basic non-integral δ_j (say δ_ρ) becomes integral or one of the integral δ_j changes its value by an integral amount, and
- (c) the objective function F_4 increases.

The selection of ξ determines the non-basic variable which is to enter the basis and the δ -variable which is to become integral (say at b_ρ). The tableau is augmented not only by the constraint (7) but by a pseudo-constraint of the form:

$$\delta_\rho \quad \{ \geq \text{ or } \leq \} b_\rho \quad (8)$$

This requires two new linearly dependent slack variables, say v^+ and v^- , so that the relationship (8) may be expressed as

$$\delta_\rho + v^+ - v^- = b_\rho \quad (8a)$$

These marking "constraints" permit changes in δ_ρ later if this suits us. They do not, of course, genuinely constrain the system. The variables v^+ and v^- are referred to as "marking slacks". The generation of these slacks may be kept under control by removing each of them that becomes basic, together with its non-basic counterpart. This can be done very easily.

At each iteration, the method usually generates a large number of feasible extreme points, using the erasable marking "constraints" referred to above. The effect on the objective function F_4 (and on the subsidiary functions, F_1 , F_2 and F_3) of a basis change to a new extreme point is calculated and used as a guide in the choice of a new basis. After the selection, the simplex method is used to transform the matrix.

Before starting the selection process on the first tableau, the problem is scaled by subtracting a constant from the objective function so that we have $\Omega = 3p$, and the parameters are set, as follows:

Define

$$\begin{aligned} \xi &= \text{decrease in } F_1 \\ \psi &= \text{increase in } F_2 \\ \beta &= \text{increase in } F_3 \end{aligned}$$

The parameter S is put equal to $5.5p$. The parameter Z is set so that for each feasible potential basis change with negative ψ and positive β , the quantity $(\psi + \beta Z)$ is non-negative. If there are no potential basis changes with negative ψ and positive β , Z is put equal to zero. The parameter Q is set so that for each basis change with positive $(\psi + \beta Z)$ and positive ξ the quantity

$$\pi = (Q - \Omega) (\psi + \beta Z) - \xi(S + F_2 + ZF_3)$$

is non negative. If there are no potential changes with positive $(\psi + \beta Z)$ and positive ξ , Q is put equal to $\Omega + 1$.

The criterion for the selection of a new basis is that we take the change giving the maximum of the non-negative π .

The algorithm consists of the following eight steps:

1. Set up the continuous optimal tableau in canonical form.
2. Calculate the feasible basis changes and label these.
3. Calculate the properties of each possible new basis.
4. Set the parameters S, Z and Q of the derived objective function,

$$F_4 = (S + F_2 + ZF_3)/(Q - F_1)$$

according to the structure of the problem.

5. Select the change to be made to give a new basis
 - (a) if we have an integer solution, and
 - (b) otherwise.
6. If no selection can be made, vary the parameters S, Z or Q or permit a selection. If no changes can be made, stop.
7. Transform the tableau.
8. Return to Step 2.

The constraints originally present, together with the lattice generated by the algorithm constitute a set of convex polyhedra whose extreme points are explored by the algorithm. Since there is a finite number of these extreme points, and since the revisiting of a basis is prohibited, the algorithm must, after a finite number of iterations, arrive at a maximum of F_4 with one of the following properties:

- (a) it is an integer solution; or
- (b) it is not an integer solution, but a change of parameters opens up some more feasible points; or

- (c) it is not an integer solution, and no change of parameters will open up more feasible points.

If an integer solution is found, the algorithm is applied to the region between this solution and the continuous optimum. This ensures that we do not finish up with a local optimum.