ON THE TYPE OF A POLYNOMIAL RELATIVE TO A CIRCLE - AN OPEN PROBLEM

JOHN J. H. MILLER,
TRINITY COLLEGE, DUBLIN.

ABSTRACT: This paper is a sequel to [2]. In it we prove a Theorem and Corollary which imply, in particular, the unproved Theorem 4 of that paper. In addition we analyze the reasons for the failure, in certain special cases, of an algebraic algorithm for determining the type of a polynomial relative to a circle. For background material the reader is referred in addition to [1], [3], [4] and the references contained in those papers.

Consider the polynomial with real or complex coefficients

$$f(z) = a_0 + a_1 z + \dots + a_n z^n$$

and assume that $a_0 \neq 0$ and $a_n \neq 0$. We say that f is of type (p_1, p_2, p_3) relative to a circle in the complex plane if, counting multiplicities, it has p_1 zeros inside, p_2 zeros on, and p_3 zeros outside the circle. To simplify the notation only, we shall assume in what follows that the circle is the unit circle centred at the origin.

We now give some simple definitions. The inverse z^* in the unit circle of any point z is defined by $z^* = 1/\bar{z}$ and that of the polynomial f by $f^*(z) = z^n \overline{f(z^*)}$. It is easy to see that f^* is the polynomial

$$f^*(z) = \bar{a}_n + \bar{a}_{n-1} z + \dots + \bar{a}_0 z^n.$$

Furthermore, if the zeros of f are denoted by z_j then the zeros of f^* are z_j^* . Thus f is of type (p_1, p_2, p_3) iff f^* is of type (p_3, p_2, p_1) . Clearly z_j lies on the unit circle iff $z_j^* = z_j$, and the zeros common to f and f^* are the zeros z_j lying on the unit circle together with any zero z_j of f whose inverse z_j^* is also a zero of f . If f and f^* have the same zeros and the same set of multiplicities then f is said to be self-inversive. It is not hard to see that a polynomial is a common factor of both f and f^* only if it is a self-inversive polynomial. Such a common factor is called a self-inversive factor of f , and the common factor of f and f^* of maximal degree is called the maximal self-inversive factor of f . It follows that f and f^* have the same maximal self-inversive factor and, in

particular, that they have the same zeros on the unit circle.

We now state and prove our main result.

THEOREM: Assume that f is a polynomial of degree n such that $f(0) \neq 0$, $|f^*(0)| - |f(0)| = 0$ and $f^*(0)f(z) - f(0)f^*(z)$ is not identically zero. Then $f^*(0)f(z) - f(0)f^*(z) = z^p g(z)$, where p is some integer satisfying $1 \leq p \leq [n/2]$ and g is a polynomial of degree $n - 2p$ such that $g(0) \neq 0$. Furthermore g is self-inversive, the maximal self-inversive factor of f is a factor of g and g is of type $(q, n - 2p - 2q, q)$ for some integer q satisfying $0 \leq q \leq [(n - 2p)/2]$. Moreover, f has at least $p + q$ zeros inside the unit circle, at least $p + q$ zeros outside the unit circle and at most $n - 2p - 2q$ zeros on the unit circle.

PROOF: If $f(z) = a_0 + a_1 z + \dots + a_n z^n$ then

$f^*(0)f(z) - f(0)f^*(z) = b_1 z + b_2 z^2 + \dots + b_n z^n$ where for convenience we put

$$b_j = \bar{a}_n a_j - a_0 \bar{a}_{n-j}$$

for $j = 0, 1, \dots, n$. We note that $b_0 = 0$ and under the hypotheses of the theorem also $b_n = 0$. Therefore, we have

$$(1) \quad \bar{b}_{n-j} = -\frac{a_n}{a_0} b_j, \quad j = 1, \dots, n-1$$

so that

$$(2) \quad b_j = 0 \text{ iff } b_{n-j} = 0, \quad j = 1, \dots, n-1$$

Furthermore, there is certainly some integer p satisfying $1 \leq p \leq [n/2]$ such that

$$b_0 = \dots = b_{p-1} = 0, \quad b_p \neq 0$$

and hence by (2) also

$$b_n = \dots = b_{n-p+1} = 0, \quad b_{n-p} \neq 0.$$

It then follows that

$$(3) \quad f^*(0)f(z) - f(0)f^*(z) = z^p g(z)$$

where

$$g(z) = b_p + \dots + b_{n-p} z^{n-2p}.$$

Moreover, by (1), it is easy to see that

$$g^*(z) = - \frac{a_n}{a_0} g(z)$$

so that g and g^* have the same zeros and hence g is self-inversive. Since the maximal self-inversive factor of f is also a factor of f^* , it follows from (3) that it is also a factor of g . Because g is self-inversive and is of degree $n - 2p$ it must be of type $(q, n - 2p - 2q, q)$ for some q satisfying $0 \leq q \leq [(n - 2p)/2]$.

To complete the proof of the theorem we put

$$h(z; \lambda) = f^*(0) f(z) - \lambda f(0) f^*(z), \quad 0 \leq \lambda \leq 1.$$

This is a polynomial of degree at most n . The coefficient of z^n is $|f^*(0)|^2 - \lambda |f(0)|^2 = (1 - \lambda) |f(0)|^2$, which is non-zero for all λ satisfying $0 \leq \lambda < 1$. Thus, for all such λ ,

$$h^*(z; \lambda) = -\lambda \overline{f(0)} f(z) + \overline{f^*(0)} f^*(z).$$

If α is a common zero of f and f^* then it is clearly also a common zero of h and h^* . Conversely, if α is a common zero of h and h^* then

$$\begin{aligned} f^*(0) f(\alpha) - \lambda f(0) f^*(\alpha) &= 0 \\ -\lambda f(0) f(\alpha) + \overline{f^*(0)} f^*(\alpha) &= 0. \end{aligned}$$

But the determinant of the coefficient matrix of this system of equations is

$$|f^*(0)|^2 - \lambda^2 |f(0)|^2 = (1 - \lambda^2) |f(0)|^2$$

which is non-zero for all λ satisfying $0 \leq \lambda < 1$. The only solution therefore is the trivial one

$$f(\alpha) = 0, \quad f^*(\alpha) = 0$$

so that α is a common zero of f and f^* . We have thus shown that f and h have the same maximal self-inversive factor for all λ satisfying $0 \leq \lambda < 1$.

We now write $f = \phi k$, where ϕ is the maximal self-inversive factor of f . Then $f^* = \phi^* k^*$ and from what we have just proved it follows that

$$h(z; \lambda)/\phi(z) = f^*(0) k(z) - \lambda f(0) k^*(z)$$

has no self-inversive factor for all λ satisfying $0 \leq \lambda < 1$. In particular then

h/ϕ has no zeros on the unit circle and is therefore of constant type for all such λ . Since ϕ is independent of λ it follows that h is also of constant type for all such λ .

We look now at the zeros of h for λ arbitrarily close to 1. Writing $\lambda = 1 - \varepsilon$, where $\varepsilon > 0$ is arbitrarily small, we have

$$\begin{aligned} h(z; 1 - \varepsilon) &= f^*(0) f(z) - f(0) f^*(z) + \varepsilon f(0) f^*(z) \\ &= z^p g(z) + \varepsilon f(0) f^*(z). \end{aligned}$$

Because the zeros of a polynomial are continuous functions of its coefficients, it is easy to see that as $\varepsilon \rightarrow 0$ exactly p zeros of h converge to the origin, $n - 2p$ zeros of h converge to the zeros of g and the remaining p zeros of h converge to infinity. But, since h is of constant type for all λ satisfying $0 < \lambda < 1$, none of these zeros may cross the unit circle as $\varepsilon \rightarrow 0$ (some may of course reach the unit circle in the limit). Furthermore, in the limit we have

$$h(z; 1) = z^p g(z)$$

which has $p + q$ zeros inside, $n - 2p - 2q$ zeros on and q zeros outside the unit circle. It follows therefore that, for all λ satisfying $0 < \lambda < 1$, $h(z; \lambda)$ has at least $p + q$ zeros inside, at least $p + q$ zeros outside and at most $n - 2p - 2q$ zeros on the unit circle. In particular, since

$$h(z; 0) = f^*(0) f(z)$$

the same is true of f . This concludes the proof of the theorem.

Notice that f and g will have the same zeros on the unit circle if no zero of $h(z; 1 - \varepsilon)/\phi(z)$ reaches the unit circle in the limit as $\varepsilon \rightarrow 0$. In this case the argument in the previous paragraph shows that f has then exactly $p + q$ zeros inside, $n - 2p - 2q$ zeros on and $p + q$ zeros outside the unit circle.

Notice, also, that g will be the maximal self-inversive factor of f if the zeros of $h(z; 1 - \varepsilon)/\phi(z)$ neither reach the unit circle nor form self-inversive pairs in the limit as $\varepsilon \rightarrow 0$. We have thus established the non-trivial part of the following corollary.

COROLLARY: Suppose that f satisfies the hypotheses of the theorem and that f and g have the same zeros on the unit circle. Then f is of type $(p + q, n - 2p - 2q, p + q)$. If, in addition, g is a factor of f then g is the maximal self-inversive factor of f .

REMARK: The above corollary shows that the structure of the polynomial $f^*(0) f(z) - f(0) f^*(z) = z^p g(z)$ determines the type of f except when g has zeros on the unit circle which either are not zeros of f or are zeros of f but have higher multiplicities.

The observation that, if $f^*(0) f(z) - f(0) f^*(z) = z^p g(z)$ then the corresponding expression for f^* is just $-z^p g(z)$, indicates that it is unlikely that the structure of $f^*(0) f(z) - f(0) f^*(z)$ determines the type of f in the exceptional cases noted in the Remark. For example, if $f(z) = 6z^3 - 31z^2 + 41z - 6$ then $f^*(0) f(z) - f(0) f^*(z) = 60z(z+1)$, so that $p = 1$, $q = 0$ and $g(z) = z + 1$. We conclude then from the Theorem that f has at least one zero inside the unit circle, at least one zero outside the unit circle and at most one zero on the unit circle. Since the zero of g is not a zero of f , the hypotheses of the Corollary are not fulfilled and we do not know from which side of the unit circle this zero has come. Thus, a priori, the type of f could be either $(2, 0, 1)$ or $(1, 0, 2)$. Now consider the polynomial $e(z) = -6z^3 + 41z^2 - 31z + 6$. Then $e^*(0) e(z) - e(0) e^*(z) = -60z(z+1)$ so that the type of e could, a priori, also be either $(2, 0, 1)$ or $(1, 0, 2)$. But $e = f^*$ and so e is of one type iff f is of the other.

The following problem therefore remains open: find an efficient algebraic algorithm to determine the type of a polynomial relative to the unit circle, which works for all polynomials satisfying the hypotheses of the Theorem. By efficient we mean that full use should be made of the relations between the coefficients of the polynomial which are implied by the hypotheses of the Theorem (there is an algebraic algorithm (see [3]) which works in all cases, but it is not efficient in the above sense). Such an algorithm could then be implemented algebraically on a computer and it could be used to determine the stability properties of polynomials. It could also be used, in conjunction with Lehmer's algorithm, to isolate and refine with exact error bounds the locations of the zeros of polynomials.

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