PENALTY METHODS AND SOME APPLICATIONS OF MATHEMATICAL PROGRAMMING

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I. The problem of minimization of the convex function $f: R^n \rightarrow R$ on the convex solid compact set $\Omega \subset R^n$ is under consideration.

Theorem I. Let functions $\Phi_k: R^n \rightarrow R, k=1,2,\dots$ be convex in the domain $\Omega_\delta = \{x: \inf_{y \in \Omega} \|x-y\| \leq \delta\}$ at some $\delta > 0$ and quasi-convex on the entire R^n ;

$$1^\circ \lim_{k \rightarrow \infty} \Phi_k(x) = 0, \text{ if } x \in \text{int } \Omega;$$

$$2^\circ \lim_{k \rightarrow \infty} \Phi_k(x) = +\infty, \text{ if } x \notin \Omega.$$

Then functions $F_k(x) = f(x) + \Phi_k(x)$ starting from the certain number achieve their absolute minima; corresponding sequence $\{x^k\}$ of minima is bounded, its any limiting point belongs to Ω and brings the minimum of function f on Ω .

It is necessary to emphasize that in literature the question of convergence of penalty method is usually considered for specific sequences of functions Φ_k . The above theorem establishes the conditions of the method's convergence in terms of the structure of penalty functions. In [2] functions of the form $\Phi_k(x) = A_k G(x)$, where $A_k \rightarrow \infty$, are investigated and the result depends only on the structural properties of function G .

Theorem 2. Let conditions of theorem I be satisfied and for the boundary points of set Ω in equalities

$$\Phi_k(x) \geq d > 0, \quad k = 1, 2, \dots$$

hold at that.

Then if K is sufficiently large, points x^k are located strictly inside Ω .

Let $\Omega = \{x : g^j(x) \leq 0, j = 1, 2, \dots, m\}$, where $g^j: R^n \rightarrow R$ are convex functions. Suppose that f, g^j are differentiable and

$\inf_{x \in R^n} \max_{1 \leq j \leq m} g^j(x) < 0$. The following statement is correct.

Theorem 3. Let all functions Φ_k be differentiable and satisfy conditions of theorem 1. Let also

$$\nabla \Phi_k(x) = \sum_{j=1}^m \Psi_k^j(x) \nabla g^j(x),$$

where functions Ψ_k^j are non-negative and continuous on R^n and for any j the sequence $\{\Psi_k^j(z^k)\}$ vanishes, if $\lim_{k \rightarrow \infty} g^j(z^k) < 0$.

Then the sequence $\{y^k\}$, where $y^k = (y_1^k, \dots, y_m^k)$, $y_j^k = \Psi_k^j(x^k)$ ($j = 1, 2, \dots, m$) is bounded, every limiting point of the sequence $\{x^k, y^k\}$ is the solution of the dual problem.

The points (x^k, y^k) mentioned in the theorems are feasible for the dual problem. And it means that starting from the certain number at every step of the process bilateral values for the optimal value of the objective function of the initial problem are automatically (calculation of values $\Psi_k^j(x^k)$ will be required only) obtained, if the conditions of theorem 2 are satisfied.

The first theorem of this type was obtained by McCormik and Fiacco [1] for functions $\Phi_k(x) = -r_k \sum_{j=1}^m \frac{1}{g^j(x)}$ ($r_k > r_{k+1} > 0, r_k \rightarrow 0$) and generalized for some special classes of penalty functions in [2].

2. Let $V \subset L_2(Q)$ be Hilbert space and the bilinear form

$$\Phi(u, v) = \int_Q \left(\sum_{i,j=1}^n d_{ij} \frac{\partial u}{\partial x_i} \frac{\partial v}{\partial x_j} + d_0 uv \right) dx$$

be continuous on V . The problem of minimization of the quadratic functional

$$J(u) = \Phi(u, u) - 2 \int_Q f u dx$$

on the convex closed set $K \subset V$ is under consideration.

Here $Q \subset R^n$ is bounded domain with piece-wise smooth boundary Γ , $d_{ij} \in C^1(\bar{Q})$, $d_0 \in C(\bar{Q})$, $f \in L_2(Q)$. It is assumed that the quadratic form $\Phi(u, u)$ is positive definite on V .

On the basis of some modification of Ritz's method (and in certain cases, of Courant's method too) the given problem may be approximately expressed as the problem of convex programming. To make this approach efficient the set K has to satisfy the following conditions (!):

(!) Systems of functions

$\{\xi_{ij}\}$, $j=1,2,\dots,k_i$; $i=1,2,\dots$; $\xi_{ij} \in V$,

and convex closed sets $N_\ell \subset R^{K_\ell}$, $\ell=1,2,\dots$, such that

1) $\sum_{\tau=1}^{K_\ell} a_\tau^\ell \xi_{\ell\tau} \in K$ at all $a^\ell \in N_\ell$, $\ell=1,2,\dots$;

2) for any element $v \in K$ for $\varepsilon > 0$ it is possible to indicate number T and vectors $a^T \in N_T$, $a^{T+i} \in N_{T+i}, \dots$ so that

$$\left\| \sum_{\tau=1}^{K_{T+i}} a_\tau^{T+i} \xi_{T+i,\tau} - v \right\|_V \leq \varepsilon;$$

3) for minimization of the convex function of K_ℓ variables on set N_ℓ usual methods of convex programming are applicable, - are known or may be efficiently constructed.

Let us note that the requirement of linear independence of coordinate functions being natural for the classic Ritz's scheme may turn rather burdening and complicating the structure of set N_ℓ in this case.

Approximated solution of the initial problem is solved in the following way.

Taking fixed ℓ , we determine element z^ℓ giving the minimum to functional J in the set of elements of the type $u^\ell = \sum_{\tau=1}^{K_\ell} a_\tau^\ell \xi_{\ell\tau}$, where $a^\ell \in N_\ell$.

Functional

$$\Psi(a^\ell) = J(u^\ell) = \sum_{\tau,\gamma=1}^{K_\ell} c_{\tau\gamma} a_\tau^\ell a_\gamma^\ell - 2 \sum_{\tau=1}^{K_\ell} f_\tau a_\tau^\ell,$$

where

$$c_{\tau\gamma} = \int_Q \left(\sum_{i,j=1}^n d_{ij} \frac{\partial \xi_{\ell\tau}}{\partial x_i} \frac{\partial \xi_{\ell\gamma}}{\partial x_j} + d_0 \xi_{\ell\tau} \xi_{\ell\gamma} \right) dx, \quad f_\tau = \int_Q f \xi_{\ell\tau} dx$$

is the sum of a non-negative-definite quadratic form and a linear function depending on K_ℓ variables. Besides $J(u^\ell) \geq \min_{u \in K} J(u)$ at all $a^\ell \in N_\ell$, so that the unknown element z^ℓ exists and is determined by vector $\bar{a}^\ell$ giving minimum of Ψ on N_ℓ (such vector $\bar{a}^\ell$ and hence element z^ℓ are generally speaking not unique).

It's clearly seen that the sequence $\{z^\ell\}$, $\ell=1,2,\dots$, is a minimizing one in the initial problem which due to the positive definiteness of $\Phi(u,u)$ results in $\|z^\ell - z\|_V \xrightarrow{\ell \rightarrow \infty} 0$.

With all this to find the approximate solution z^ℓ it is necessary to solve the problem of convex programming in space R^{K_ℓ} .

Three examples of the specific accomplishment of the mentioned scheme are considered below.

It is supposed that $K_1=1, K_2=2, \dots$, $\xi_{\ell i} = \xi_i$ ($\ell=1,2,\dots$) which in case $K=V$ corresponds to the classical scheme of Ritz's method.

First of the problems considered - the problem of elastic-plasticity - has the following variation wording (more detailed wording of problems under consideration is in [3]):

$$J(u) - \min_{u \in K \equiv K' = \{v \in W_2^1(Q) : |\nabla v(x)| \leq 1 \text{ at } Q\}}.$$

To solve it the complete system of linear-independent functions $\{\xi_k\}$ is introduced in the space $C_0^1(\bar{Q})$. The problem of functional J minimization on the set of linear combinations $u^\ell(x) = \sum_{i=1}^{\ell} a_i \xi_i(x)$ looks like:

$$J(u^\ell) - \min_{|\nabla u^\ell(x)| \leq 1 \text{ at } x \in \bar{Q}}.$$

The latter inequality can be put down in the following way:

$$\sum_{k,p=1}^{\ell} a_k a_p \sum_{i=1}^n \frac{\partial \xi_k(x)}{\partial x_i} \frac{\partial \xi_p(x)}{\partial x_i} \leq 1 \text{ at } x \in \bar{Q}.$$

To give the set N_ℓ in the above way point 3 of the condition (!) is violated. However due to the choice of the specific penalty function of the penalty method it is possible to avoid the preliminary approximation of set N_ℓ .

Namely, under the natural supposition that at certain p

$$\max_{x \in \bar{Q}} \min_{1 \leq i \leq n} \left| \frac{\partial \xi_p(x)}{\partial x_i} \right| > 0,$$

set N_ℓ is a solid compactum it is possible to prove that the sequence of penalty functions

$$\Phi_k(a) = \int_{\bar{Q}} e^{A_k (|\nabla u^\ell(x)|^2 - 1)} dx$$

at $A_k \geq 0$, $A_k \rightarrow \infty$ satisfies the conditions of theorems 1 and 2. It is necessary to emphasize that the smoothness degree of an integrated function completely depends on the smoothness of coordinate functions so that with the appropriate choice of the latter ones for calculating $\Phi_k(a)$ cubature formulas of the high degree of accuracy may be applied.

The approach described related to the construction of penalty integral function is universal enough.

In the two variation problems that follow the specificity of set K is essentially taken into consideration which permits us to come to simpler problems of convex programming.

In the second problem

$$V = W_2'(Q), \quad K \equiv K^2 = \{u \in V : u|_{\Gamma} \geq 0\}.$$

To solve it we shall introduce into consideration the systems of functions $\{\varphi_k\}$ and $\{\psi_k\}$ possessing the following properties:

- a) $\psi_k \in C_0'(\bar{Q})$ and the system $\{\psi_k\}$ is complete in space $C_0'(\bar{Q})$
 b) $\varphi_k \in C'(\bar{Q})$, $\varphi_k|_{\Gamma} \geq 0$ and the system $\{\varphi_k\}$ is complete on the boundary in the following sense: for any function $u \in C(\bar{Q})$, $u|_{\Gamma} \geq 0$ and $\varepsilon > 0$ there exist m and non-negative $\alpha_1, \alpha_2, \dots, \alpha_m$ such that $\|u - \sum_{k=1}^m \alpha_k \varphi_k\|_{C(\Gamma)} \leq \varepsilon$ holds.

Let us note that requirements concerning coordinate functions may be weakened by substitution of $C_0'(\bar{Q})$ by $\dot{W}_2'(Q)$, $C'(\bar{Q})$ by $W_2'(Q)$.

Fixing t and p we look for the minimum of functional J on the set of linear combinations

$$u^{t,p}(x) = \sum_{i=1}^t a_i \psi_i(x) + \sum_{k=1}^p b_k \varphi_k(x)$$

under the condition of non-negativity of coefficients b_k .

With all this the sequence of coordinate functions is obtained by mixing sequences $\{\psi_i\}$ and $\{\varphi_i\}$ (for the purpose of simplicity we shall consider $\xi_i = \psi_{i/2}$ if i is even, $\xi_i = \varphi_{(i+1)/2}$ if i is odd) and sets

$$N_\ell = \{a^\ell \in R^\ell : a_{2\tau-1}^\ell \geq 0, \tau = 1, 2, \dots, \lfloor \frac{\ell+1}{2} \rfloor\}$$

correspond to the sequence $\{\xi_i\}$ formed by it.

It means that to find the approximated solution $\mathcal{Z}^\ell$ it will be necessary in this case to solve the simplest problem of quadratic programming. Without loss of generality we may believe that $\bar{Q}$ is located strictly inside unit cube $S = \{x : 0 \leq x_i \leq 1, i = 1, 2, \dots, n\}$.

Then functions of the type

$$x_1^{k_1} x_2^{k_2} \dots x_n^{k_n} (1-x_1)^{z_1} (1-x_2)^{z_2} \dots (1-x_n)^{z_n} \quad (*)$$

with all possible non-negative indices may be taken as φ_i . System $\{\psi_i\}$ is subordinate to standard requirements of Ritz's classic scheme.

In the third problem

$$V = \dot{W}_2'(Q), \quad K \equiv K^3 = \{u \in V : u(Q) \geq 0\}.$$

The construction of minimization sequence results in the solution of the simplest quadratic programs too. The system of differentiable in Q functions $\{\xi_i\}$, $\xi_i \in C_0(\bar{Q})$, is constructed such that for any non-negative function $u \in C_0'(\bar{Q})$ and $\varepsilon > 0$ there exist m and non-negative $\alpha_1, \alpha_2, \dots, \alpha_m$ for which

$$\|u - \sum_{i=1}^m \alpha_i \xi_i\|_{C(\bar{Q})} \leq \varepsilon, \left\| \frac{\partial u}{\partial x_k} - \sum_{i=1}^m \alpha_i \frac{\partial \xi_i}{\partial x_k} \right\|_{L_2(Q)} \leq \varepsilon, k=1,2,\dots,n$$

holds. Further we set $N_\varepsilon = \{a' \in R^l : a' \geq 0\}$. Under the condition $Q \subset S$ as ξ_i functions of type

$$\omega(x) x_1^{k_1} x_2^{k_2} \dots x_n^{k_n} (1-x_1)^{r_1} (1-x_2)^{r_2} \dots (1-x_n)^{r_n} \quad (**)$$

may be taken where $\omega \in C(\bar{Q})$, ω is differentiable and positive on Q .

It is possible to prove that if the boundary Γ is not too bad (i.e. in case of convex or Lipschitz domains) the abovementioned suppositions ensure the correctness of the conditions (!) in all problems.

Theorem 4. The system of functions (*) satisfies condition b). The system of functions (**) is complete in the sense mentioned above.

It is necessary to indicate that for the solution of the last two problems within the framework of the scheme described it is possible to use functions which are usually applied in variation-difference approximation. However, dimension of the problems of quadratic programming obtained will naturally be very large and this may turn out an obstacle for their practical solution.

On the other hand the negative effect related to the bad definiteness of matrices when using Ritz's classic scheme will not evidently be very significant as the conditions of non-negativity which by the meaning of the problem are essential, cut off the vicinity of the absolute minimum of the quadratic functional.

References.

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