ON THE NUMERICAL APPROXIMATION OF PROBLEMS OF IMPULSE CONTROLS

J.L. LIONS

COLLEGE DE FRANCE and IRIA-LABORIA

Lecture at I.F.I.P. Meeting, NOVOSIBIRSK, July 1974.

INTRODUCTION

Problems of impulse control arise in a large number of situations in inventory theory (cf. A.F. VEINOTT Jr. [1]).

From a general point of view, they correspond to problems of optimal control when the state is finite dimensionnal ⁽¹⁾ and it is given by an Ito's differential equation (or an ordinary differential equation) with jumps and where actually these jumps are indeed the control variable ⁽²⁾.

In a number of recent publications with A. BENSOUSSAN, (cf. bibliography), we have observed that, by using dynamic programming, one is lead, in order to solve these classes of impulse control problems, to partial differential inequalities.

The partial differential inequalities correspond to new free boundary problems and in order to solve these problems (both from a theoretical and a numerical viewpoint), we have introduced with Bensoussan a new tool : the quasi-variational inequalities (Q.V.I. for short).

The aim of this lecture is to recall some general results relative to the Q.V.I., in the evolution and in the stationary cases and to give some ideas

⁽¹⁾ If the state is infinite dimensional, one is lead to partial differential inequalities in infinite dimensional spaces.

⁽²⁾ We have to choose in an optimal manner (if possible) the total number of jumps, the times where the jumps arise, and the "amount" of the jumps (with some constraints on these "amounts").

on the numerical approximation of the solution of Q.V.I. ; we also report briefly on some numerical experiments which have been made, mainly in connection with applications in inventory theory (cf. BENSOUSSAN-GOURSAT-MAAREK[1]), or in energy policies related to optimal use of dams (cf. BRETON-LEGUAY[1]).

The numerical experiments reported in this paper have been conducted by GOURSAT, LEGUAY, MAURIN at IRIA-LABORIA.

Let us notice that the Q.V.I. turn out to be useful in a classical free boundary problem in infiltration theory, (cf. BAIocchi[1] ; for earlier work using Variational Inequalities (V.I.), cf. BAIocchi-MAGENES[1]).

The Q.V.I. also appear in a natural context in the optimal control of systems governed by V.I. ⁽¹⁾; cf. F. MIGNOT[1].

They also appear in the search for Nash points of equilibrium (cf. BENSOUSSAN-LIONS[2]).

The plan is as follows :

1. Problems of impulse controls.
 2. Formulation in terms of Q.V.I.
 3. Numerical approximation of the stationary case (I).
 - 3.1. General remarks.
 - 3.2. An example.
 4. Numerical approximation of the evolution case.
 - 4.1. General remarks.
 - 4.2. An example.
 5. Numerical approximation of the stationary case (II).
 - 5.1. The problem of the dimension.
 - 5.2. Decomposition methods.
 6. Problems with delays.
- Bibliography.

⁽¹⁾ And therefore in the optimal control of systems whose state is given by the solution of a free boundary problem.

1. - PROBLEMS OF IMPULSE CONTROL

We suppose that the state $y = y(s;v)$ of the system at time s when one applies the control v (to be defined below) is a vector of $\mathbb{R}^n$; we can think of y as being the amount of goods we have in our inventory. We suppose that

$$(1.1) \quad y(t;v) = x.$$

The state equation is an Ito's differential equation with jumps, which corresponds to the following situation : there is a demand, which in the interval t, s is given by

$$(1.2) \quad D(t,s) = -\mu(s-t) - \sigma(w(s) - w(t))$$

where

$$(1.3) \quad \mu = \{\mu_1, \dots, \mu_n\} \in \mathbb{R}^n \text{ (the deterministic part of the demand)}$$

and where, to simplify, we assume that

$$(1.4) \quad \sigma = \sigma \text{ (Identity),}$$

and where $w(t)$ is a Standard Wiener process in $\mathbb{R}^n$ ⁽¹⁾.

The control v (or the policy) is of the following form :

$$(1.5) \quad v = \{\theta^1, \xi^1; \theta^2, \xi^2; \dots; \theta^N, \xi^N; N\}$$

where

$$(1.6) \quad \left| \begin{array}{l} t \leq \theta^1 < \theta^2 < \dots < \theta^N \leq T = \text{given number, finite or} \\ \text{infinite (the horizon of the problem),} \end{array} \right.$$

$$(1.7) \quad \xi^i = \text{amount of goods ordered at time } \theta^i,$$

$$(1.8) \quad N = \text{number of orders ;}$$

in (1.5) the θ^i, ξ^i, N are at our disposal, with some restrictions on the ξ^i 's,

⁽¹⁾ We can also take care of the case when we have moreover a Poisson process.

Cf. BENSOUSSAN and the A.

that we can express by :

$$(1.9) \quad \xi^i \in \mu, \mu = \text{closed set of } \mathbb{R}^n;$$

the θ^i , ξ^i and N are random functions.

The state is now given, assuming that the delivery is made without delays ⁽¹⁾
by

$$(1.10) \quad \begin{cases} dy = -\mu ds - \sigma db(s) = \sum_{i=1}^N \xi^i \delta(s-\theta^i) ds \\ y(0) = x. \end{cases}$$

The pay-off, or cost function, is supposed to be given by

$$(1.11) \quad J(v) = E \left\{ \int_t^T f(y(s;v)) ds + kN(v) \right\},$$

where E = expectation.

Remark 1.1.

In the deterministic case, $\sigma = 0$, and one has the same kind of pay-off, without of course taking the expectation in (1.11). ■

In (1.11), f is a given ≥ 0 function, which corresponds to a storage cost or to a shortage cost, and k is a given > 0 constant : the set-up cost.

Remark 1.2.

We refer to the bibliography, for much more general ordering costs, cf. in particular, Remark 2.2. Other cases will also be considered below. ■

The problem is now to characterize

$$(1.12) \quad \inf_v J(v) = u(x,t)$$

and to obtain the (or one) optimal policy, if it exists.

⁽¹⁾ For the case of deliveries with delays, cf. BENSOUSSAN and the A., in the bibliography and Section 6 below.

2. - FORMULATION IN TERMS OF QUASI-VARIATIONAL INEQUALITIES

One can prove that the function u defined by (1.12) satisfies the following set of inequalities and equalities

$$(2.1) \quad \left| \begin{array}{l} -\frac{\partial u}{\partial t} + \sum \mu_i \frac{\partial u}{\partial x_i} - \frac{\sigma^2}{2} \Delta u - f \leq 0, \\ u - M(u) \leq 0 \\ (-\frac{\partial u}{\partial t} + \sum_{i=1}^n \mu_i \frac{\partial u}{\partial x_i} - \frac{\sigma^2}{2} \Delta u - f) (u - M(u)) = 0, \end{array} \right.$$

where

$$(2.2) \quad Mu(x, t) = k + \inf_{\xi \in \mu} u(x + \xi, t),$$

and where u is subject to

$$(2.3) \quad u(x, T) = 0.$$

If the state is subject to stay in a set $\bar{\mathcal{O}} \in \mathbb{R}^n$, where $\mathcal{O}$ is an open set of $\mathbb{R}^n$, assumed to be bounded (this being unessential), then one has to add to (2.1) (where $x \in \mathcal{O}$)⁽¹⁾, the boundary conditions

$$(2.4) \quad \left| \begin{array}{l} \frac{\partial u}{\partial \nu} \leq 0, \quad u - M(u) \leq 0, \quad \frac{\partial u}{\partial \nu} (u - M(u)) = 0 \\ \text{on } \Gamma = \partial \mathcal{O}, \quad t < T \end{array} \right.$$

($\frac{\partial}{\partial \nu}$ = normal derivative directed toward the exterior of $\mathcal{O}$). ■

Formulation as a Q.V.I. of evolution.

For $u, v \in H^1(\mathcal{O})$ ⁽²⁾, we define

$$(2.5) \quad a(u, v) = \frac{\sigma^2}{2} \int_{\mathcal{O}} \frac{\partial u}{\partial x_i} \frac{\partial v}{\partial x_i} dx + \int_{\mathcal{O}} \mu_i \frac{\partial u}{\partial x_i} v dx.$$

⁽¹⁾ In definition (2.2) x and $x + \xi$ should be in $\bar{\mathcal{O}}$.

⁽²⁾ Sobolev's space of order 1. Cf. SOBOLEV[1].

For $f, v \in L^2(\mathcal{O})$, $(f, v) = \int_{\mathcal{O}} f v \, dx$ as usual.

It is now a simple matter to show that u is characterized by the Q.V.I. of evolution

$$(2.6) \quad \left\{ \begin{array}{l} - \left(\frac{\partial u}{\partial t}, v-u \right) + a(u, v-u) - (f, v-u) \geq 0 \quad \forall v \in H^1(\mathcal{O}) \\ \text{such that } v \leq Mu(x, t), \\ u - Mu \leq 0, \\ u(x, T) = 0. \end{array} \right.$$

It is this formulation that we shall use in the numerical approximation. ■

The stationary case.

Let us consider now the case when

$$T = +\infty$$

and let us assume that $w(t+h)-w(t)$ has a probability law which depends only on h . One considers then the pay-off :

$$(2.7) \quad J(v) = E \left\{ \int_0^\infty e^{-\alpha s} f(y(s; v)) ds + k \sum_{i=1}^\infty e^{-\alpha \theta_i} \right\}$$

where $\alpha > 0$ is the discount rate.

If we set

$$(2.8) \quad u(x) = \inf. J(v),$$

then one can show that u is characterized by the (stationary) Q.V.I.

$$(2.9) \quad a(u, v-u) \geq (f; v-u) \quad \forall v \leq M(u), \quad u \leq M(u);$$

let us notice that now $a(u, v)$ is given by :

$$(2.10) \quad a(u, v) = \frac{\sigma^2}{2} \sum \int_{\mathcal{O}} \frac{\partial u}{\partial x_i} \frac{\partial v}{\partial x_i} \, dx + \sum u_i \int_{\mathcal{O}} \frac{\partial u}{\partial x_i} v \, dx + \alpha \int_{\mathcal{O}} u v \, dx. \quad \blacksquare$$

Remark 2.1.

Assuming that $f \geq 0$ belongs to $L^\infty(\mathcal{O})$, one can prove that (2.9) admits a unique solution $u \geq 0$. (Cf. BENSOUSSAN-GOURSAT-LIONS [1], L. TARTAR [1], Th. LAETSCH [1]).

One can also prove that (2.6) admits a unique solution (cf. BENSOUSSAN-LIONS) by arguments using simultaneously functional analysis and probability theory; a direct proof using only (2.6) is not known yet. ■

Remark 2.2.

The structure of M , as given by (2.2), corresponds to the set up cost k being fixed and independent from the goods which are ordered and of the amount which is ordered. But actually the method is entirely general; to fix ideas, let us assume that we have two goods ($n = 2$) and that we pay k_1 if we order only the i^{th} item ($i=1,2$) and that we pay $k (< k_1 + k_2)$ if we order simultaneously the two goods. Then :

$$(2.11) \quad \left\{ \begin{array}{l} M(v) = \inf(M_1(v), M_2(v), M_3(v)) , \\ M_1(v)(x) = k_1 + \inf.v(x_1 + \xi_1, x_2), \quad \xi_1 \geq 0, \\ M_2(v)(x) = k_2 + \inf.v(x_1, x_2 + \xi_2), \quad \xi_2 \geq 0, \\ M_3(v)(x) = k + \inf.v(x_1 + \xi_1, x_2 + \xi_2), \xi_i \geq 0, \end{array} \right.$$

and one has the same formulations (and the same results) than above with this new function M .

For an "axiomatic" study of M , we refer to BENSOUSSAN-LIONS, loc. cit., TARTAR [1]. Cf. also JOLY-MOSCO [1]. ■

3. - NUMERICAL APPROXIMATION OF THE STATIONARY CASE (I).3.1. General remarks.

Let u^0 be a given constant, satisfying

$$(3.1) \quad u^0 \geq \frac{1}{\alpha} \sup. f.$$

Let us introduce λ such that

$$(3.2) \quad a(v, v) + \lambda \int_{\mathcal{O}} v^2 dx \geq c \|v\|_{H^1(\mathcal{O})}^2, \quad c > 0, \quad v \in H^1(\mathcal{O})$$

and let us inductively define u^n by the solution of the V.I. ⁽¹⁾

$$(3.3.) \quad \left| \begin{array}{l} a(u^n, v - u^n) + \lambda(u^n, v - u^n) \geq (f + \lambda u^{n-1}, v - u^n), \\ \forall v \leq M(u^{n-1}), \quad u^n \leq M(u^{n-1}). \end{array} \right.$$

One proves that

$$(3.4) \quad u^n \rightarrow u \quad \text{in } L^p(\mathcal{O}) \text{ strongly and in } H^1(\mathcal{O}) \text{ weakly.} \quad \blacksquare$$

The next step is to approximate u^n ; one uses the standard methods for the numerical approximation of V.I. (Cf. CEA-GLOWINSKI [1] GLOWINSKI-LIONS-TREMOLIERES [1] and the bibliography of these works); in the numerical experiments which have been made so far, one has used the method of finite differences, and not the finite element methods (cf. Remark 3.1 below); let us denote by u_h^n the solution of :

$$(3.5) \quad \left| \begin{array}{l} a_h(u_h^n, v_h - u_h^n) + \lambda(u_h^n, v_h - u_h^n) \geq (f + \lambda u_h^{n-1}, v_h - u_h^n), \\ \forall v_h \leq M(u_h^{n-1}), \quad u_h^n \leq M(u_h^{n-1}), \end{array} \right.$$

where v_h belongs to the space V_h generated by the characteristic functions of non overlapping cubes corresponding to standard finite differences approximations : $a_h(u_h, v_h)$ denotes the approximation of $a(u, v)$ obtained after replacing $\frac{\partial}{\partial x_i}$ by their approximation ∇_{ih} with "mesh" h ; for the approximation of $\Sigma \mu_i \frac{\partial}{\partial x_i}$ one "follows the characteristics" (which are the trajectories of the state, if $\sigma = 0$).

Remark 3.1.

By virtue of the fact that if $u_h, v_h \in V_h$ as defined above, then $\sup(u_h, v_h) \in V_h$, one can prove that

$$(3.6) \quad \left| \begin{array}{l} u_h^n \rightarrow u_h \quad (\text{as } n \rightarrow \infty) \text{ in } L^p(\mathcal{O}) \text{ strongly,} \\ \nabla_{ih} u_h^n \rightarrow \nabla_{ih} u_h \text{ in } L^2(\mathcal{O}) \text{ weakly} \end{array} \right.$$

where u_h is the solution of the "discrete Q.V.I."

(1) Variational Inequality, which admits a unique solution by virtue of LIONS-STAMPACCHIA [1] and of (3.2).

$$(3.7) \quad \left| \begin{array}{l} a_h(u_h, v_h - u_h) \geq (f, v_h - u_h), \\ \forall v_h \in V_h, \quad v_h \leq M(u_h), \quad u_h \leq M(u_h). \end{array} \right.$$

It is an open problem to know if this result still holds true if one replaces finite differences by suitable finite elements. ■

Remark 3.2.

If we start the iterative algorithm (3.3) by

$$u_0 = 0$$

and if we denote by u_n the corresponding result, then

$$(3.8) \quad u_0 \leq u_1 \leq \dots \leq u_{n-1} \leq u_n \leq \dots \leq u.$$

It is not known if, in general $u_n \rightarrow u$, but this seems to be true in all the numerical computations made so far.

Remark 3.3

It has not been proved that $u_h \rightarrow u$ as $h \rightarrow 0$. But this is conjectured as highly probable and it is "proved" in all the numerical experiments which have been made. The conjecture would be proved if we could prove that

$$(3.9) \quad u_h \geq u_{h'}, \quad \text{if } h \geq h'$$

which is also a conjecture

Let us notice that all these questions are related to one sided approximations of V.I. : cf. G. Strang[1], U. Mosco and G. Strang[1]. They are also related to the discrete maximum principle, which is a non trivial question for the approximation by finite elements. (cf. Ph. CIARLET and P.A. RAVIART[1]).

3.2. An example

We briefly report on an example of GOURSAT [1]; the open set $\mathcal{O} \subset \mathbb{R}^2$ is given by $\mathcal{O} =]-\frac{1}{2}, \frac{1}{2}[\times]-\frac{1}{2}, \frac{1}{2}[$;

$$\mu_1 = 12, \mu_2 = 8, \sigma_1 = 2,5 \sqrt{2} 10^{-2} \mu_1, \sigma_2 = 3 \sqrt{2} 10^{-2} \mu_2 \quad (1);$$

M is given by (2.11), where $k_1 = k_2 = 0.7, k_3 = 1$;
the function f is given by :

$$f(x) = f_1(x_1) + f_2(x_2), f_i(x) = 8x \text{ if } x > 0; -60x \text{ if } x < 0.$$

The function u is minimum at one point denoted by $S = \{x_{1s}, x_{2s}\}$ (cf. Fig. 1).
It is sufficient to represent the solution for $x_1 \leq x_{1s}, x_2 \leq x_{2s}$ (2).

There is one region (the so-called saturated region) where $u = M(u)$; this region is divided into three parts :

$$(3.10) \quad \left| \begin{array}{l} C_1 \text{ where } u = M_1(u), C_2 \text{ where } u = M_2(u), \\ C_3 \text{ where } u = M_3(u) ; \end{array} \right.$$

numerical experiments show that (at least in the case under study) these regions do not overlap : they are represented on Fig. 1 ; the "internal" boundary $(s_1) \cup (s) \cup (s_2)$ is a free surface.

One also needs the curves (S_i) where (S_1) for instance denotes the set of minima of the function :

$$x_1 \rightarrow u(x_1, x_2)$$

which is enough to consider for $x_2 \in$ projection on the x_2 axis of C_1 ;
 (S_2) is defined in a similar manner exchanging the roles of x_1 and of x_2 .
The curves (S_1) and (S_2) are represented on Fig. 1.

Let us now explain on this example how to derive the optimal policy from the knowledge of u and more precisely of $(s_1) \cup (s) \cup (s_2)$ and of $(S_1) \cup (S_2)$.

(1) The operator $-\frac{\sigma^2}{2} \Delta$ is replaced by $-\frac{\sigma_1^2}{2} \frac{\partial^2}{\partial x_1^2} - \frac{\sigma_2^2}{2} \frac{\partial^2}{\partial x_2^2}$ (the variance of the demand depends on the goods that are considered). The values given here correspond to situations arising in practice; cf. BENSOUSSAN-GOURSAT-MAAREK, loc. cit.

(2) But the problem being global in Φ , one has to compute u everywhere in order to obtain S.

If the inventory $y \notin (s_1) \cup (s) \cup (s_2)$, do not order.

If the inventory reaches (s) at point P , order $P\vec{S}$.

If the inventory reaches (s_1) (resp. (s_2)) at P_1 (resp. P_2), order $P_1\vec{Q}_1$ (resp. $P_2\vec{Q}_2$), i.e. only the first (resp. 2nd) item in the amount P_1Q_1 (resp. P_2Q_2).

To obtain such results takes about 10' on CII 10070.

Remark 3.4.

The numbers σ are "small" against the μ_j 's but with sufficient care in the discretization of the first order derivatives, this fact does not lead to severe numerical difficulties. ■

Remark 3.5.

Specific applications to management problems are given in BENSOUSSAN-GOURSAT-MAAREK [1]. ■

Figure 1.

Figure 2.

4. - NUMERICAL APPROXIMATION OF THE EVOLUTION CASE.

4.1. General remarks

It is a simple matter to derive semi-discrete schemes (corresponding to the time discretization) associated to (2.6). For instance, starting with $u^0 = 0$, one defines u^n for $n \geq 1$ by the solution of the Q.V.I. ⁽¹⁾.

$$(4.1) \quad \left| \begin{array}{l} \left(\frac{u^n - u^{n-1}}{\Delta t}, v - u^n \right) + a(u^n, v - u^n) \geq (f, v - u^n) \\ v \leq M(u^n), \quad u^n \leq M(u^n); \end{array} \right.$$

in (4.1) we assume that f does not depend on t (cf. BENSOUSSAN-LIONS, loc. cit. for more general cases).

One can prove (BENSOUSSAN, LIONS, loc. cit.) that the function $u_{\Delta t} = u^n$ in $((n-1)\Delta t, n\Delta t)$ converges (in a suitable topology) to the solution u of (2.6).

Remark 4.1.

Of course the solution of (4.1) is in turn computed by a space-approximation as in Section 3 - and with analogous open problems. ■

Remark 4.2.

Numerical experiments show good results with an "implicit-explicit" method of the type

$$(4.2) \quad \left| \begin{array}{l} \left(\frac{u^n - u^{n-1}}{\Delta t}, v - u^n \right) + a(u^n, v - u^n) \geq (f, v - u^n) \\ \forall v \leq M(u^{n-1}), \quad u^n \leq M(u^{n-1}). \end{array} \right. \quad \blacksquare$$

Remark 4.3.

In case (as above) where f does not depend on t , the preceding schemes are also used for solving the stationary problems; convergence of u^n to the state of equilibrium is quite fast. ■

4.2. An example

The example given here is taken from GOURSAT [1]. We consider one item (space dimension = 1) ; μ and σ depend on t ⁽²⁾ :

⁽¹⁾ We reverse the order with respect to (2.6), in order to work with the usual notations of numerical analysis; (4.1) is an implicit scheme.

⁽²⁾ Everything which has been said extends to this case.

$$(4.3) \quad \mu(t) = 5(2.4 + 1.4 \sin \pi t), \sigma(t) = 3\sqrt{2} \cdot 10^{-2} \mu(t).$$

The set-up cost depends on the amount which is ordered :

$$(4.4) \quad k(\xi) = k_0 + k_1 \xi, \quad k_0 = k_1 = 3 \cdot 10^{-1};$$

this leads to the same problem than above, with

$$(4.5) \quad \mu(x, t) = \inf_{\xi \geq 0} [k_0 + k_1 \xi + u(x + \xi, t)];$$

we suppose that f depends on x and t :

$$(4.6) \quad f(x, t) = (1 - 0.125t) f(x), \quad f(x) = 8x \text{ if } x > 0, \quad -60x \text{ if } x < 0.$$

The set $\mathcal{D}$ is taken to be $]-\frac{1}{2}, 3[$.

The "saturated region" where $u = M(u)$ is shaded on Fig. 3; the boundary (s) is the free surface; the set of points where $x \rightarrow u(x, t)$ is minimum is represented by (S).

When the stock (taking into account the stochastic demand) reaches P (cf. Fig. 3) then we order PQ .

5. - NUMERICAL APPROXIMATION OF THE STATIONARY CASE (II).

5.1. The problem of the dimension

In cases where the space dimension is large (say ≥ 4) one faces the very difficult problem of solving Q.V.I. in high (or very high) dimension.

The only reasonable hope seems to obtain some "sub-optimal" policies of "simple" structure.

In the management literature, it seems that the method usually advocated is essentially to decompose the problem in considering the different goods as being independent.

It is at this point natural to try to extend to Q.V.I. the splitting-up methods; these methods have been introduced (cf. G.I. MARCHUK[1], N.N. YANENKO [1] and the bibliography of these books) for the solution of large systems of partial differential equations; they have been extended, in LIONS-TEMAM[1] to V.I. and in BENSOUSSAN-LIONS-TEMAM[1] to more general variational problems. We are now going to indicate how these methods can be extended to Q.V.I.; one obtains in this manner rather simple algorithms.

Figure 3.

5.2. Decomposition methods.

We follow here unpublished work of BENSOUSSAN, MAURIN, TEMAM and the A. In order to simplify the exposition (but what follows is entirely general), we suppose that $n = 2$ and that

$$(5.1) \quad f(x) = f_1(x_1) + f_2(x_2) \quad .$$

We consider

$$(5.2) \quad \left\{ \begin{array}{l} a(u,v) = a_1(u,v) + a_2(u,v), \\ a_i(u,v) = \frac{\sigma_i^2}{2} \int_{\mathcal{O}} \frac{\partial u}{\partial x_i} \frac{\partial v}{\partial x_i} dx + \mu_i \int_{\mathcal{O}} \frac{\partial u}{\partial x_i} v dx + \alpha_i \int_{\mathcal{O}} uv dx, \\ i = 1, 2, \alpha_1 + \alpha_2 = \alpha \quad . \end{array} \right.$$

We want to solve the stationary problem (2.9).

We introduce $u^0 = \max \{ \frac{1}{\alpha_1} \sup f_1, \frac{1}{\alpha_2} \sup f_2 \}$, and we define u^1 in three steps. The first two steps are done in parallel and are decomposed if $\mathcal{O} = \mathcal{O}_1 \times \mathcal{O}_2$ (which is not restrictive for the applications) : $u^{1/3}$ and $u^{2/3}$ are given by the solution of the Q.V.I.

$$(5.3) \quad \left\{ \begin{array}{l} a_1(u^{1/3}, v - u^{1/3}) + \frac{1}{\Delta t} (u^{1/3} - u^0, v - u^{1/3}) \geq (f_1, v - u^{1/3}) \\ \forall v \leq M_1(u^{1/3}), u^{1/3} \leq M_1(u^{1/3}), \end{array} \right.$$

$$(5.4) \quad \left\{ \begin{array}{l} a_2(u^{2/3}, v - u^{2/3}) + \frac{1}{\Delta t} (u^{2/3} - u^0, v - u^{2/3}) \geq (f_2, v - u^{2/3}) \\ \forall v \leq M_2(u^{2/3}), u^{2/3} \leq M_2(u^{2/3}). \end{array} \right.$$

We notice that in (5.3) (resp. in (5.4)) the variable x_2 (resp. x_1) plays the role of a parameter, so that (5.3) and (5.4) are really unidimensional Q.V.I.

We then define (third step) u^1 by :

$$(5.5) \quad u^1 = \inf \left(\frac{1}{2} (u^{1/3} + u^{2/3}), M \left(\frac{1}{2} (u^{1/3} + u^{2/3}) \right) \right).$$

We inductively proceed :

$$(5.6) \quad \left[\begin{array}{l} a_1(u^{n+1/3}, v-u^{n+1/3}) + \frac{1}{\Delta t}(u^{n+1/3} - u^n, v-u^{n+1/3}) \geq (f_1, v-u^{n+1/3}), \\ \forall v \in M_1(u^{n+1/3}), u^{n+1/3} \in M_1(u^{n+1/3}), \end{array} \right.$$

$$(5.7) \quad \left[\begin{array}{l} a_2(u^{n+2/3}, v-u^{n+2/3}) + \frac{1}{\Delta t}(u^{n+2/3} - u^n, v-u^{n+2/3}) \geq (f_2, v-u^{n+2/3}), \\ \forall v \in M_2(u^{n+2/3}), u^{n+2/3} \in M_2(u^{n+2/3}), \end{array} \right.$$

$$(5.8) \quad u^{n+1} = \inf \left(\frac{1}{2} (u^{n+1/3} + u^{n+2/3}), M \left(\frac{1}{2} (u^{n+1/3} + u^{n+2/3}) \right) \right).$$

One can prove that :

$$\begin{aligned} u^0 &\geq u^{1/3} \geq u^{1+1/3} \geq \dots \geq u^{n-1+1/3} \geq u^{n+1/3} \dots \geq 0, \\ u^0 &\geq u^{2/3} \geq u^{1+2/3} \geq \dots \geq u^{n-1+2/3} \geq u^{n+2/3} \dots \geq 0, \\ u^0 &\geq u^1 \geq u^2 \geq \dots \geq u^{n-1} \geq u^n \geq \dots \geq 0. \end{aligned}$$

It seems highly probable (but it has not been proved) and it is verified on the numerical experiments that

$$(5.9) \quad u^n \downarrow u.$$

Remark 5.1

In (5.3) ... (5.7), one can replace $\frac{1}{\Delta t}$ by a parameter ρ_n depending on the order of the iteration; the optimal choice of ρ_n is not known.

Remark 5.2.

If in (5.3) (resp. (5.4)), one replaces $M_1(u^{1/3})$ (resp. $M_2(u^{2/3})$) by $M_1(u^0)$ (resp. $M_2(u^0)$), one obtains V.I. instead of Q.V.I.

The same remark applies when replacing in (5.6) $M_1(u^{n+1/3})$ by $M_1(u^n)$ and in (5.7) $M_2(u^{n+2/3})$ by $M_2(u^n)$.

The numerical experiments which have been made (MAURIN[1]) are satisfactory by both methods.

6. - PROBLEMS WITH DELAYS

Suppose now that when one places an order, the delivery is made **after** a fixed delay h and let us assume that, while expecting a delivery, we do not place new orders ⁽¹⁾.

(1) Other cases are considered in a series of notes of BENSOUSSAN and the A. to appear.

Then $u(x,t)$ being still defined by (1.12), it is characterized by the Q.V.I.⁽¹⁾

$$(6.1) \quad - \left(\frac{\partial u}{\partial t}, v-u(t) \right) + a(u(t)), v-u(t) \geq (f, v-u(t))$$

$$\forall v \leq Mu(t-h),$$

$$(6.2) \quad u(t) \leq Mu(t-h),$$

$$(6.3) \quad u(T) = 0.$$

It is not difficult to introduce semi-discrete schemes analogous to those introduced in Section 4.

An example of this kind, arising in connection with the optimal use of dams is now under study; cf. BRETON and LEGUAY[1], LEGUAY[1]; one is lead to a system of four Q.V.I. of evolution, with a fixed delay : the numerical results are reported in BRETON and LEGUAY, loc. cit.

7. - A REMARK

It is also possible to apply the methods of Q.V.I. to problems of optimal control with L^1 constraints on the control variable, as those considered by other methods in Cernousko[1], Bratous and Cernousko[1].

(2) We write $u(t)$ for $x \rightarrow u(x,t)$.

BIBLIOGRAPHY

- C. BAIOCCHI [1] C.R. Acad. Sc. Paris, 278, (1974).
- C. BAIOCCHI and E. MAGENES [1] These proceedings.
- A. BENSOUSSAN and J.L. LIONS [1] C.R. Acad. Sc. Paris :
 1) 276 (1973), pp. 1189-1193 ;
 2) 276 (1973), pp. 1333-1337 ;
 3) 278 (1974), pp. 675-679 ;
 4) 278 (1974),
 5) 278 (1974).
- [2] Lecture in Grenoble. To appear in Springer Lectures Notes.
- [3] Applied Mathematics and Optimization, Int. J. (1974).
- [4] Ouspechi Mat. Nauk (Dedicated to Prof. Petrowski) (1973).
- [5] Book in preparation.
- A. BENSOUSSAN, M. GOURSAT and J.L. LIONS [1] C.R. Acad. Sc. Paris, 276 (1973), pp. 1279-1283.
- A. BENSOUSSAN, J.L. LIONS and R. TEMAM [1] Cahier IRIA n° 11 June 1972, pp. 5-189.
- A. BENSOUSSAN, M. GOURSAT and G. MAAREK [1] Report LABORIA, 1974.
- A.S. BRATOUS and F.L. CERNOUSKO [1] Journal of Num. Math. and Math. Physics 14 (1974), pp. 68-78.
- A. BRETON and C. LEGUAY [1] Report LABORIA, 1974.
- J. CEA and R. GLOWINSKI [1] Méthodes numériques pour l'écoulement laminaire d'un fluide rigide viscoplastique incompressible. Int. J. Comp. Mech. Vol B (1973) pp. 225-255.
- F.L. CERNOUSKO [1] Applied Math. and Mech. 35 (1971), 333-342.
- Ph. CIARLET and P.A. RAVIART [1] Comp. Methods in Appl. Mech and Eng. 2 (1973), 17-31.
- R. GLOWINSKI, J.L. LIONS and R. TREMOLIERES [1] Book, Dunod. Paris (1974).
- M. GOURSAT [1] Report LABORIA, 1974.
- M. JOLY - U. MOSCO [1] C.R. Acad. Sc. Paris, 1974.
- Th. LAETSCH [1] To appear.
- C. LEGUAY [1] Report LABORIA, 1974.
- J.L. LIONS and G. STAMPACCHIA [1] Comm. Pure and Applied Math. XX (1967) pp. 493-519.
- J.L. LIONS and R. TEMAM [1] C.R. Acad. Sc. Paris, 263 (1966), pp. 563-565.
- G.I. MARCHUK [1] Lecture International Congress of Mathematics, Nice, 1970;
 [2] Numerical Methods in Meteorology, Leningrad 1967 (Translation in French, A. Colin, Paris, 1969).
- S. MAURIN [1] Report LABORIA, 1974.
- F. MIGNOT [1] C.R. Acad. Sc. Paris, 1974.

- U. MOSCO and G. STRANG [1] One-sided approximation and variational inequalities.
- S.L. SOBOLEV [1] Applications of functional analysis to Mathematical Physics. Leningrad, 1950.
- G. STRANG [1] Proceedings of the Conference on Numerical Analysis, Dundee, July 3-7, 1973.
- L. TARTAR [1] C.R. Acad. Sc. Paris, 278 (1974).
- A. VEINOTT [1] Management Science. 12(11) (1966), pp. 745-777.
- N.N. YANENKO [1] Fractional steps Methods. Translation in French, A. Colin (1968).