EQUILIBRIUM AND PERTURBATIONS
IN PLASMA-VACUUM SYSTEMS

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INTRODUCTION.

High temperature plasmas of toroidal forms necessary to obtain fusion reactions between light elements (thermonuclear fusion) must be maintained far from all material contact. In order to realise this condition one uses, in the vacuum surrounding the plasma, a metal wall supposed perfectly conducting and currents I_i whose positions and intensities have to be suitably chosen. These currents are particularly important in the case of non-circular cross-section plasmas and in the case of discharges of long duration where the role of the wall diminishes with the passage of time since in reality the wall conductivity is not infinite.

EQUILIBRIUM WITH EXTERNAL CONDUCTORS. /1/ /2/ /3/

The problem of equilibrium consists of finding a toroidal solution of the system of equations

$$\begin{aligned} \mathbf{J} \times \mathbf{B} &= \text{grad } P \\ \text{div } \mathbf{B} &= 0 \\ \mathbf{J} &= \text{rot } \mathbf{B} \end{aligned}$$

$\mathbf{B}$, $\mathbf{J}$ and P are respectively the magnetic field, the current intensity and the plasma pressure. The pressure must be constant over nested tori and zero on the outermost torus.

This problem can be solved in symmetry of revolution. No toroidal-type solution is known for other geometries. In the following we will consider therefore symmetry of revolution and we will use in general the cylindrical coordinates (X, θ, Y) . In this case the configuration is defined by the flux functions $F(r)$ and $f(F)$ which satisfy the following equations

$$(I) \quad \begin{cases} \mathcal{L}F_P = -\frac{1}{2} \frac{df^2}{dF} - x^2 \frac{dP}{dF} & \text{in the plasma} \\ \mathcal{L}F_V = x \sum_i I_i \delta(X-X_i) \delta(Y-Y_i) & \text{in vacuo.} \end{cases}$$

At the plasma-vacuum interface Γ_P we have

$$\begin{cases} F_P = 0 \\ F_V = \gamma = \text{const.} \\ \frac{1}{x} \frac{\partial F_V}{\partial n} = \frac{1}{x} \frac{\partial F_P}{\partial n} = B_{m\Gamma_P} \end{cases}$$

and at the wall Γ_V

$$F_V = 0$$

with the operator $\mathcal{L} \equiv \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} - \frac{1}{x} \frac{\partial}{\partial x}$ and $\frac{\partial}{\partial n}$ represents the normal derivative. The currents I_i are placed $P_i(X_i, Y_i)$. The functions $f(F)$ and $P(F)$ characterise the plasma and are supposed known.

First of all one solves the internal problem by specifying the external cross-section Γ_P of the plasma. To resolve the problem in the vacuum, only a knowledge of the meridian magnetic field $B_{m\Gamma_P}(\ell)$ is necessary which depends on the curvilinear abscissa ℓ of Γ_P .

Schematically, the method used to solve the problem consists of defining a complete set of functions $W_{n\Gamma_P}(\ell)$ which are orthogonal and periodic over the cross-section of the plasma. The given field $B_{m\Gamma_P}(\ell)$ is expanded over the cross-section of the plasma in terms of these functions.

$$B_{m\Gamma_P}(\ell) = \frac{1}{x} \frac{\partial F}{\partial n} \Big|_{\Gamma_P} = \sum_n a_n W_{n\Gamma_P}(\ell)$$

where

$$a_n = \oint \frac{W_{n\Gamma_P}}{x} \frac{\partial F}{\partial n} d\ell$$

with the norm

$$\oint W_{n\Gamma_P}^2 d\ell = 1.$$

Let us now define $W_n(X, Y)$ by

$$\mathcal{L}W_n = 0$$

with

$$W_n = 0 \text{ on } \Gamma_V$$

$$W_n = W_{n_{T_P}}(\ell) \text{ on } \Gamma_P.$$

Using the relation of Green's type for the operator $\mathcal{L}$

$$\iiint_V (W \mathcal{L}U - U \mathcal{L}W) \frac{dV}{X^2} = \iint_{\Gamma_P + \Gamma_V} (W \frac{\partial U}{\partial n} - U \frac{\partial W}{\partial n}) \frac{d\sigma}{X^2},$$

with $U = F$ and $W = W_n$, we obtain :

$$\sum_i I_i W_n(X_i, Y_i) + \gamma \int_{\Gamma_P} \frac{d\ell}{X} \frac{\partial W_n}{\partial n} = a_n.$$

One therefore attempts to resolve optimally this infinite system of equations with γ and I_i as unknowns, the number of currents being limited and their position chosen to have at the same time a sum $\sum |I_i|$ a minimum (or some other similar condition) and, if possible, the wall currents very weak.

Such a program has been perfected by SOUBBARAMAYER (from the C.E.A.), BOUJOT and MORERA (from C.I.S.I.) and is now operational. The technique employed is the method of finite elements.

Fig.1- Numerical values and plasma cross-section forms for the example treated.

Fig.2- An example of the meridian field for the internal model.

Fig.3- The currents in the conductors producing a given from of plasma cross-section.

As an illustration of the method, we have treated some cases whose results are presented in Figs.1-3. Fig.1 shows the different forms Γ_p of the plasma studied. The wall remains the same in all cases. The solution has been sought with 36 conductors placed as shown in the figure (taking into account the symmetry). The meridian field is calculated at the same time as the internal problem by supposing a parabolic form for the current density which is zero at the plasma boundary. This field is shown in Fig.2. The results are indicated in Fig.3 which gives the current intensities according to their position around the plasma.

PERTURBED EQUILIBRIA.

The preceding problem is solved numerically only to within a certain precision. Further, the arrangement and the intensity of the currents found will not be exactly realised due to, for example, technical reasons. Consequently this first problem of equilibrium will be considered as a first approximation. One therefore seeks the configuration which will be obtained under imposed real conditions. One can in addition suppose that the functions $f(F)$ and $P(F)$ characterising the plasma are only known approximatively.

Mathematically the problem is posed as follows. An approximate solution of the system (I) is known which satisfies

$$(II) \quad \left\{ \begin{array}{l} \mathcal{L} \cdot \bar{F}_{OP} = -\frac{1}{2} \frac{d \bar{f}^2}{d \bar{F}_0} - x^2 \frac{d \bar{P}}{d \bar{F}_0} \\ \mathcal{L} \cdot \bar{F}_{OV} = x \sum_i I_i \delta(x-x_i) \delta(y-y_i) \end{array} \right.$$

The plasma-vacuum interface $\bar{\Gamma}_p$ is close to Γ_p and the intensity of the external currents is given together with their position. The wall Γ_v remains unchanged. The condition on $\bar{\Gamma}_p$ and Γ_v are given by

$$\left\{ \begin{array}{l} \bar{F}_{OP} = \bar{F}_{OV} = 0 \\ \frac{1}{x} \frac{\partial \bar{F}_{OV}}{\partial n} - \frac{1}{x} \frac{\partial \bar{F}_{OP}}{\partial n} = \delta B_{m,ie} \quad \text{on } \bar{\Gamma}_p \\ \bar{F}_{OV} = 0 \quad \text{on } \Gamma_v \end{array} \right.$$

It is supposed that the quantities δB_{mie} , $\delta f^2 = f^2 - \bar{f}^2$, $\delta P = P - \bar{P}$ are small. The solution of the neighbouring system (I) will be obtained by introducing a vector field $\vec{\xi}$ which transforms every point of the plasma-vacuum configuration

sought for (defined by system (I)) into points of the approximated plasma-vacuum configuration (defined by system (II)). Points on the boundaries correspond as well as points of each medium.

In order to specify the transformations, we call X, Y the coordinates of a point P in the space E defined by the solution of (I), and $\bar{X}, \bar{Y}$ the corresponding point in the space $\bar{E}$ defined by the solution of (II). Let S denote this transformation. It is an infinitesimally small transformation.

$$S = 1 + s = 1 - \vec{\xi} \cdot \nabla \quad S^{-1} = 1 - \bar{s} \quad .$$

The system (I) can thus be transformed in the space $\bar{E}$. For example

$$\begin{aligned} \mathcal{L} &\implies S \cdot \mathcal{L} \cdot S^{-1} \\ F &\implies S \cdot F(XY) = \bar{F}(\bar{X} \bar{Y}) \\ \Gamma &\implies \bar{\Gamma} \quad . \end{aligned}$$

Consequently in the known space $\bar{E}$, $\bar{F}(\bar{X}, \bar{Y})$ is approximated by the solution $F_0(X, Y)$ of system (II) :

$$F(\bar{X} \bar{Y}) = \bar{F}_0(\bar{X} \bar{Y}) + \bar{F}_1(\bar{X} \bar{Y}) \quad .$$

We now introduce two functions G and g whose knowledge will easily permit the solution of the problem posed.

$$G(\bar{X} \bar{Y}) = \bar{F}_{P_1} - s \bar{F}_{P_0} \quad g(\bar{X} \bar{Y}) = \bar{F}_{V_1} - \bar{s} \bar{F}_{V_0} \quad .$$

These functions satisfy

$$\left\{ \begin{aligned} \mathcal{L}.G(\bar{X}, \bar{Y}) &= - \left(\frac{1}{2} \frac{d^2 f^2}{dF^2} + X^2 \frac{d^2 P}{dF^2} \right) G(\bar{X} \bar{Y}) - \frac{1}{2} \left(\frac{d \delta f^2}{dF} \right) - X^2 \frac{d \delta P}{dF} \\ \mathcal{L}.g &= 0 \\ G &= g \\ \frac{\partial G}{\partial n} - \frac{\partial}{\partial n} g &= - \bar{X} \delta B_{m,ie} \frac{G}{\bar{X} B_{m,ie}} \quad \left(\frac{1}{2} \frac{df^2}{dF} + \bar{X}^2 \frac{dP}{dF} \right) \quad \text{on } \bar{\Gamma}_P \end{aligned} \right.$$

$g = \text{constant}$ on $\bar{\Gamma}_V$.

The expressions $\frac{d^2 f^2}{dF^2}$, $\frac{d^2 P}{dF^2}$, $\frac{d \delta f^2}{dF}$ and $\frac{d \delta P}{dF}$ are taken with $F = F_0(\bar{X} \bar{Y})$

$$\delta I = I - \bar{I}_1 = \int_{\bar{\Gamma}_P} \frac{1}{\bar{X}} \frac{\partial g}{\partial n} d\bar{s} - \int_{\bar{\Gamma}_P} \delta B_{m,ie} d\bar{s} \quad ,$$

where I_1 is the total current in the approximated configuration defined by system (II) and (I) the desired current. Thus

$$F(X, Y) = \bar{F}_0(X, Y) + G(X, Y) ,$$

and in particular the boundary Γ_p is defined with respect to the boundary $\bar{\Gamma}_p$ approximated by the normal component of $\vec{\xi}$

$$\xi_n = \frac{G}{X B_m} \Big|_{\bar{\Gamma}_p} .$$

The perfection of this numerical program is in progress.

NEIGHBOURING EQUILIBRIA.

The equations satisfied by the functions G and g also allow of the study of the bifurcation in the neighbouring equilibria, these neighbouring equilibria being defined by the new perturbed conditions. These bifurcations are found where certain perturbations no longer correspond to a solution of the equations. They appear instead as the solutions of a problem of homogeneous partial differential equations with given conditions or continuity at the plasma-vacuum interface. They indicate in general the onset of unstable equilibria, whence their interest. In our case, the equilibria remain in symmetry of revolution.

Certain simple cases can as examples be treated analytically. The cylindrical case has been consequently studied for elliptical cross-section and length L which is considered as the limiting case of a torus of the same length. The calculation is carried out in elliptic cylinder coordinates. The limits of stability are determined according to the eccentricity of the ellipse and the proximity of the metal wall. These limits also depend upon the current profile in the plasma. The results show that the instability increases with increasing ellipticity and for current profiles which are more peaked at the centre of the plasma.

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