ON THE PROBLEM OF AN OPTIMAL THREE-DIMENSIONAL
AIRCRAFT MOTION

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The optimization problem of aircraft control during three-dimensional maneuver is considered. The equations for three-dimensional motion of a vehicle in a spherical speed system have the following form

$$\frac{dV}{dt} = \frac{C_p \cos(\alpha + \epsilon) - C_x}{m} \cdot q S - g(H) \sin \Theta, \quad (1.1)$$

$$\frac{d\lambda}{dt} = \frac{V}{r} \cos \Theta \frac{\cos \eta}{\cos \varphi}, \quad (1.2)$$

$$\frac{d\varphi}{dt} = \frac{V}{r} \cos \Theta \sin \eta, \quad (1.3)$$

$$\frac{d\eta}{dt} = \frac{C_p \sin(\alpha + \epsilon) + C_y}{m V \cos \Theta} \cdot q S \sin \gamma - \frac{V}{r} \cos \Theta \cos \eta \operatorname{tg} \varphi, \quad (1.4)$$

$$\frac{d\mu}{dt} = -\bar{C}_e C_p \frac{q S}{m}, \quad \bar{C}_e = \frac{C_e}{3600 g_0}. \quad (1.5)$$

$$\frac{d\Theta}{dt} = \frac{C_p \sin(\alpha + \epsilon) + C_y}{m V} \cdot q S \cos \gamma - \frac{g(H)}{V} \cos \Theta \cdot \varkappa, \quad (1.6)$$

$$\frac{dH}{dt} = V \sin \Theta, \quad (1.7)$$

where V , Θ , H , m are instant values for speed, trajectory inclination angle, altitude, mass; λ - longitude, φ - latitude, C_p , C_x , C_y , C_e - the coefficients of thrust, aerodynamic drag, lift and fuel consumption; α , S - angle of attack, vehicle characteristic area; $\mu = \ln G$, $G = m g_0$ - aircraft weight, $\varkappa = 1 - \frac{V^2}{g(H)r}$. The control functions are assumed to be: $\bar{R}$ - the thrust throttling coefficient, $C_p = C_{pR} \bar{R}$, C_y - lift coefficient, γ - roll angle, measured from the trajectory plane. Now we shall consider the case, when the control $\bar{u} = (\bar{R}, C_y, \gamma)$ lies within of the closed fixed parallelepiped

$$0 \leq \bar{r}_{\min} \leq \bar{r} \leq \bar{r}_{\max}, \quad (2.1)$$

$$0 < c_{y_{\min}} \leq c_y \leq c_{y_{\max}}, \quad (2.2)$$

$$-\frac{\pi}{2} < \gamma_{\min} \leq \gamma \leq \gamma_{\max} \quad (2.3)$$

(By linear transformation of control functions

$$U = U_{\min}(x) + [U_{\max}(x) - U_{\min}(x)] u$$

it is possible to turn from convex constraints on control, depending on phase coordinates to unit cube $0 \leq u \leq 1$).

It is necessary to find an optimal control $\bar{u}_* = (\bar{r}_*, c_{y*}, \gamma_*)$ which transfers the system (1) from the initial position, $x_0 \in G^0$, $x_0 = x(0) = (V_0, \lambda_0, \varphi_0, \eta_0, \mu_0, H_0, \Theta_0)$ into the end state

$$x_1 = x(T) \in G^1 \quad (3)$$

/ G^0 and G^1 are initial and end varieties/, some functional I reaching the maximum value under the constraints (2). Below the optimization problem for the end mass value $I = m(T)$ is considered as an example. Let us find an optimal control for the formulated problem, using the maximum principle [1] under the following assumptions:

1. The sum of all forces projections to the trajectory normal is equal to zero

$$\frac{d\Theta}{dt} = 0. \quad (4.1)$$

Hence

$$\frac{qS}{m} = \frac{g(H)x}{C_y \cos \gamma}.$$

2. The angles of attack α , engine incidence ε and trajectory inclination Θ are considered small values of the second order:

$$\begin{aligned} \sin(\alpha + \varepsilon) &\approx 0, & \cos(\alpha + \varepsilon) &\approx 1, \\ \sin \Theta &\approx 0, & \cos \Theta &\approx 1. \end{aligned} \quad (4.2)$$

The system of optimal motion equations has the first integrals [2]

$$p_\eta \frac{\cos \lambda}{\cos \varphi} + p_\varphi \sin \lambda - p_\lambda \operatorname{tg} \varphi \cos \lambda = C_1, \quad (5.1)$$

$$p_\eta \frac{\sin \lambda}{\cos \varphi} - p_\varphi \cos \lambda - p_\lambda \operatorname{tg} \varphi \sin \lambda = C_2, \quad (5.2)$$

$$p_\lambda = C_3. \quad (5.3)$$

The assumptions (4), relations (5) make it possible to exclude the variables Θ , r , p_Θ , p_r out of consideration and substitute

known functions of coordinates and constants C_1, C_2, C_3 for $P_\eta, P_\varphi, P_\lambda$.

As the Hamiltonian function

$$\mathcal{H} = p_v \frac{C_p \cos(\alpha + \varepsilon) - C_x}{C_y \cos \gamma} \cdot g \alpha + p_\eta \left[\frac{C_p \sin(\alpha + \varepsilon) + C_y}{V} \cdot \frac{g \alpha \operatorname{tg} \gamma}{C_y} - \right. \\ \left. - \frac{V}{r} \cos \eta \operatorname{tg} \varphi \right] + p_\lambda \frac{V \cos \eta}{r \cos \varphi} + p_\varphi \frac{V}{r} \sin \eta - p_\mu \frac{\bar{C}_e C_p g \alpha}{C_y \cos \gamma}.$$

does not contain μ explicitly, then

$$p_\mu(t) = p_{\mu_0} = \text{const} \quad (6)$$

and the conjugate variable p_v is still unknown.

Let C_x and C_y be connected by a quadratic polar $C_x = C_{x_0} + AC_y^2$. Taking (4.2) and (6.1) into consideration, transform the Hamiltonian into the following form

$$\mathcal{H} = \frac{a}{\cos \gamma} + b \operatorname{tg} \gamma + c, \quad (7.1)$$

where

$$a(p_v, C_y, \bar{R}) = g \alpha \left[\frac{\tilde{\varphi}(p_v, \bar{R})}{C_y} - p_v AC_y \right] \quad (7.2)$$

$$\tilde{\varphi}(p_v, \bar{R}) = p_v (C_R \bar{R} - C_{x_0}) - p_\mu \bar{C}_e C_R \bar{R}. \quad (7.3)$$

The coefficients b and c do not depend on the control

$$b = p_\eta \frac{g \alpha}{V}, \quad (7.4)$$

$$c = \frac{V}{r} \left(p_\lambda \frac{\cos \eta}{\cos \varphi} + p_\varphi \sin \eta - p_\eta \cos \eta \operatorname{tg} \varphi \right). \quad (7.5)$$

For the case of stationarity under consideration the first integral takes place

$$\mathcal{H} = \frac{a}{\cos \gamma} + b \operatorname{tg} \gamma + c = h, \quad (8)$$

and the constant $h = 0$, when motion time T is not fixed. It is not difficult to prove, that the problem of defining the control values, minimizing the Hamiltonian function $\min_{\bar{R}, C_y, \gamma} \mathcal{H}$, comes to successive solution of three onedimensional problems for minimum point

calculation

$$\min_{\gamma_{\min} \leq \gamma \leq \gamma_{\max}} \left(\frac{a}{\cos \gamma} + b \operatorname{tg} \gamma + c \right), \quad (9)$$

$$\min_{\bar{R}_{\min} \leq \bar{R} \leq \bar{R}_{\max}} [p_v (C_R \bar{R} - C_{x_0}) - p_\mu C_R \bar{C}_e(\bar{R}) \bar{R}], \quad (10)$$

$$\min_{C_{y_{\min}} \leq C_y \leq C_{y_{\max}}} \left(\frac{\tilde{\varphi}}{C_y} - p_v A C_y \right). \quad (11)$$

Using (8) while defining γ it is possible to write corresponding optimality conditions in terms of values of b and c , depending on constants C_1 , C_2 , C_3 and phase coordinates

$$\gamma_* = \begin{cases} \arcsin \frac{-b}{\sqrt{b^2 + c^2}} = \gamma_{\text{OPT}}, & \gamma_{\min} \leq \gamma_{\text{OPT}} \leq \gamma_{\max}, \\ \gamma_{\max}, & \mathcal{H}(\gamma_{\max}) < \mathcal{H}(\gamma_{\min}), \\ \gamma_{\min}, & \mathcal{H}(\gamma_{\min}) < \mathcal{H}(\gamma_{\max}). \end{cases} \quad (12)$$

During calculation of $\bar{R}_*$ it is necessary to find all the roots $\bar{R}_i$ of the equation (p_v is considered known):

$$\frac{\partial}{\partial \bar{R}} [p_v (C_R \bar{R} - C_{x_0}) - p_\mu C_R \bar{C}_e(\bar{R}) \bar{R}] = 0, \quad (13)$$

which satisfy the condition $\bar{R}_{\min} \leq \bar{R}_i \leq \bar{R}_{\max}$. Add $\bar{R}_{\min}$ and $\bar{R}_{\max}$ to the obtained roots and select the meaning, to which the minor value of the function $\tilde{\varphi}$ corresponds. From the equality $\frac{\partial}{\partial C_y} \left(\frac{\tilde{\varphi}}{C_y} - p_v A C_y \right) = 0$ we derive

$$C_y = C_{y_{\text{OPT}}} = \sqrt{-\frac{\tilde{\varphi}}{A p_v}}. \quad (14)$$

Let us investigate the function $\alpha(p_v, C_y, \bar{R})$, (22), versus C_y .

a) If $p_v < 0$ and $\tilde{\varphi} > 0$, then among three values of lift $C_{y_{\text{OPT}}}$, $C_{y_{\min}}$, $C_{y_{\max}}$ choose that one, to which minimum α corresponds;

b) if $p_v < 0$, $\tilde{\varphi} < 0$ then $C_{y_*} = C_{y_{\min}}$;

c) if $p_v > 0$, $\tilde{\varphi} > 0$ then $C_{y*} = C_{y\max}$;
 d) if $p_v > 0$ and $\tilde{\varphi} < 0$, choose out of boundary values of $C_{y\min}$ or $C_{y\max}$ the meaning, to which minimum α corresponds. It remains to find p_v ; to this end we use the equation

$$\alpha(p_v, C_y, \bar{R}) = - (b \sin \delta_* + c \cos \delta_*). \quad (15)$$

As a result we obtain the following algorithm for optimal control determination. Let constants C_i , defined by the solution of a boundary value problem, be known. During every integration step of differential equations system (I) the conjugate variables p_γ , p_φ , p_λ are calculated from solution of algebraic equations (5.1) - (5.3); then from the formulae (7.4), (7.5) coefficients b and c are defined; according to conditions of optimal roll control the value of δ_* and right-hand side of (15) are found. After this the value of a conjugate value p_v is determined as a root of the equation (15), the values of thrust $\bar{R}_*$ and lift coefficient C_{y*} , obtained by the method described above, are substituted into its left-hand side. The values, corresponding to the obtained root, are assumed to be optimal control, that is they are used as instant controls. The procedure just described makes it possible to obtain an external in a sense of maximum principle [1]. In addition, to solve a given problem it is necessary to write boundary conditions and construct algorithm, which permits defining constants C_1 , C_2 , C_3 , corresponding to these boundary conditions, by using the features of varieties G^0 and G^1 .

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