MULTIDIMENSIONAL LINDENMAYER ORGANISMS

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Summary

Most cellular organisms in the real world are not one-dimensional. How can we model the global development of such organisms by rules that are local to each cell? It is not unreasonable to suppose that the development of an individual cell depends on the cell state and the cell context, the state and position of the neighbours of the cell. In this paper we present models in which this cell information and the attendant local developmental rules can be represented discretely. The precise mathematical description of the models is left to an appendix.

von Neumann models

The first class of models we shall discuss have been much studied (e.g., Cellular automata, ed. E.F. Codd, Academic Press, 1968; Essays on cellular automata, ed. A.W. Burks, 1970). The distinguishing feature of models in this class is that the positions of the cells in an organism are fixed once and for all. As an organism develops the states of the neighbours of a cell may change, but the number and positions of its neighbours may not. The local developmental rules need only specify a new cell state for each of the finitely many possible cell contexts.

In figure 1 we compare a von Neumann model for the development of the red algae *Callithamion roseum* with the corresponding OL-system. We suppose that the cells of the organism lie on the points with integer coordinates in 2-dimensional Euclidean space so each cell has 8 neighbours. The seed, the initial configuration of the organism, has the cell at the origin in state a and all other cells in the "vacuous" state. The context independent local developmental rules are:

$$a \rightarrow b, b \rightarrow b, c \rightarrow b, d \rightarrow e, e \rightarrow f, f \rightarrow g, h \rightarrow h, h' \rightarrow h'.$$

The context dependent rules are given in figure 2. If the reader continues the development of the algae in figure 1, he will appreciate the following disadvantages of the von Neumann model:

- growth and cell division can only occur when there are cells in the vacuous state;

- unwanted interaction because of the severe restrictions on growth direction;
- the inflexibility caused by having a fixed global limit on the number of neighbours a cell can have.

Time	Nature	VN model	L-model
0		a	a
1		bc	bc
2		bbd	bbd
3		bbed	bbed
4		bbfed	bbfed
5		bbgfd	bbgfd
6		bbhgfed a	bbh(a)gfed
7		bbhhgfed a b c	bbh(bc)h(a)gfed
8		bbhhhgfed b a b d	bbh(bbd)h(bc)h(a)gfed

Figure 1. The development of *Callithrix*^m on Roseum

Papers by Szilard (this volume) and Lindenmayer [in 45] have indicated how the von Neumann model can be modified so that these disadvantages are somewhat mitigated.

Web models

Our second class of models was invented by J.L. Pfaltz and A. Rosenfeld (Web grammars, Proc. Joint Int. Conf. on Artificial Intelligence, Washington, 1969) for other purposes. The distinguishing feature of models in this class is that the positions of the cells in an organism can be represented by an unordered graph, where there is an edge between two vertices if and only if the corresponding cells are neighbours. The price we pay for removing the restriction on the possible number of neighbours is that cell contexts give no indication of the relative positions of cell neighbours.

Suppose we can apply a local developmental rule to a particular cell with a particular context. What should it give us? The natural answer is a latent organ-

A problem at time 12.

d
 e
 f d
 g e
 b b c
 b b b
 b b b b b h h h h h h g f e d
 b b b a
 b b b
 a h f d
 g e
 f d
 e
 d

Figure 2. Details of the VN model in figure 1.

ism, a labelled graph just like that which represents the whole organism. In this way every vertex of the labelled graph for the whole organism produces its own labelled graph and these must be joined by some embedding algorithm.

Let us look at figure 3 which shows the web model for our red algae. The seed, is represented by the graph consisting of a single vertex labelled by a. All the local developmental rules are context independent, and to be found in the tableau:

State	a	b	c	d	e	f	g	h
Latent organism	b-c	b	b-d	e-d	f	g	h-a	h

Figure 3. The web model for *Callithamion Roseum*.

But what is the embedding algorithm? The literature on web grammars shows the choice of embedding algorithm may influence greatly the expressive power of our class of web models. Experimenting with various possibilities has shown that the following embedding algorithm is a reasonably simple and powerful choice. For a particular web model we have a forbidden list of state pairs. Using this and a collection of latent organisms, one for each vertex in a labelled graph, we build a new graph by 1) replacing the vertices by the corresponding latent organisms; 2) for each edge (v, v') in the original graph, joining each vertex in the latent organism for v with each vertex in the latent organism for v' ; 3) dropping the edges added at step 2 which would give a state pair in the forbidden list. For our red algae the forbidden list is:

(b, d), (f, d), (a, b), (a, g), (a, h), (c, h)

In this particular example we can also use the concept of the skin of a latent organism to give a simpler embedding algorithm [69].

Figure 4 illustrates a web model for the development of a leaf. Again the seed is represented by the graph consisting of a single vertex labelled by a. Again all the local developmental rules are context independent. This time the rule tableau is:

State	a	b	c	d	e	f
Latent organism	b-a	c	d-e	d	f	c-c

and the forbidden list is:

(a, d), (a, e), (b, e), (c, e), (d, e).

We notice, that states a and d represent the primary and tertiary cells of Nägeli [61].

Web models have their weaknesses. On learning about them Aristid Lindenmayer challenged the author to reproduce the way nature creates a closed surface. It so happens that web models have no zip mechanism – if two cells in an organism are disconnected, then there is no way for a descendant of one of them to be linked to a descendant of the other at a later stage in the development. It is possible to introduce the needed context sensitivity in many ways – e.g. by allowing rules to apply to subgraphs not just vertices – but, as yet, none of these ways can be justified by a convincing biological argument. Notice that the kind of context sensitivity needed to make a box is different from that which is allowed in web models. To see the allowable kind of context sensitivity consider the fictive organism, the IMIAD,

Figure 4. The web model for Phascum Cuspidatum.

Time 6.

OL-representation: $\{d(de)(de)\}(d(c)(c))(d(f))(de)(c)ba$

Time 6 1/2.

Time 7.

Figure 5. An epoch in the life of Phascum Cuspidatum.

models by allowing the edges of the underlying graph to be ordered. This influences the model in two ways:

- the ordering in the context can be used to restrict the choice of applicable local developmental rules;
- the ordering in the graph influences the interconnection of the latent organisms produced by the local developmental rules.

Figure 7 illustrates the second of these. The point to note is that any subregion of France that touches Sp, It, Be must also touch H. This restriction cannot be expressed if the underlying graph is unordered.

It is very difficult to give a precise rule for connecting several ordered graphs into one large ordered graph. Furthermore, we shall later want to consider cells which touch one another in several places and even cells which enclose other cells. Ordered graphs cannot express this, and we have a problem. The way out was suggested to the author by J. Thatcher: to consider an ordered graph as a function from vertices to circular words on the vertices. A circular word on an alphabet A is an ordinary word A on A except that it consists of one or more subwords that are "cycles" in the sense: the first (last) letter is the right (left) neighbour of the last (first) letter. The circular words are used in the appendix to give a precise description of the embedding algorithm for map models. Figure 8 shows an application of this embedding algorithm. The restriction on the rule is met by splitting the context of 4 into the two words, 56∞ and $\infty 5$, then substituting these words for ∞ in the two cells in the skin of the latent organism.

In order to check that the embedding algorithm is correctly formulated, a computer program has been written that draws a picture of the life stages in an organism that is described by a map model. Figure 9 shows two such pictures. The smooth curves and straight boundary edges in the computer plots are due to the fact that the computer represents the organism as a list of lists from which it can generate a triangular network before plotting (the internal cell boundaries are the perpendicular bisectors of triangle edges). In the computer the local developmental rules of the web model become rules to change the list of lists that represent the current life stage.

Before leaving map models let us consider two more examples. Figure 10 shows the map model for *Phascum Cuspidatum* in action. It should be compared with figure 5 and also figure 11, which shows the map model for a fictive organism UACER that develops internal organs.

Query: Can France divide so that one part has only 11 neighbours and the other has only 4 neighbours?

In answering this problem it helps to consider the context of France as the "circular" word.

Figure 7. The religious war problem.

Rule

Name	Condition	Latent Organism	Restriction
BUD	state p	<u>p / s</u>	Split touches ∞
More precise description of the latent organism.			
cell 1	state: p	context: $\begin{pmatrix} \infty \\ 2 \end{pmatrix}$	
cell 2	state: s	context: $\begin{pmatrix} \infty \\ 1 \end{pmatrix}$	
skin	$\begin{pmatrix} 1 \\ 2 \end{pmatrix}$		

An application of the rule.

Before:

cell 5	old context: $\begin{pmatrix} 4 \\ 6 \end{pmatrix}$	new context: $\begin{pmatrix} 1-2 \\ 6 \end{pmatrix}$
cell 6	old context: $\begin{pmatrix} 4 \\ 5 \end{pmatrix}$	new context: $\begin{pmatrix} 1 \\ 5 \end{pmatrix}$
cell 1	l. o. context: $\begin{pmatrix} 2 \\ \infty \end{pmatrix}$	new context: $\begin{pmatrix} 5 \\ 2 \end{pmatrix}$
cell 2	l. o. context: $\begin{pmatrix} 1 \\ \infty \end{pmatrix}$	new context: $\begin{pmatrix} 1 \\ 5 \end{pmatrix}$

After:

Figure 8. Rule application in map models.

Splitting of cell 4 produces cell 8

Splitting of cell 7 produces cell 9

Splitting of cell 6 produces cell 10

Figure 9. An epoch in the life of a leaf.

Rules:

Name	Condition	Lat. org.	Restriction
BUD	state p		split touches ∞ once
ANTI	state s		split touches ∞ once
PERI	state s		s but not split touches ∞
STABLE	state t		NONE

Life history. Figure 10. Map model for Phascum Cuspidatum

Rules

Name	Condition	Lat. org.	Restriction
BABY	state p		none
SPLIT	state s		none
HOLE	state s		none
STABLE	state t		none

Life history

Figure 11. Internal organs: growth of a UACER

Globe models

At last we can indicate how to model three dimensional organisms. Imagine a demon inside one of the cells of a two dimensional organism. It would see the outside world as a circle, divided into segments by the neighbouring cells, and it could model this world as a circular word. Now imagine a demon inside one of the cells of a three dimensional organism. It would see the outside world as a sphere, divided into clouds (regions, contact areas) by the neighbouring cells. It could model this world using spherical words if only we had a suitable definition of spherical words. Revive the two-dimensional demon for a moment. It could split a circle into two segments by picking two points, then convert each segment into a circle by adjoining a new segment (consult figure 8 and the appendix). The analogous process in three dimensions is to split a sphere into two clouds by picking a circle, then convert each cloud to a sphere by adjoining a new cloud.

How do we make all this discrete? Just as the circular words of our map models can be defined in terms of ordinary words that represent circle segments, so the spherical words can be defined in terms of flat words that represent clouds. But what is a flat word. Nothing but a map of the kind we have discussed in the previous section. There we described two possible discrete representations: labelled ordered graphs and functions into circular words. All this is discussed in more detail in [69]. We favour a more schematical representation of the model here because:

- for very few three dimensional biological organisms is the development known in detail;
- the precise mathematical description is complicated;
- the author cannot draw three dimensional organisms.

Appendix

The mathematical description of the models

We define a model class to be a triple consisting of:

- 1) a set of configuration spaces;
- 2) a set of rule repertoires;
- 3) an algorithm Ψ which, given a rule repertoire R and a configuration space Γ , provides a binary relation on Γ .

A model in such a class is (Γ, R, y_0) where Γ is a configuration space, R is a rule repertoire, and y_0 is an element of Γ called the seed of the model. If y and y' are configurations in Γ , we write:

$$y \rightarrow y' \text{ for } \Psi(R, \Gamma) \text{ holds on } (y, y') ;$$

$$y \Rightarrow y' \text{ for the transitive closure of } \rightarrow .$$

We define the language of the model to be the set of configurations y such that $y_0 \Rightarrow y$.

In all our models we will have sets:

Structure, State, Context, Latent Organism

and these sets will be such that:

- 4) State is finite;
- 5) the configuration space Γ of the model is the set of linear words on State $\times$ Structure;
- 6) the rule repertoire R of the model is a finite subset of

$$\text{State} \times 2^{\text{Context}} \times \text{Latent Organism}.$$

The configurations are denoted by $(B, L \times D)$, and the rules are denoted by (S, P, I) . For this to make sense B must be a subset of integers, L must be a function from B to State, D must be a function from B to Structure, S must be in State, P must be a property of contexts, and I must be a latent organism.

Before we fill in this general frame work, let us introduce a few concepts we shall need. A circular word on a set X is a triple (B, π, F) , where B is a finite set of integers, π is a permutation of B , and F is a function from B into X . We denote the empty circular word on X by Φ . Suppose $C = (B, \pi, F)$ is a circular word on X . If π is the identity permutation, we call C a linear word on X . If G is

a function from X to Y , then we write $G * C$ for $(B, \pi, G \cdot F)$, a circular word on Y . Now we can compare the four model classes discussed in the paper:

model class	von Neumann	Web	Map	Globe
Structure	Integer^m	subset of Integer	circular words on Integer	circular words on circular words on Integer
Context	State^m	multiset on State	circular words on State	circular words on circular words on State
Latent Organism	State	Configuration Space	Configuration Space $\times$ Restriction	Configuration Space $\times$ Restriction

The von Neumann model class

In a model of this class we have an integer m , and a finite set State with a designated element s_0 such that: $\text{Structure} = \text{Integer}^m$, $\text{Context} = \text{State}^m$, $\text{Latent organism} = \text{State}$. In addition we have designated functions $\delta_1, \delta_2 \dots \delta_n$ from Integer to Integer. Contrary to what one might expect, m is the number of elements in the context and this is totally unrelated to the spatial dimensionality of the organism that we are trying to model. Another unusual feature of our formulation of a von Neumann model is the definition of $(B, L \times D) \rightarrow (B^1, L^1 \times D^1)$. For this we require

- 1) $B^1 = B \cup [y \mid x = \delta_i(y) \text{ for some } x \in B \text{ and some } i]$;
- 2) $D^1(y) = (\delta_1(y), \delta_2(y), \dots, \delta_n(y))$ for $y \in B^1$;
- 3) for each $y \in B^1$ there is a rule $(L^{11}(y), P(y), L^1(y))$ such that $P(y)$ holds of the context $(L^{11} \circ \delta_1(y), L^{11} \circ \delta_2(y), \dots, L^{11} \circ \delta_n(y))$ where $L^{11}(y) = \underline{\text{if } y \in B \text{ then } L(y) \text{ else } s_0}$.

Example 1

$$m = 1, \delta_1(n) = n-1.$$

There are three states: a, b, s_0 .

The seed is $[[17], L_0 \times D_0]$ where $L_0(17) = a, D_0(17) = 16$. The Rule repertoire is: $(a, \text{true}, b), (b, \text{true}, b), (s_0, \text{true}, a), (s_0, \text{true}, s_0)$. There are two configurations y such that $\text{seed} \rightarrow y$:

- $[[17, 18], L_1 \times D_1]$ where $L_1(17) = b, L_1(18) = a, D_1(17) = (16), D_1(18) = (17)$;
- $[[17, 18], L_2 \times D_2]$ where $L_2(17) = b, L_2(18) = s_0, D_2(17) = (16), D_2(18) = (17)$.

Example 2 (Conway's life)

Suppose the points with integer coefficients in two-dimensional Euclidean space are numbered in some way. Then we have a 1:1 function J from such points onto Integer, and we can define $\delta_1, \delta_2, \dots, \delta_8$ as the functions from Integer to Integer given by:

$$\begin{aligned}\delta_1(J(x, y)) &= J(x-1, y) & \delta_2(J(x, y)) &= J(x+1, y) \\ \delta_3(J(x, y)) &= J(x-1, y-1) & \delta_4(J(x, y)) &= J(x+1, y-1) \\ \delta_5(J(x, y)) &= J(x-1, y+1) & \delta_6(J(x, y)) &= J(x+1, y+1) \\ \delta_7(J(x, y)) &= J(x, y+1) & \delta_8(J(x, y)) &= J(x, y-1)\end{aligned}$$

Suppose there are two states: s_1, s_0 . Suppose that the rule repertoire is: $(s_0, \text{BORN}, s_1), (s_0, \neg \text{BORN}, s_1), (s_1, \text{LIVE}, s_1), (s_1, \neg \text{LIVE}, s_0)$ where BORN (LIVE) is true of contexts containing 3 (2 or 3) occurrences of the state s_1 . Let us define the seed as $(B_0, L_0 \times D_0)$ where

$$\begin{aligned}B_0 &= [J(0, 0), J(0, 1), J(0, 2), J(1, 0), J(1, 1), J(1, 2), J(2, 1)]. \\ L_0(x) &= s_1 \text{ for } x \in B_0; \\ D_0(x) &= (\delta_1(x), \delta_2(x), \dots, \delta_8(x)) \text{ for } x \in B_0.\end{aligned}$$

This time there is only one y such that $\text{Seed} \rightarrow y$.

It is $(B_1, L_1 \times D_1)$ where:

$$\begin{aligned}B_1 &= [J(i, j) \mid -1 \leq i, j \leq 3] - [J(3, -1), J(3, 3)]; \\ L_1(x) &= \underline{\text{if}} \ x = J(0, 0) \vee x = J(0, 2) \vee x = J(-1, 1) \\ &\quad \vee x = J(2, 0) \vee x = J(2, 1) \vee x = J(2, 2) \ \underline{\text{then}} \ s_1 \ \underline{\text{else}} \ s_0; \\ D_1(x) &= (\delta_1(x), \delta_2(x), \dots, \delta_8(x)) \text{ for } x \in B_1.\end{aligned}$$

The Web model class.

In a model of this class we have a finite set **State** such that:
Structure = a subset of Integer, **Context** = a set of functions from **State** to Integer, **Latent Organism** = **Configuration Space**. In addition we have a finite subset of **State**² called the forbidden set. By definition we have:

$$(B, L \times D) \rightarrow (B', L' \times D')$$

if and only if there is a function ρ from B to **Latent Organism** and a function τ from B to subsets of $\text{Integer}^{\text{Context}}$ such that:

- 1) if $x, y \in B$, then $y \notin \tau(x)$ and $\tau(y)$ is disjoint from $\tau(x)$;

2) for each $x \in B$ there is a rule $(L(x), P(x), \rho(x))$ such that $P(x)$ holds of the context $C(x)$ given by:

$$C(x)(s) = \text{number of } y \in D(x) \text{ such that } L(y) = s;$$

3) $\tau(x)$ has the same cardinality as the first component of $\rho(x)$;

4) $(B', L' \times D') = \mathcal{W}(B, D, \rho, \tau, \text{forbidden set})$.

Clearly we must define the function $\mathcal{W}$. For each $x \in B$ we have a latent organism $\rho(x) = (B_x, L_x \times D_x)$. By condition (3) there is a uniquely defined order preserving function h from $\tau(x)$ onto B_x . We define:

$$\begin{aligned} B'' &= \text{union of } \tau(x) \text{ for } x \in B \\ L''(y) &= L_x \circ h_x(y) \text{ for } y \in \tau(x) \\ D''(y) &= h_x^{-1} \circ D_x \circ h_x(y) \text{ for } y \in \tau(x) \\ D'''(y) &= [y' \mid (\exists x') (x' \in D(x) \wedge y' \in \tau(x') \\ &\quad \wedge (L''(y), L''(y')) \notin \text{forbidden set})] \end{aligned}$$

$$\mathcal{W}(B, D, \rho, \tau, \text{forbidden set}) = (B'', L'' \times (D'' \cup D''')).$$

Example 3

There are two states: a, b . The forbidden set consists of: $(a, b), (b, a)$. The Seed is $([1], L_0 \times D_0)$ where $L_0(1) = a, D_0(1) = \emptyset$. We shall need two other configurations:

$$\begin{aligned} \alpha &= ([2, 3], L_\alpha \times D_\alpha) \text{ where } L_\alpha(2) = b, D_\alpha(2) = [3] \\ &\quad L_\alpha(3) = a, D_\alpha(3) = [2]; \\ \beta &= ([4], L_\beta \times D_\beta) \text{ where } L_\beta(4) = b, D_\beta(4) = \emptyset. \end{aligned}$$

Now we can give the Rule repertoire: $(a, \text{true}, \alpha), (b, \text{true}, \beta)$. To find a new member of the language, we choose ρ and τ such that $\rho(1) = \alpha, \tau(1) = [7, 9]$. Our function h_1 becomes: $h_1(7) = 2, h_1(9) = 3$, and we get

$$\begin{aligned} B'' &= [7, 9] \\ L''(7) &= L_\alpha(2) = b, L''(9) = L_\alpha(3) = a \\ D''(7) &= h_1^{-1} \circ D_\alpha(2) = [9], D''(9) = h_1^{-1} \circ D_\alpha(3) = [7] \\ \text{Seed} &\rightarrow (B'', L'' \times D''). \end{aligned}$$

Example 4

There are three states: a, b, e . The forbidden set consists of $(a, b), (b, a)$. The Seed is $([1, 2, 3, 4], L_0 \times D_0)$ where:

$$\begin{aligned} L_0(1) &= b, L_0(2) = b, L_0(3) = e, L_0(4) = a \\ D_0(1) &= [2, 3], D_0(2) = [1, 3], D_0(3) = [1, 2, 4], D_0(4) = [3]. \end{aligned}$$

We shall need three other configurations:

$$\alpha = ([1], L_\alpha \times D_\alpha) \text{ where } L_\alpha(1) = a, D_\alpha(1) = \emptyset$$

$$\beta = ([1], L_\beta \times D_\beta) \text{ where } L_\beta(1) = b, D_\beta(1) = \emptyset$$

$$\epsilon = ([1, 2], L_\epsilon \times D_\epsilon) \text{ where } L_\epsilon(1) = L_\epsilon(2) = b, D_\epsilon(1) = [2], D_\epsilon(2) = [1].$$

Now we can give the Rule repertoire: (a, true, α), (b, true, β), (e, true, ϵ). To find a new member of the language, we choose ρ and τ such that:

$$\begin{aligned} \rho(1) &= \beta & \rho(2) &= \beta & \rho(3) &= \epsilon & \rho(4) &= \alpha \\ \tau(1) &= [5] & \tau(2) &= [6] & \tau(3) &= [7, 8] & \tau(4) &= [9]. \end{aligned}$$

Our functions h_1, h_2, h_3, h_4 become: $h_1(5) = 1 = h_2(6), h_3(7) = 1, h_3(8) = 2, h_4(9) = 1$, and we get:

$$B'' = [5, 6, 7, 8, 9]$$

$$\begin{array}{lll} L''(5) = L_\beta(1) = b & D''(5) = \emptyset & D'''(5) = [6, 7, 8] \\ L''(6) = L_\beta(1) = b & D''(6) = \emptyset & D'''(6) = [5, 7, 8] \\ L''(7) = L_\epsilon(1) = b & D''(7) = [8] & D'''(7) = [5, 6] \\ L''(8) = L_\epsilon(2) = b & D''(8) = [7] & D'''(8) = [5, 6] \\ L''(9) = L_\alpha(1) = a & D''(9) = \emptyset & D'''(9) = \emptyset \end{array}$$

$$\text{Seed} \rightarrow ([5, 6, 7, 8, 9], L'' \times (D'' \cup D''')).$$

The Map model class

In a model of this class we have a finite set State such that: Structure = circular words on Integer, Context = circular words on State, Latent Organism = Configuration space $\times$ Restriction where

$$\text{Restriction} = \text{subsets of Integer}^2 \times \text{State}.$$

By definition we have $(B \cup \{0\}, L \times D) \rightarrow (B' \cup \{0\}, L' \times D')$ if and only if there is a function ρ from B to Latent Organism, a function σ from Integer^2 to $(\text{Integer}^*)^*$, and a function τ from B to subsets of Integer such that:

- 1) if $x, y \in B$, then $0, y \notin \tau(x)$ and $\tau(x)$ is disjoint from $\tau(y)$;
- 2) for each $x \in B$ there is a rule $(L(x), P(x), \rho(x))$ such that $P(x)$ holds of the context $L * D(x)$;
- 3) $\tau(x) \cup \{0\}$ has the same cardinality as the first component of the first component of $\rho(x)$;
- 4) σ is consistent with respect to (B, L, D, ρ, τ) (see note below);
- 5) $(B' \cup \{0\}, L' \times D') = \mathcal{M}(B, L, D, \rho, \sigma, \tau)$.

Clearly we must define the function $\mathcal{M}$. For each $x \in B$ we have a latent organism $\rho(x) = ((B_x, L_x \times D_x), r_x)$. By condition (3) there is a uniquely defined 1:1 order-preserving function h_x from $\tau(x) \cup \{0\}$ onto B_x . We define:

$$\begin{aligned} B^{II} &= \text{union of } \tau(x) && \text{for } x \in B \\ L^{II}(y) &= L_x \circ h_x(y) && \text{for } y \in \tau(x) \\ D^{II}(y) &= h_x^{-1} * (D_x \circ h_x(y)) && \text{for } y \in \tau(x). \end{aligned}$$

Now we have to define the patching together of a number of configurations. For each $x \in B$, $y \in \tau(x)$ we define (see note below):

$$\begin{aligned} w(y) &= \text{Flatten } (\sigma_y * D(x)) \text{ where } \sigma_y(x') = \sigma(y, x') \\ D^{III}(y) &= \text{Substitute } (w(y), D^{II}(y)) \end{aligned}$$

If we set $D^{III}(0) = \Phi$ and $L^{II}(0) = L(0)$, then we can define $\mathcal{M}(B, L, D, \rho, \sigma, \tau)$ to be $(B^{III} \cup \{0\}, L^{II}, D^{III})$.

Note

After a careful study of the two examples we are about to give, the eager reader should be able to give a precise definition of: Flatten, Substitute, σ is consistent with respect to (B, L, D, ρ, τ) . Here we will only mention:

- the most interesting consistency requirement on σ is that we may not have $(h_x^{-1}(i), h_x^{-1}(j), L(y))$ in r_x when we have $i, j \in \tau(x) \wedge \sigma(i, j) = y$ for $x, y \in B$;
- Flatten reads around a circular word, and it produces linear words as it proceeds;
- Substitute (w, c) replaces occurrences of 0 in the circular word c by linear words from w .

Example 5 is the discrete version of figure 8. Let us begin by inviting the reader to draw the following circular words on Integer:

$$\begin{aligned} D_0(0) &= D^{III}(0) = \Phi \\ D_0(4) &= ([0, 1, 2], \pi_0, v_4) \\ &\quad \text{where } \pi_0(0) = 1, \pi_0(1) = 2, \pi_0(2) = 0 \\ &\quad \quad v_4(0) = 0, v_4(1) = 5, v_4(2) = 6 \\ D_0(5) &= ([0, 1, 2], \pi_0, v_5) \text{ where } v_5(0) = 0, v_5(1) = 5, v_5(2) = 4 \\ D_0(6) &= ([0, 1, 2], \pi_0, v_6) \text{ where } v_6(0) = 0, v_6(1) = 4, v_6(2) = 5 \\ D_\alpha(0) &= ([0, 1], \pi_\alpha, v_0) \\ &\quad \text{where } \pi_\alpha(0) = 1, \pi_\alpha(1) = 0, v_0(0) = 1, v_0(1) = 2 \\ D_\alpha(1) &= ([0, 1], \pi_\alpha, v_1) \text{ where } v_1(0) = 0, v_1(1) = 2 \\ D_\alpha(2) &= ([0, 1], \pi_\alpha, v_2) \text{ where } v_2(0) = 0, v_2(1) = 1 \end{aligned}$$

$$\begin{aligned}
D_{\beta}(0) &= ([1], \text{identity}, \text{identity}) \\
D_{\beta}(1) &= ([0], \text{identity}, \text{identity}) \\
D^{III}(1) &= ([0, 1, 2], \pi_0, v_1^I) \text{ where } v_1^I(0) = 0, v_1^I(1) = 2, v_1^I(2) = 8 \\
D^{III}(7) &= ([0, 1, 2], \pi_0, v_7^I) \text{ where } v_7^I(0) = 0, v_7^I(1) = 8, v_7^I(2) = 2 \\
D^{III}(2) &= ([0, 1, 2, 3], \pi^I, v_2^I) \\
&\quad \text{where } \pi^I(0) = 1, \pi^I(1) = 2, \pi^I(2) = 3, \pi^I(3) = 0 \\
&\quad v_2^I(0) = 0, v_2^I(1) = 7, v_2^I(2) = 8, v_2^I(3) = 1 \\
D^{III}(8) &= ([0, 1, 2, 3], \pi^I, v_8^I) \\
&\quad \text{where } v_8^I(0) = 0, v_8^I(1) = 1, v_8^I(2) = 2, v_8^I(3) = 7.
\end{aligned}$$

Now we can give the model. There are four states: p , s , t , s_0 . The Seed is $([0, 4, 5, 6], L_0 \times D_0)$ where $L_0(0) = s_0$, $L_0(4) = p$, $L_0(5) = t$, $L_0(6) = t$. We shall need two other configurations:

$$\begin{aligned}
\alpha &= ([0, 1, 2], L_{\alpha} \times D_{\alpha}) \text{ where } L_{\alpha}(0) = s_0, L_{\alpha}(1) = p, L_{\alpha}(2) = s \\
\beta &= ([0, 1], L_{\beta} \times D_{\beta}) \text{ where } L_{\beta}(0) = s_0, L_{\beta}(1) = t.
\end{aligned}$$

The rule repertoire is: $(p, \text{true}, (\alpha, [(2, 1, s_0)])),$
 $(t, \text{true}, (\beta, \emptyset)).$

To find a new member of the language, we can choose the functions ρ, τ given by: $\rho(4) = (\alpha, [(2, 1, s_0)]), \rho(5) = \rho(6) = (\beta, \emptyset), \tau(4) = [1, 2], \tau(5) = [7], \tau(6) = [8]$. Our functions h_4, h_5, h_6 become $h_4(1) = 1, h_4(2) = 2, h_5(7) = 1, h_6(8) = 1$, and we get:

$$\begin{aligned}
B^{II} &= [1, 2, 7, 8] & L^{II}(0) &= s_0 \\
L^{II}(1) &= L_{\alpha}(1) = p & D^{II}(1) &= D_{\alpha}(1) \\
L^{II}(2) &= L_{\alpha}(2) = s & D^{II}(2) &= D_{\alpha}(2) \\
L^{II}(7) &= L_{\beta}(1) = t & D^{II}(7) &= h_5^{-1} * D_{\beta}(1) = ([0], \text{id}, \text{id}) \\
L^{II}(8) &= L_{\beta}(1) = t & D^{II}(8) &= h_6^{-1} * D_{\beta}(1) = ([0], \text{id}, \text{id})
\end{aligned}$$

Now we patch the structures together using a function σ satisfying: $\sigma(1, 2) = 0$, $\sigma(2, 1) = 6$, $\sigma(1, 5) = \Lambda$, $\sigma(1, 6) = 8$, $\sigma(2, 5) = 7$, $\sigma(2, 6) = 8$, $\sigma(7, 6) = 8$, $\sigma(7, 4) = 2$, $\sigma(8, 4) = 12$, $\sigma(8, 5) = 7$. We note that the only applied rule with a non-empty restriction is satisfied because $L(6) \neq s_0$, and continue by computing:

$$w(1) = 8, w(2) = 78, w(3) = 82, w(4) = 127.$$

Substituting these in D^{II} gives:

$$\text{Seed} \rightarrow ([0, 1, 2, 7, 8], L^{II}, D^{II}).$$

Example 6 shows other features of map models. Again we begin by inviting the reader to draw the following circular words in integer:

$$\begin{aligned}
D_0(0) &= D'''(0) = \Phi \\
D_0(11) &= ([0, 12, 13, 14], \pi_{11}, \text{identity}) \\
&\quad \text{where } \pi_{11}(0) = 14, \pi_{11}(14) = 12, \pi_{11}(12) = 13, \pi_{11}(13) = 0 \\
D_0(12) &= ([11, 13, 14], \pi_{12}, \text{identity}) \\
&\quad \text{where } \pi_{12}(11) = 14, \pi_{12}(14) = 13, \pi_{12}(13) = 11 \\
D_0(13) &= ([0, 11, 12, 14], \pi_{13}, \text{identity}) \\
&\quad \text{where } \pi_{13}(0) = 11, \pi_{13}(11) = 12, \pi_{13}(12) = 14, \pi_{13}(14) = 0 \\
D_0(14) &= ([0, 11, 12, 13], \pi_{14}, \text{identity}) \\
D_\alpha(0) &= ([4, 5], \pi_{\alpha_0}, \text{identity}) \text{ where } \pi_{\alpha_0}(4) = 5, \pi_{\alpha_0}(5) = 4 \\
D_\alpha(4) &= ([0, 5], \pi_{\alpha_4}, \text{identity}) \text{ where } \pi_{\alpha_4}(0) = 5, \pi_{\alpha_4}(5) = 0 \\
D_\alpha(5) &= ([0, 4], \pi_{\alpha_5}, \text{identity}) \text{ where } \pi_{\alpha_5}(0) = 4, \pi_{\alpha_5}(4) = 0 \\
D_\beta(0) &= ([1, 2, 3], \pi_{\beta_0}, \text{identity}) \\
&\quad \text{where } \pi_{\beta_0}(1) = 2, \pi_{\beta_0}(2) = 3, \pi_{\beta_0}(3) = 1 \\
D_\beta(1) &= ([0, 2, 3], \pi_{\beta_1}, \text{identity}) \\
&\quad \text{where } \pi_{\beta_1}(0) = 2, \pi_{\beta_1}(2) = 3, \pi_{\beta_1}(3) = 0 \\
D_\beta(2) &= ([0, 1, 3], \pi_{\beta_2}, \text{identity}) \\
&\quad \text{where } \pi_{\beta_2}(0) = 3, \pi_{\beta_2}(3) = 1, \pi_{\beta_2}(1) = 0 \\
D_\beta(3) &= ([0, 1, 2], \pi_{\beta_3}, \text{identity}) \\
&\quad \text{where } \pi_{\beta_3}(0) = 1, \pi_{\beta_3}(1) = 2, \pi_{\beta_3}(2) = 0 \\
D_\epsilon(0) &= ([7, 8], \pi_{\epsilon_0}, \text{identity}) \text{ where } \pi_{\epsilon_0}(7) = 8, \pi_{\epsilon_0}(8) = 7 \\
D_\epsilon(7) &= ([0, 8], \pi_{\epsilon_7}, \text{identity}) \text{ where } \pi_{\epsilon_7}(0) = 8, \pi_{\epsilon_7}(8) = 0 \\
D_\epsilon(8) &= ([0, 7], \pi_{\epsilon_8}, \text{identity}) \text{ where } \pi_{\epsilon_8}(0) = 7, \pi_{\epsilon_8}(7) = 0 \\
D_{\delta_0}(0) &= ([6, 9], \pi_{\delta_0}, \text{identity}) \text{ where } \pi_{\delta_0}(6) = 9, \pi_{\delta_0}(9) = 6 \\
D_{\delta_0}(6) &= ([0, 9], \pi_{\delta_6}, \text{identity}) \text{ where } \pi_{\delta_6}(0) = 9, \pi_{\delta_6}(9) = 0 \\
D_{\delta_0}(9) &= ([0, 6], \pi_{\delta_9}, \text{identity}) \text{ where } \pi_{\delta_9}(0) = 6, \pi_{\delta_9}(6) = 0 \\
D'''(1) &= ([2, 3, 4, 5, 7, 8], \pi_1^1, \text{identity}) \\
&\quad \text{where } \pi_1^1(8)=7, \pi_1^1(7)=4, \pi_1^1(4)=5, \pi_1^1(5)=2, \pi_1^1(2)=3, \pi_1^1(3)=8 \\
D'''(2) &= ([1, 3, 5], \pi_2^1, \text{identity}) \\
&\quad \text{where } \pi_2^1(5)=3, \pi_2^1(3)=1, \pi_2^1(1)=5 \\
D'''(3) &= ([1, 2, 5, 6, 7, 8], \pi_3^1, \text{identity}) \\
&\quad \text{where } \pi_3^1(5)=6, \pi_3^1(6)=7, \pi_3^1(7)=8, \pi_3^1(8)=1, \pi_3^1(1)=2, \pi_3^1(2)=5 \\
D'''(4) &= ([0, 1, 5, 7], \pi_4^1, \text{identity}) \\
&\quad \text{where } \pi_4^1(0)=5, \pi_4^1(5)=1, \pi_4^1(1)=7, \pi_4^1(7)=0 \\
D'''(5) &= ([0, 1, 2, 3, 4, 6], \pi_5^1, \text{identity}) \\
&\quad \text{where } \pi_5^1(0)=6, \pi_5^1(6)=3, \pi_5^1(3)=2, \pi_5^1(2)=1, \pi_5^1(1)=4, \pi_5^1(4)=0 \\
D'''(6) &= ([0, 3, 5, 7, 9, 10], \pi_6^1, \text{if } i=10 \text{ then } 0 \text{ else } i) \\
&\quad \text{where } \pi_6^1(0)=7, \pi_6^1(7)=3, \pi_6^1(3)=5, \pi_6^1(5)=10, \pi_6^1(10)=9, \pi_6^1(9)=0 \\
D'''(7) &= ([0, 1, 3, 4, 6, 8], \pi_7^1, \text{identity}) \\
&\quad \text{where } \pi_7^1(0)=4, \pi_7^1(4)=1, \pi_7^1(1)=8, \pi_7^1(8)=3, \pi_7^1(3)=6, \pi_7^1(6)=0 \\
D'''(8) &= ([1, 3, 7], \pi_8^1, \text{identity}) \\
&\quad \text{where } \pi_8^1(3)=7, \pi_8^1(7)=1, \pi_8^1(1)=3
\end{aligned}$$

$$D'''(9) = D_\delta(9).$$

Now we can give the model. There are five states: a, b, c, d, s_0 . The Seed is $([0, 11, 12, 13, 14], L_0 \times D_0)$ where $L_0(0) = s_0$, $L_0(11) = a$, $L_0(12) = b$, $L_0(13) = c$, $L_0(14) = d$. We shall need four other configurations:

$$\begin{aligned} \alpha &= ([0, 4, 5], L_\alpha \times D_\alpha) & \text{where } L_\alpha(0) = s_0, L_\alpha(4) = L_\alpha(5) = a \\ \beta &= ([0, 1, 2, 3], L_\beta \times D_\beta) & \text{where } L_\beta(0) = s_0, L_\beta(1) = L_\beta(2) = L_\beta(3) = b \\ \epsilon &= ([0, 7, 8], L_\epsilon \times D_\epsilon) & \text{where } L_\epsilon(0) = s_0, L_\epsilon(7) = L_\epsilon(8) = c \\ \delta &= ([0, 6, 9], L_\delta \times D_\delta) & \text{where } L_\delta(0) = s_0, L_\delta(6) = L_\delta(9) = d. \end{aligned}$$

The rule repertoire is: $(A, \text{true}, (\alpha, \emptyset))$, $(b, \text{true}, (\beta, \emptyset))$, $(c, \text{true}, (\epsilon, \emptyset))$, $(d, \text{true}, (\delta, \emptyset))$. To find a new member of the language, we can choose the functions ρ and τ given by:

$$\begin{aligned} \rho(11) &= (\alpha, \emptyset) & \rho(12) &= (\beta, \emptyset) & \rho(13) &= (\epsilon, \emptyset) & \rho(14) &= (\delta, \emptyset) \\ \tau(11) &= [4, 5] & \tau(12) &= [1, 2, 3] & \tau(13) &= [7, 8] & \tau(14) &= [6, 9]. \end{aligned}$$

With this choice $h_{11}, h_{12}, h_{13}, h_{14}$ are identity mappings and we get:

$$\begin{array}{ll} B'' = [1, 2, 3, 4, 5, 6, 7, 8, 9] & L''(0) = s_0 \\ L''(1) = L_\beta(1) = b & D''(1) = D_\beta(1) \\ L''(2) = L_\beta(2) = b & D''(2) = D_\beta(2) \\ L''(3) = L_\beta(3) = b & D''(3) = D_\beta(3) \\ L''(4) = L_\alpha(4) = a & D''(4) = D_\alpha(4) \\ L''(5) = L_\alpha(5) = a & D''(5) = D_\alpha(5) \\ L''(6) = L_\delta(6) = d & D''(6) = D_\delta(6) \\ L''(7) = L_\epsilon(7) = c & D''(7) = D_\epsilon(7) \\ L''(8) = L_\epsilon(8) = c & D''(8) = D_\epsilon(8) \\ L''(9) = L_\delta(9) = d & D''(9) = D_\delta(9) \end{array}$$

Now we patch the structures together using a function σ satisfying: $\sigma(1, 2) = 11$, $\sigma(2, 3) = 11$, $\sigma(1, 3) = 13$, $\sigma(4, 5) = 0$, $\sigma(5, 4) = 12$, $\sigma(6, 9) = 0$, $\sigma(9, 6) = 0$, $\sigma(7, 8) = 12$, $\sigma(8, 7) = 12$, $\sigma(1, 11) = 45$, $\sigma(1, 14) = \Lambda$, $\sigma(1, 13) = 87$, $\sigma(2, 11) = 5$, $\sigma(2, 14) = \Lambda$, $\sigma(2, 13) = \Lambda$, $\sigma(3, 11) = 5$, $\sigma(3, 14) = 6$, $\sigma(3, 13) = 78$, $\sigma(4, 14) = \Lambda$, $\sigma(4, 12) = 1$, $\sigma(4, 13) = 7$, $\sigma(5, 14) = 6$, $\sigma(5, 12) = 321$, $\sigma(5, 13) = \Lambda$, $\sigma(6, 13) = 7$, $\sigma(6, 12) = 3$, $\sigma(6, 11) = 5$, $\sigma(7, 11) = 4$, $\sigma(7, 12) = 1$, $\sigma(7, 14) = 6$, $\sigma(8, 11) = \Lambda$, $\sigma(8, 12) = 13$, $\sigma(8, 14) = \Lambda$, $\sigma(9, 13) = \Lambda$, $\sigma(9, 12) = \Lambda$, $\sigma(9, 11) = \Lambda$. We note that all rules have empty restrictions and continue by computing:

$$\begin{aligned}
 w(1) &= 8745 & , \quad w(2) &= 5, \quad w(3) = 5678 \\
 w(4) &= 17 & , \quad w(5) &= 6321 \\
 w(6) &= 735 & , \quad w(9) &= \Lambda \\
 w(7) &= 41, 36 & , \quad w(8) &= 31.
 \end{aligned}$$

Substituting these in D^{II} gives:

$$Seed \rightarrow ([0, 1, 2, 3, 4, 5, 6, 7, 8, 9], L^{II}, D^{III}).$$