A Generalized Preimage for the Standard and Supercover Digital Hyperplane Recognition

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Abstract. A new digital hyperplane recognition method is presented. This algorithm allows the recognition of Standard and Supercover hyperplanes by incrementally computing in a dual space the generalized preimage of a given hypervoxel set. Each point in this preimage corresponds to a Euclidean hyperplane which intersects all given hypervoxels. An advantage of the generalized preimage is that it does not depend on the hypervoxel locations. Moreover, the proposed recognition algorithm does not require the hypervoxels to be connected or ordered in any way.

1 Introduction

In digital geometry, objects are usually considered as digital point or hypervoxel (*pixels* in 2D and *voxels* in 3D) sets. Indeed, this is the structural decomposition mostly used to store digital information. A drawback of this kind of representation is that it does not provide any information on the shape or topology of digital objects. Another way of obtaining the description of digital objects is the hyperplane decomposition. This process, called *digital hyperplane recognition*, consists of determining if a digital point set forms a hyperplane segment, that is a hyperplane bounded region.

The recognition problem has so far mainly been studied in dimensions 2 and 3 (see [1] for an overview on 2D recognition algorithms), with various approaches such as linear programming techniques [2, 3], computational geometry methods [4, 5, 6] or *preimage* computation based algorithms [7]. Very few papers handle the problem in arbitrary dimensions [8, 9]. Computational and efficiency aspects of digital hyperplane recognition problems are investigated in [10].

In this paper, we are interested in preimage based approaches for the recognition of Standard and Supercover analytical hyperplanes. Informally, the preimage [11] of a hypervoxel set consists of all Euclidean hyperplanes the digitization of which contains the given hypervoxels. More precisely, the preimage of a hypervoxel set is computed in a dual space where each point is mapped onto a Euclidean hyperplane. Preimage computation algorithms depending on the hypervoxel locations have been proposed in dimensions 2 and 3 [7, 12].

In this work, we perform the recognition of Standard and Supercover digital hyperplanes by computing the set of Euclidean hyperplanes which intersect a

given hypervoxel set. In order to do that, we incrementally compute the *generalized preimage* of the hypervoxels, which is a preimage defined in any dimension and independent of the hypervoxel connectivity and location. This preimage is computed from the dual of each hypervoxel. Indeed, each point in this dual object corresponds to a Euclidean hyperplane which cuts the hypervoxel. Hence, a major part of this paper is devoted to determining the formulas describing the dual of a hypervoxel. First, a positive and a negative extrusion are defined. Then, we show that the dual of a hypervoxel can be computed from the extrusions of the dual of its vertices. Finally, the intersection of all hypervoxel duals forms the generalized preimage. The recognition process consists therefore simply in computing the generalized preimage of a hypervoxel set (i.e. computing the dual of a hypervoxel set). More precisely, we start with a hypervoxel dual and add hypervoxel duals as long as the generalized preimage is not empty.

In Section 2, we introduce our notations and definitions as well as the Standard and Supercover analytical hyperplane descriptions. In Section 3, we determine the dual of a hypervoxel. The generalized preimage of a hypervoxel set is introduced in Section 4. Then, we explain how the Standard and Supercover hyperplane recognition algorithm works. We also provide some hints on how the computational efficiency can be increased thanks to a particular hypervoxel dual description. Conclusion and future works are proposed in Section 5.

2 Preliminaries

In this section, we first propose some notations and give the definition of a hypercube. Then, we present the two digitization models considered in this work that are the Standard model and the Supercover model.

2.1 Notations and Definitions

Let $n \in \mathbb{Z}$, $n > 0$. In the following, we will denote by $\mathcal{E}_n$ the classical n -dimensional Euclidean space, and by $\llbracket 1, k \rrbracket$ the subset of integer values $\{1, \dots, k\} \subset \mathbb{Z}$. We define an α -*hypercube*, $\alpha \in \mathbb{R}$, as follows:

Definition 1 (α -Hypercube). *The hypercube (or n -dimensional cube) centered on $(c_1, \dots, c_n) \in \mathbb{R}^n$, with size $\alpha \in \mathbb{R}$, is the set of points $(x_1, \dots, x_n) \in \mathbb{R}^n$ verifying*

$$\forall i \in \llbracket 1, n \rrbracket, c_i - \frac{\alpha}{2} \leq x_i \leq c_i + \frac{\alpha}{2}$$

Let $p \in \mathbb{Z}^n$ be a point with integer-valued coordinates, also called *digital point*. We call *hypervoxel* a unit-size hypercube centered on a digital point. Hypervoxels in dimensions 2 and 3 are respectively called *pixels* and *voxels*.

2.2 The Standard and Supercover Analytical Models

The *Standard model* [13] and *Supercover model* [14, 15] are both digital analytical models, defined in any dimension, which provide a digitization of Euclidean

objects. Moreover, the Standard model is the only one that allows, in dimension n , the $(n - 1)$ -connected digitization of any linear object of $\mathbb{R}^n$.

Standard and Supercover hyperplanes (or n -dimensional planes) are defined analytically as follows (see Figure 1):

Definition 2 (Standard Hyperplane [13]). *The Standard hyperplane with parameters $(c_0, \dots, c_n) \in \mathbb{R}^{n+1}$ is the set of points $(x_1, \dots, x_n) \in \mathbb{Z}^n$ verifying*

$$-\frac{\sum_{i=1}^n |c_i|}{2} \leq c_0 + \sum_{i=1}^n c_i x_i < \frac{\sum_{i=1}^n |c_i|}{2}$$

where $c_1 \geq 0$, or $c_1 = 0$ and $c_2 \geq 0$, or ..., or $c_1 = c_2 = \dots = c_{n-1} = 0$ and $c_n \geq 0$.

Definition 3 (Supercover Hyperplane [15]). *The Supercover hyperplane with parameters $(c_0, \dots, c_n) \in \mathbb{R}^{n+1}$ is the set of points $(x_1, \dots, x_n) \in \mathbb{Z}^n$ verifying*

$$-\frac{\sum_{i=1}^n |c_i|}{2} \leq c_0 + \sum_{i=1}^n c_i x_i \leq \frac{\sum_{i=1}^n |c_i|}{2}$$

Fig. 1. Standard and Supercover hyperplane examples in dimensions 2 and 3: (a) The Supercover and Standard lines with parameters $(0, 3, -7)$. The pixel colored in white does not belong to the Standard line (pixels colored in grey) but belongs to the Supercover one, (b) The Standard plane with parameters $(0, 3, -1, 2)$

Remark 1. The Supercover digitization of a Euclidean hyperplane also consists of all hypervoxels which are intersected by the hyperplane, whereas the Standard one consists of all hypervoxels cut by the hyperplane except when a hypervoxel vertex is intersected (see Figure 1a). In this case, several hypervoxels adjacent to this vertex do not belong to the Standard digitization. This is due to the fact that one inequality in Definition 2 is strict.

3 Dual of a Hypervoxel

We use a dual transformation similar to the well known Hough transform which is an efficient tool usually used in image processing to recognize parametric shapes in an image. A review on existing variations of this method is presented in [16].

In the two following sections, we first define the parameter space in which our dual transformation is performed. Then, we describe the dual of a hypervoxel, which is the basis of the recognition algorithm presented in Section 4.

3.1 Parameter Space: Definition and Properties

In this work, we use the n -dimensional parameter space $\mathcal{P}_n \subset \mathbb{R}^n$, and define the two functions $\mathcal{D}_{\mathcal{E}} : \mathcal{E}_n \rightarrow \mathcal{P}_n$ and $\mathcal{D}_{\mathcal{P}} : \mathcal{P}_n \rightarrow \mathcal{E}_n$ by:

$$\mathcal{D}_{\mathcal{E}}(x_1, \dots, x_n) = \left\{ (y_1, \dots, y_n) \in \mathcal{P}_n \mid y_n = - \sum_{i=1}^{n-1} x_i y_i + x_n \right\}$$

$$\mathcal{D}_{\mathcal{P}}(y_1, \dots, y_n) = \left\{ (x_1, \dots, x_n) \in \mathcal{E}_n \mid x_n = \sum_{i=1}^{n-1} y_i x_i + y_n \right\}$$

Informally, each point in $\mathcal{E}_n$ (resp. $\mathcal{P}_n$) is transformed by $\mathcal{D}_{\mathcal{E}}$ (resp. $\mathcal{D}_{\mathcal{P}}$) into a hyperplane in $\mathcal{P}_n$ (resp. $\mathcal{E}_n$). In the rest of this paper, we will generically write *Dual* for $\mathcal{D}_{\mathcal{E}}$ or $\mathcal{D}_{\mathcal{P}}$.

Let O be a subset of $\mathbb{R}^n$. Then, $Dual(O) = \bigcup_{p \in O} Dual(p)$ is called the *dual* of O . Let O_1 and O_2 be two subsets of $\mathbb{R}^n$ such that $O_1 \subseteq O_2$. It is clear that $Dual(O_1) \subseteq Dual(O_2)$. Moreover, the following properties can be deduced from our definition of the duality.

Proposition 1. *Let O_1 and O_2 be two subsets of $\mathbb{R}^n$. Then,*

$$Dual(O_1 \cup O_2) = Dual(O_1) \cup Dual(O_2) \quad (1)$$

$$Dual(O_1 \cap O_2) \subseteq Dual(O_1) \cap Dual(O_2) \quad (2)$$

Proof. (1): $Dual(O_1 \cup O_2) = \bigcup_{p \in O_1 \cup O_2} Dual(p) = \left[\bigcup_{p \in O_1} Dual(p) \right] \cup \left[\bigcup_{p \in O_2} Dual(p) \right] = Dual(O_1) \cup Dual(O_2)$.

(2): Since $O_1 \cap O_2 \subseteq O_1$ and $O_1 \cap O_2 \subseteq O_2$, we deduce that $Dual(O_1 \cap O_2) \subseteq Dual(O_1)$ and $Dual(O_1 \cap O_2) \subseteq Dual(O_2)$. Thus, $Dual(O_1 \cap O_2) \subseteq Dual(O_1) \cap Dual(O_2)$. $\square$

Property 1. Let $p \in \mathbb{R}^n$ be a point. The dual of each point which lies in $Dual(p)$ is a hyperplane which passes through p .

3.2 Hypervoxel Dual Representation

In this work, we need to define the dual of a hypervoxel. We first define the positive and negative extrusions of a point.

Definition 4 (Positive and Negative Extrusions). *Let $p = (x_1, \dots, x_n) \in \mathbb{R}^n$ be a point. The positive extrusion of p is defined by:*

$$p^+ = \{p' = (x'_1, \dots, x'_n) \in \mathbb{R}^n \mid \forall i \in \llbracket 1, n-1 \rrbracket, x_i = x'_i \text{ and } x_n \leq x'_n\}$$

In the same way, the negative extrusion of p is defined by:

$$p^- = \{p' = (x'_1, \dots, x'_n) \in \mathbb{R}^n \mid \forall i \in \llbracket 1, n-1 \rrbracket, x_i = x'_i \text{ and } x_n \geq x'_n\}$$

Let O_1 and O_2 be two subsets of $\mathbb{R}^n$ such that $O_1 \subseteq O_2$. Then, $O_1^+ \subseteq O_2^+$ and $O_1^- \subseteq O_2^-$. Moreover, the following properties can be deduced from Definition 4.

Proposition 2. Let O_1 and O_2 be two subsets of $\mathbb{R}^n$. Then,

$$(O_1 \cup O_2)^+ = O_1^+ \cup O_2^+$$

In the same way, $(O_1 \cup O_2)^- = O_1^- \cup O_2^-$.

Proof. $(O_1 \cup O_2)^+ = \bigcup_{p \in O_1 \cup O_2} p^+ = \left[\bigcup_{p \in O_1} p^+ \right] \cup \left[\bigcup_{p \in O_2} p^+ \right] = O_1^+ \cup O_2^+$. The proof of $(O_1 \cup O_2)^- = O_1^- \cup O_2^-$ is obtained in the same way. $\square$

Proposition 3. Let $p \in \mathbb{R}^n$ be a point. Then,

$$\text{Dual}(p)^+ = \text{Dual}(p^+)$$

In the same way, $\text{Dual}(p)^- = \text{Dual}(p^-)$.

Proof. Let us consider $p = (x_1, \dots, x_n) \in \mathcal{E}_n$. Then, $\text{Dual}(p^+) = \mathcal{D}_{\mathcal{E}}(p^+) =$

$$\bigcup_{p' \in p^+} \text{Dual}(p') = \bigcup_{p' = (x'_1, \dots, x'_n) \in p^+} \{(y_1, \dots, y_n) \in \mathcal{P}_n \mid y_n = -\sum_{i=1}^{n-1} x'_i y_i + x'_n\} =$$

$$\{(y_1, \dots, y_n) \in \mathcal{P}_n \mid y_n \geq -\sum_{i=1}^{n-1} x_i y_i + x_n\} = \bigcup_{p' \in \mathcal{D}_{\mathcal{E}}(p)} p'^+ = \mathcal{D}_{\mathcal{E}}(p)^+ = \text{Dual}(p)^+.$$

The proof of $\text{Dual}(p)^- = \text{Dual}(p^-)$ can be obtained in the same way. $\square$

Proposition 3 is illustrated in Figure 2.

Fig. 2. Positive and negative extrusions of a point p (half-lines) and their dual object: a half-space, (a) Positive extrusion of p , (b) Negative extrusion

Proposition 4. *Let H be a hypervoxel. Then,*

$$\text{Dual}(H) = \text{Dual}(H)^+ \cap \text{Dual}(H)^-$$

Proof. In the following, we assume that $H \in \mathcal{E}_n$. Since $\text{Dual}(H) \subseteq \text{Dual}(H)^+$ and $\text{Dual}(H) \subseteq \text{Dual}(H)^-$, we deduce that $\text{Dual}(H) \subseteq \text{Dual}(H)^+ \cap \text{Dual}(H)^-$.

We now prove that $\text{Dual}(H)^+ \cap \text{Dual}(H)^- \subseteq \text{Dual}(H)$. Consider a point $p = (x_1, \dots, x_n) \in \text{Dual}(H)^+ \cap \text{Dual}(H)^-$. Then, there exists $p' = (x'_1, \dots, x'_n) \in \text{Dual}(H)$ such that $p \in p'^+$ and there exists $p'' = (x''_1, \dots, x''_n) \in \text{Dual}(H)$ such that $p \in p''^-$. We deduce that $\forall i \in \llbracket 1, n-1 \rrbracket$, $x'_i = x_i = x''_i$ and $x'_n \leq x_n \leq x''_n$.

Next we prove that $\text{Dual}(p) \cap H \neq \emptyset$, which would imply $p \in \text{Dual}(H)$. Since $p' \in \text{Dual}(H)$ and $p'' \in \text{Dual}(H)$, we have $\text{Dual}(p') \cap H \neq \emptyset$ and $\text{Dual}(p'') \cap H \neq \emptyset$. Let $q' = (q'_1, \dots, q'_n) \in \text{Dual}(p') \cap H$ and $q'' = (q''_1, \dots, q''_n) \in \text{Dual}(p'') \cap H$. Then, we have

$$q'_n = \sum_{i=1}^{n-1} x_i q'_i + x'_n \text{ and } q''_n = \sum_{i=1}^{n-1} x_i q''_i + x''_n$$

Since $x'_n \leq x_n \leq x''_n$, we deduce that

$$q'_n \leq \sum_{i=1}^{n-1} x_i q'_i + x_n \text{ and } q''_n \geq \sum_{i=1}^{n-1} x_i q''_i + x_n$$

Thus, $\text{Dual}(p) \cap [q', q''] \neq \emptyset$. Finally, since H is convex we know that $[q', q''] \subset H$. We then deduce that $\text{Dual}(p) \cap H \neq \emptyset$. $\square$

Let us now describe the dual of a hypervoxel from its vertices. Let H be a hypervoxel centered on $(c_1, \dots, c_n) \in \mathbb{Z}^n$ and let $\mathcal{V}$ be the set of its 2^n vertices. We define two subsets of $\mathcal{V}$ with cardinality 2^{n-1} as follows:

$$\begin{aligned} \mathcal{V}_+ &= \left\{ v = (x_1, \dots, x_n) \in \mathcal{V} \mid x_n = c_n + \frac{1}{2} \right\} \\ \mathcal{V}_- &= \left\{ v = (x_1, \dots, x_n) \in \mathcal{V} \mid x_n = c_n - \frac{1}{2} \right\} \end{aligned}$$

The dual of a hypervoxel can then be defined by:

Theorem 1 (Dual of a Hypervoxel). *Let H be a hypervoxel, $\mathcal{V}_+$ and $\mathcal{V}_-$ the two vertex sets defined previously. Then:*

$$\text{Dual}(H) = \left[\bigcup_{v \in \mathcal{V}_-} \text{Dual}(v)^+ \right] \cap \left[\bigcup_{v \in \mathcal{V}_+} \text{Dual}(v)^- \right]$$

Proof. We assume that H is centered on $(c_1, \dots, c_n) \in \mathbb{Z}^n$. Let us first prove that $Dual(H^+) = Dual(\mathcal{V}_+^+)$, and then that $Dual(H)^+ = Dual(\mathcal{V}_-^+)$. The proof of $Dual(H^-) = Dual(\mathcal{V}_-^+)$ can be obtained in the same way.

The convex hull of the vertices in $\mathcal{V}_-$ is defined by

$$\mathcal{C} = \{(x_1, \dots, x_n) | \forall i \in \llbracket 1, n-1 \rrbracket, c_i - \frac{1}{2} \leq x_i \leq c_i + \frac{1}{2} \text{ and } x_n = c_n - \frac{1}{2}\}.$$

We can deduce from Definition 4 that $H^+ = \mathcal{C}^+$. Then, let us prove that $Dual(\mathcal{C}^+) = Dual(\mathcal{V}_+^+)$. Specifically, we know that $\mathcal{V}_- \subset \mathcal{C}$. Then, $\mathcal{V}_-^+ \subset \mathcal{C}^+$ and thus $Dual(\mathcal{V}_-^+) \subseteq Dual(\mathcal{C}^+)$. Next we show that $Dual(\mathcal{C}^+) \subseteq Dual(\mathcal{V}_+^+)$.

Let $p \in Dual(\mathcal{C}^+)$. Then, $Dual(p) \cap \mathcal{C}^+ \neq \emptyset$. We need to prove that there exists $v \in \mathcal{V}_-$ such that $Dual(p) \cap v^+ \neq \emptyset$. We proceed by contradiction and assume that $\forall v \in \mathcal{V}_-, Dual(p) \cap v^+ = \emptyset$. Since $\mathcal{C}$ is the convex hull of the vertices in $\mathcal{V}_-$, we can deduce that $\mathcal{C}^+$ is the convex hull of the points in $\mathcal{V}_-^+$. Hence, since $Dual(p)$ is a hyperplane, we deduce that $Dual(p) \cap \mathcal{C}^+ = \emptyset$. $\square$

An illustration of Proposition 1 and Theorem 1 is given in Figure 3.

Fig. 3. (a) A pixel H , (b) Illustration of $Dual(H)^- (= Dual(H^-))$, (c) Illustration of $Dual(H)^+ (= Dual(H^+))$, (d) Dual of H

Theorem 1 allows us to compute the dual of a hypervoxel from its vertices. The dual of a hypervoxel can also be described as the union of several open polytopes.

Let $\mathcal{S} = \{-1, 1\}$ and let $\mathcal{R} : \mathcal{S} \rightarrow \{\mathbb{R}^-, \mathbb{R}^+\}$ be the function defined by: $\mathcal{R}(-1) = \mathbb{R}^-$ and $\mathcal{R}(1) = \mathbb{R}^+$. Moreover, let $s = (s_1, \dots, s_{n-1}) \in \mathcal{S}^{n-1}$. We denote by $\mathcal{R}^s$ the Cartesian product $\mathcal{R}(s_1) \times \dots \times \mathcal{R}(s_{n-1})$. A partition of $\mathbb{R}^n$ can thus be defined by:

$$\mathbb{R}^n = \bigcup_{s \in \mathcal{S}^{n-1}} (\mathcal{R}^s \times \mathbb{R})$$

Let H be a hypervoxel centered on $(c_1, \dots, c_n) \in \mathbb{Z}^n$. We denote by v_-^s the vertex of H with coordinates $(c_1 + \frac{1}{2}s_1, \dots, c_1 + \frac{1}{2}s_{n-1}, c_n - \frac{1}{2})$ and by v_+^s the vertex with coordinates $(c_1 - \frac{1}{2}s_1, \dots, c_1 - \frac{1}{2}s_{n-1}, c_n + \frac{1}{2})$. Then, we have the following property:

Corollary 1 (Hypervoxel Dual Decomposition). *Let H be a hypervoxel. Then,*

$$Dual(H) = \bigcup_{s \in \mathcal{S}^{n-1}} [(\mathcal{R}^s \times \mathbb{R}) \cap Dual(v_-^s)^+ \cap Dual(v_+^s)^-]$$

Proof. Let us assume that $H \subset \mathcal{E}_n$. We know that $\forall s \in \mathcal{S}^{n-1}, [(\mathcal{R}^s \times \mathbb{R}) \cap \text{Dual}(v_-^s)^+ \cap \text{Dual}(v_+^s)^-] \subset \text{Dual}(v_-^s)^+ \cap \text{Dual}(v_+^s)^-$. Hence, since $\text{Dual}(v_-^s)^+ \cap \text{Dual}(v_+^s)^- \subset \text{Dual}(\mathcal{V}_-)^+ \cap \text{Dual}(\mathcal{V}_+)^-$, we deduce from Theorem 1 that $\forall s \in \mathcal{S}^{n-1}, [(\mathcal{R}^s \times \mathbb{R}) \cap \text{Dual}(v_-^s)^+ \cap \text{Dual}(v_+^s)^-] \subset \text{Dual}(H)$. Let us prove the reverse inclusion.

Let $p = (x_1, \dots, x_n) \in \text{Dual}(H)$. Then, there exists $s = (s_1, \dots, s_{n-1}) \in \mathcal{S}^{n-1}$ such that $p \in \mathcal{R}^s$. Moreover, by Theorem 1 it follows that $\forall v = (v_1, \dots, v_n) \in \mathcal{V}_-, x_n \geq \sum_{i=1}^{n-1} -v_i x_i + v_n$, and $\forall v = (v_1, \dots, v_n) \in \mathcal{V}_-, x_n \leq \sum_{i=1}^{n-1} -v_i x_i + v_n$. However, since $p \in \mathcal{R}^s$, and thus $x_i \in \mathcal{R}(s_i)$, we deduce that $-(c_i + \frac{1}{2}s_i)x_i \leq -(c_i - \frac{1}{2}s_i)x_i$. Hence, p verifies the following inequalities: $x_n \leq \sum_{i=1}^{n-1} -(c_i - \frac{1}{2}s_i)x_i + c_n + \frac{1}{2}$ and $x_n \geq \sum_{i=1}^{n-1} -(c_i + \frac{1}{2}s_i)x_i + c_n - \frac{1}{2}$. We get that $p \in \text{Dual}(v_-^s)^+ \cap \text{Dual}(v_+^s)^-$. $\square$

Corollary 1 allows the decomposition of the dual of a hypervoxel into 2^{n-1} open polytopes bounded by the hyperplanes which are the duals of the hypervoxel vertices (see Figure 4).

Fig. 4. Hypervoxel dual examples. Numbering shows the correspondence between the vertices of a pixel (resp. voxel) and the lines (resp. planes) forming the border of its dual: (a) Dual of the pixel centered on $(0, 0)$, (b) Dual of the voxel centered on $(0, 0, 0)$.

4 Standard and Supercover Hyperplane Recognition

In this section, we present our Standard and Supercover hyperplane recognition algorithm. The aim is to determine if a hypervoxel set belongs to a Standard or Supercover digital hyperplane. The idea of our algorithm is to compute the set of Euclidean hyperplanes (if it exists) which cross the given hypervoxels. Each hyperplane in this set is the dual of a point in the parameter space which belongs to a particular polytope. We call this polytope the *generalized preimage* of the hypervoxels. Then, based on the shape (empty or not) of this preimage, we can deduce if the hypervoxel set belongs or not to a Standard or Supercover hyperplane. In the following, we first give the definition of the generalized preimage.

Then, we detail our recognition algorithm, and give some simplifications we can apply on it in order to improve its complexity.

4.1 The Notion of Generalized Preimage

As said previously, each point in the generalized preimage of a hypervoxel set $\mathcal{H}$ is the dual of a Euclidean hyperplane which cuts all hypervoxels of $\mathcal{H}$. We define the generalized preimage of a hypervoxel set as follows:

Definition 5 (Generalized Preimage). *Let $\mathcal{H} = (H_1, \dots, H_k)$ be a set of k hypervoxels, and let $Dual(H_i)$, $i \in \llbracket 1, k \rrbracket$, be the dual of H_i in the parameter space. The generalized preimage $\mathbb{G}_P$ of $\mathcal{H}$ is defined by:*

$$\mathbb{G}_P(\mathcal{H}) = \bigcap_{i=1}^k Dual(H_i)$$

Note that the Standard digitization of many hyperplanes in the dual of the generalized preimage does not contain the given hypervoxels (see Remark 1). However, we know that incorrect hyperplanes are located on the border of the generalized preimage (because these hyperplanes cross hypervoxel vertices). Thus, in order to obtain a correct hyperplane, it is sufficient to choose a point which is not on the generalized preimage border.

In the following, we will assume that a generalized preimage which is not empty is not either reduced to m -dimensional subspace segments, $m < n$.

4.2 Recognition Algorithm

Let $\mathcal{H} = \{H_1, \dots, H_k\}$ be a set of k hypervoxels. The Standard and Supercover hyperplane recognition (see Algorithm 1) is simply performed by computing the generalized preimage $\mathbb{G}_P$ of $\mathcal{H}$. First, $\mathbb{G}_P(H_1)$, i.e. the dual of H_1 , is computed according to the hypervoxel dual definition given by Theorem 1. Then, $\mathbb{G}_P(\{H_1, H_2\})$ is computed from the intersection of $\mathbb{G}_P(H_1)$ and $Dual(H_2)$. And so on until $\mathbb{G}_P(\{H_1, \dots, H_k\})$ is computed or $\mathbb{G}_P$ becomes empty. Note that the hypervoxels can be considered in any order, and do not need to be connected. Figure 5 illustrates the recognition process in dimension 2. A 3D generalized preimage example is shown in Figure 6.

To perform the intersection operations, a first naive approach is to intersect directly the generalized preimage and each hypervoxel dual. It is not an efficient method since the dual of a hypervoxel is an open concave polytope. However, Corollary 1 allows us to compute the generalized preimage of a hypervoxel set by simply computing intersections of convex polytopes and hyperplanes.

Let k be the number of given hypervoxels. These improvements lead to a complexity for our algorithm of $\mathcal{O}(k)$ in dimension 2 when applied on 4-connected curves [17]. Indeed, the generalized preimage of a pixel set is a polygone with at most four edges [11, 18]. In dimension 3 [19, 20] and higher, the complexity is $\mathcal{O}(k^2)$ in the worst case.

Algorithm 1. Standard and Supercover hyperplane recognition algorithm

Data: A set $\mathcal{H}$ of k hypervoxels $H_1, \dots, H_k$.

begin

$GP \leftarrow Dual(H_1);$

$i \leftarrow 2;$

while $GP \neq \emptyset$ **and** $i \leq n$ **do**

$GP \leftarrow GP \cap Dual(H_i);$

$i \leftarrow i + 1;$

if $GP \neq \emptyset$ **then**

$\mathcal{H}$ belongs to a Standard and Supercover hyperplane.

else

$\mathcal{H}$ does not belong to a Standard or Supercover hyperplane.

end

Fig. 5. Example of 2D generalized preimage computation

Fig. 6. Example of 3D generalized preimage: (a) A voxel set V , (b) Corresponding generalized preimage $\mathbb{G}_P(V)$ and its barycenter B (back point), (c) and (d) Result of the intersection operation between V and $Dual(B)$. Remark that $Dual(B)$ does not cross any voxel vertex of V since B does not lie on the border of $\mathbb{G}_P(V)$

5 Conclusion and Future Works

In this article, a new Standard and Supercover hyperplane recognition algorithm in arbitrary dimension has been presented. This algorithm determines if a given hypervoxel set belongs to a Standard or Supercover hyperplane by providing the set of Euclidean hyperplanes which cut all hypervoxels. This set is deduced from the computation in a dual space of the generalized preimage of the hypervoxels. This preimage is defined as the intersection of the duals of the hypervoxels. A description of the dual of a hypervoxel is given in order to increase the preimage computation efficiency. The recognition algorithm does not require given hypervoxels to be connected. Moreover, during the recognition process, hypervoxels can be considered in any order.

The results proposed in this paper are very general. They allow many extensions. For instance, the hypercubes in the recognition process are not required to have the same size. This can easily lead to recognition algorithms in multi-scale grids or heterogeneous grids. We focused in this paper on the dual of a hypervoxel but as long as the cell is convex we can compute its dual object. We can therefore also propose recognition algorithms in grids which are not based on hypercubes. The recognition algorithm can also be adapted to recognize other types of hyperplanes than Standard or Supercover analytical hyperplanes. Indeed, for each digital analytical model, there is a distance and/or a unit ball associated to the definition. For instance, for the Naive model [21] in 3D, it is the distance d_1 and the ball is an octahedron. This makes the general definition of the dual in dimension n a bit more difficult but nonetheless feasible.

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