Order Independence in Binary 2D Homotopic Thinning

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Abstract. This paper investigates binary homotopic 2D thinning in view of its independence of the order of processing image pixels. Pixel removal conditions are provided leading to an order independent thinning. They are introduced for various types of connectivity. Two kinds of pixels to be removed are considered: simple and b-simple. Use of each of those pixels yields to different types of order independent thinnings: homotopic marking and local-SKIZ.

1 Introduction

A widely used approach for computing discrete skeletons consists in iteratively thinning the input pattern with a series of homotopic structuring elements until no more pixels can be removed. *Homotopic marking* is obtained when removable pixels are those which can be removed without modifying the homotopy of the input image. These removable pixels are called *simple pixels*. Another type of thinning result is obtained when the notion of homotopy that is the basis of the definition of simple pixels, is replaced by the notion of *background homotopy*, i.e., homotopy of connected components of the background. This notion leads to a different kind of simple pixels which we call the *b-simple* pixels. Thinning with b-simple pixels leads to kind of a skeleton which resembles the skeleton of influence zones of the complement of the input image, but is based on local pixel characterisation. It will be called in this paper *local-SKIZ*. The main developments of the paper refer to the *order independence* of the thinning. When this property is satisfied, the computed skeleton is the same no matter the order in which the image pixels are processed. Unfortunately, most algorithms produce different results for forward and backward scans of the image pixels. This also occurs when computing the skeleton of an object and its mirrored version or rotation by a multiple of 90 degrees. Recently [1, 2], the concept of order independent homotopic thinning was introduced. It was also shown that it leads to order independent skeletonisation when iterated until the algorithm reaches stability. In this paper, we introduce a general framework for order independent thinning leading to various types of result. Order independence is achieved

by introducing a supplementary condition that a removable pixel must meet, and thus guarantees its removal whatever the scanning order. These conditions refer to a neighborhood of removable pixels. All the developments described in this paper are presented for both the 8-foreground with 4-background and 4-foreground with 8-background connectivities. A case when removable pixels must be simultaneously simple (resp. b-simple) for both connectivities is also described.

The paper is organised as follows. In section 2, background notions are presented. Section 3 looks at the order independence. Section 4 illustrates the results. Finally, the conclusions are given in section 5.

2 Preliminaries

2.1 Adjacency, Connectivity, and Homotopy

A binary image f is represented as the foreground (set) $F = \{p : f(p) = 1\}$ consisting of foreground pixels. The complement of F , referred to as background (set), is defined as $\overline{F} = \mathcal{D} \setminus F = \{p : f(p) = 0\}$ and consists of background pixels¹. $\mathcal{D}$ is referred to as the definition domain of the image f .

Two pixels $p, q \in \mathcal{D}$ can be either 4- or 8-adjacent. If, moreover, they have the same value, we say that they are 4- or 8-connected respectively. The k -neighborhood of a pixel p , denoted by $\mathcal{N}_k(p)$, is the set of all pixels q such that q and p are k -adjacent. Let $\mathcal{N}_k^0(p) = \mathcal{N}_k(p) \cap \overline{F}$ and $\mathcal{N}_k^1(p) = \mathcal{N}_k(p) \cap F = \mathcal{N}_k(p) \setminus \mathcal{N}_k^0(p)$. Let also $\mathcal{CC}_k(X)$ be the set of all k -connected components of a set X and $|\mathcal{CC}_k(X)|$ be the number of its elements (also called the cardinal number). In order to avoid the well-known connectivity paradoxes (see e.g. [3]) two combinations of connectivity are possible for the square grid: $\mathcal{G} = 8$ (and $\mathcal{G}' = 4$) or $\mathcal{G} = 4$ (and $\mathcal{G}' = 8$). Homotopy describes the adjacency relations between the elements of $\mathcal{CC}_{\mathcal{G}}(F)$ and $\mathcal{CC}_{\mathcal{G}'}(\overline{F})$. Two sets are said to be homotopic when they have the same homotopy tree. In this paper, we are interested in characterising homotopy between a set and an arbitrary thinning of this set so that we can restrict our attention to the notion of homotopy between ordered sets or functions due to the anti-extensivity of the thinning transformation. In this context, it was shown in [4] that two ordered sets F and G (in the sense of set inclusion) are homotopic if and only if there exists a one to one correspondence between the connected components of these sets as well as those of their background. We propose therefore to split the notion of homotopy between ordered sets into background and foreground homotopy:

Definition 1. Background and foreground homotopy. Let F and G be two ordered sets for the inclusion relationship, then F is background homotopic (resp. foreground homotopic) to G if and only if there exists a one to one correspondence (i.e., bijective mapping) between the elements of $\mathcal{CC}_{\mathcal{G}}(F)$ and those of $\mathcal{CC}_{\mathcal{G}}(G)$ (resp. between $\mathcal{CC}_{\mathcal{G}'}(\overline{F})$ and $\mathcal{CC}_{\mathcal{G}'}(\overline{G})$).

¹ Both kinds of notation are used in this paper to refer to binary images. An 'image F ' refers to the set of foreground pixels, while an 'image f ' refers to the mapping that generates F and $\overline{F}$.

Definition 2. Homotopy. *Two ordered sets are homotopic if and only if they are both background and foreground homotopic.*

2.2 Pixel Configurations

Topological numbers were introduced in [5] to characterise simple pixels. They are defined as: $\mathcal{T}_k(p)$ is the number of k -connected components of the 8-adjacent foreground neighbours of p that are k -adjacent to p . $\overline{\mathcal{T}}_k(p)$ is obtained by replacing foreground with background in the previous definition. We can define them also as: $\mathcal{T}_4(p) = |\{S \in \mathcal{CC}_4(\mathcal{N}_8^1(p)) : S \cap \mathcal{N}_4^1(p) \neq \emptyset\}|$, $\mathcal{T}_8(p) = |\mathcal{CC}_8(\mathcal{N}_8^1(p))|$, $\overline{\mathcal{T}}_4(p) = |\{S \in \mathcal{CC}_4(\mathcal{N}_8^0(p)) : S \cap \mathcal{N}_4^0(p) \neq \emptyset\}|$, $\overline{\mathcal{T}}_8(p) = |\mathcal{CC}_8(\mathcal{N}_8^0(p))|$. Using the topological numbers we can define the basic configurations of foreground pixels as follows: $\overline{\mathcal{T}}_{\mathcal{G}'}(p) = 0$ for interior (inner) pixels, $(\overline{\mathcal{T}}_{\mathcal{G}'}(p) > 0 \wedge \mathcal{T}_{\mathcal{G}}(p) > 0)$ for boundary pixels, and $\mathcal{T}_{\mathcal{G}}(p) = 0$ for isolated pixels. A pixel $p \in F$ is *simple* [6, 3] if and only if its removal from the image does not change the homotopy of F (according to Definition 2). If p is simple, F and $F \setminus \{p\}$ are homotopic, as well as $\overline{F}$ and $\overline{F} \cup \{p\}$. It was proved in [6, 3] that the simpleness of a pixel can be determined by analysing its 3×3 neighborhood. A definition of simple pixels based on topological numbers is as follows:

Definition 3. Simple pixel [5]. *A pixel p is simple $\Leftrightarrow \mathcal{T}_{\mathcal{G}}(p) = \overline{\mathcal{T}}_{\mathcal{G}'}(p) = 1$.*

If $p \in F$ the above definition is equivalent to:

Definition 4. Simple pixel [6]. *A pixel $p \in F$ is simple if and only if it satisfies the following three conditions:*

1. $\mathcal{N}_{\mathcal{G}}^1(p) \neq \emptyset$, 2. $\mathcal{N}_{\mathcal{G}'}^0(p) \neq \emptyset$, 3. $\exists S \in \mathcal{CC}_{\mathcal{G}}(\mathcal{N}_8^1(p))$ such that $\mathcal{N}_{\mathcal{G}}^1(p) \subseteq S$.

The simpleness property depends on the connectivity used for foreground and background. In [7], another type of simple pixels was considered: $\{4, 8\}$ -simple, which are simple according to both $\mathcal{G} = 4, \mathcal{G}' = 8$ and $\mathcal{G} = 8, \mathcal{G}' = 4$. This case will be referred to later in the text as $\mathcal{G} = \{4, 8\}$ and can be checked according to the following proposition².

Proposition 1. *A pixel p is $\{4, 8\}$ -simple if and only if $\mathcal{T}_4(p) = \overline{\mathcal{T}}_4(p) = 1$.*

When substituting homotopy with background homotopy in the definition of a simple pixel we get the *b-simple* ('b' refers to the background homotopy) pixels:

Definition 5. b-simple pixel. *A pixel p is b-simple if and only if $\overline{\mathcal{T}}_{\mathcal{G}'}(p) = 1$.*

The relationship between b-simple³ and simple pixels is expressed by the following proposition:

Proposition 2. b-simple pixel. *A pixel p is b-simple if and only if one of the following is true: 1. p is simple or 2. p is isolated.*

Finally, we define a $\{4, 8\}$ -b-simple pixel as a pixel which is b-simple for both $\mathcal{G} = 4$ and $\mathcal{G} = 8$.

² Proofs of all propositions are presented in the appendix.

³ By analogy, a *f-simple* pixels can be defined as those p for which $\mathcal{T}_{\mathcal{G}}(p) = 1$. Simple pixel can be thus defined as a pixel which is both b-simple and f-simple. The f-simpleness is however out of scope of this paper.

2.3 Thinning

In the simplest case, during the thinning process, all simple pixels are iteratively removed. This is usually done in two-stage iterative process. During the first stage the simple pixels are detected while during the second one they are removed. The removal of a simple pixel, by definition, preserves the homotopy of the input image F . This type of thinning is called a *homotopic marking* [8, 2]. It reduces binary objects to single pixels or to loops surrounding the holes inside the figures (preserving the homotopy). Homotopic marking can be thus considered as a skeleton satisfying the topological constraints (preserving the homotopy) but not the geometrical ones [7]⁴.

The replacement of the simple pixels by the b-simple pixels in the thinning procedure leads to a (non-homotopic) skeleton preserving only those connected components of the homotopic marking which are not simply connected. Contrary to the previous method, in this one connected components without holes disappear. That is this type of a skeleton resembles the skeleton of influence zones (SKIZ) of the complement of the input image. Comparing to the distance-based SKIZ the result of thinning with b-simple pixels contains some lines which does not separate different influence zones, but are located inside the same one. This is due to the fact that the actual SKIZ cannot be obtained using local characterisation of pixel neighborhood [9]. To stress this difference the thinning by removal of b-simple pixels will be called *local-SKIZ* (SKIZ based on local pixel characterisation)⁵.

3 Order Independence

When the property of *order independence* is satisfied, the result of thinning is the same no matter the order in which the image pixels were processed. Let us assume a generic two stage iterative thinning scheme. In the first (detection) phase, the image is scanned to find all simple pixels. In the second (removal) phase the values of the simple pixels should be removed. However, all cannot be removed at once because modification of one pixel can result in the neighbour becoming non-simple. A solution to this problem consists in finding a supplementary characterisation of the simple pixels. It is able to detect in the detection phase, only those pixels which can be removed no matter what scanning order is considered. These pixels are called *order independent simple pixels*. The order

⁴ In order to get a skeleton that better characterises the input shape, an *anchored skeletonisation* can be performed [1, 2]. It allows to pre-define a set of pixels which, by definition, cannot be removed during the thinning process.

⁵ In order to get the actual SKIZ (which - contrary to local-SKIZ - is not b-homotopic to the input image) one needs to label the connected components of the background in advance and add one additional test when testing the b-simpleness. According to this test, if a non-b-simple pixel has among its background neighbours all pixels with the same label it can be removed during the thinning process. This approach requires also that when a pixel is removed, the closest background label must be propagated.

independence should be then incorporated into this generic scheme of thinning. In view of this scheme, order independence refers to the removal phase.

A separate subject of research available in the literature is *parallel thinning* [10]. Most of these studies aim at defining for binary images a minimal set of pixels which can be removed at once (in parallel) without modifying the homotopy of the original image [4, 7, 11]. The parallel approach itself is not sufficient to get a fully order independent algorithm, which requires also a two-stage scheme which guarantees that pixels are removed iteratively 'layer' by 'layer'. In an order independent algorithm, when processing in a given iteration, all pixels that are flagged as removable can be removed in parallel. In [1, 2] the concept of order independent homotopic thinning was introduced, which leads to order independent skeletonisation when iterated until the algorithm reaches stability.

3.1 Parallel Removal of Pixels

Useful analysis of the set of simple binary pixels that can be removed in parallel has been proposed by Ronse [7]. This approach aims at finding the condition which a set of pixels has to meet to be *deletable*, i.e., to be able to be removed without changing the homotopy of the image. According to the results presented in [7] such a set cannot contain a *minimal non deletable set*. This set is defined as a minimal set of simple pixels which cannot be modified in parallel:

Proposition 3. Minimal non deletable set [7] (for proof see [7])

A set $U \in F$ is a minimal non deletable set if and only if one of the following conditions holds:

- 1. U consists of an isolated pixel, 2. U is a pair of 8-adjacent simple pixels which is not deletable, 3. for $\mathcal{G} = 8$, U is a triple or quadruple of pairwise 8-adjacent simple pixels, and U is an 8-connected component of F .*

Proposition 3 allows us to define a set of conditions that a pixel must meet to be modified without influencing the simpleness of its neighbours. It is based on the assumption that such a pixel cannot belong to a minimal non deletable set. We can thus define an *order independent simple pixel*:

Definition 6. Order independent simple pixel. *A pixel p is order independent simple if and only if the three following conditions are met:*

- 1. p is simple, 2. p does not belong to any non deletable pair of 8-adjacent simple pixels, 3. for $\mathcal{G} = 8$, p belongs neither to an isolated (i.e., surrounded by background pixels) triple nor to an isolated quadruple of pairwise 8-adjacent simple pixels.*

Comparing Definition 6 with the simpleness test (according to Definitions 4 or 3), two more conditions must be checked when analysing the given pixel: the deletability of a pair of simple pixels as well as whether they are part of the triple and quadruple configurations. Rules for finding these configurations are described further in this paper.

Fig. 1. An example of binary order independent and order dependent simple (a) and b-simple pixels (b) in different types of connectivity. Dots represent background, digits stand for: '1' - a non-simple (resp. non-b-simple) foreground pixel, '2' - an independent simple (resp. b-simple) pixel, '3' - a dependent (resp. b-simple) simple pixel. Underlined pixels belong to the final thinning result.

In case of b-simpleness, a similar reasoning to the one described above leads to the formulation of the conditions a pixel must meet to be order independent b-simple. The difference between simpleness and b-simpleness relates to the fact that in the latter case we should also consider the configurations of isolated pixels. In the local-SKIZ transformation case, isolated pixels are removable and therefore removal of all configurations leading to an isolated pixel are allowed. Such configurations are possible based on the third item of Proposition 3: isolated triple and quadruple of simple pixels. Since they are isolated and have no 'holes' they do not belong to the local-SKIZ. For the same reason, an isolated pair of simple pixels can be removed. Consequently, the following definition can be formulated:

Definition 7. Order independent b-simple pixel. A pixel p is order independent b-simple if and only if the following conditions are met:

1. p is b-simple, 2. p does not belong to any deletable pair of b-simple pixels, which is not an isolated connected component.

The foreground image pixels can be classified into the following three classes: non-simple (resp. non-b-simple) pixels, dependent simple (resp. b-simple) pixels, and order independent simple (resp. b-simple) pixels. Only the pixels belonging to the third group can be removed by an order independent thinning algorithm. In order to determine whether a pixel is independent some additional tests must be performed in addition to the simpleness (resp. b-simpleness) tests. All the necessary tests are described below. Figure 3.1 shows a binary test image and the classification of its pixels.

3.2 A Pair of Simple and b-Simple Neighbours

First, let us consider the case when a simple pixel (resp. b-simple pixel) is adjacent to only one other simple (resp. b-simple) pixel. These pixels are pairwise independent if removal of one of them will not influence the removability of the second. The following proposition holds for two adjacent simple pixels.

Proposition 4. Order independent pair of simple pixels. Let p and q be two 8-adjacent simple pixels. Let $\mathcal{G}$ be the foreground connectivity. p is order

independent of q if and only if: $|\mathcal{CC}_8(S \cap \mathcal{N}_8(q))| = 1$, where $S = \mathcal{N}_8^0(p)$ if $\mathcal{G} = 4$ and $S = \mathcal{N}_8^1(p)$ if $\mathcal{G} = 8$.

Contrary to the simple pixel case, in the case of b-simple pixels, an isolated pixel can be removed. Thus we have the following proposition:

Proposition 5. Order independent pair of b-simple pixels. *Let p and q be two 8-adjacent b-simple pixels. Let $\mathcal{G}$ be the foreground connectivity. Pixel p is order independent of q if and only if $|\mathcal{CC}_8(S \cap \mathcal{N}_8(q))| = 1$ or $|\mathcal{CC}_8(\mathcal{N}_8^1(p) \cap \mathcal{N}_8(q))| = 0$, where $S = \mathcal{N}_8^0(p)$ if $\mathcal{G} = 4$ and $S = \mathcal{N}_8^1(p)$ if $\mathcal{G} = 8$.*

When simple pixels for $\mathcal{G} = \{4, 8\}$ are investigated, p should be independent from q for $\mathcal{G} = 4$ and for $\mathcal{G} = 8$ simultaneously.

*0q	*0q	*1q	*00*	*01*	*10*	*11*	*01*	*01*	*10*	*01*	*10*	*11*
*p0	*p1	*p1	*pq*	*pq*	*pq*	*pq*	*pq*	*pq*	*pq*	*pq*	*pq*	*pq*
***	***	***	*00*	*00*	*00*	*10*	*01*	*10*	*11*	*11*	*11*	*11*
a	b	c	d	e	f	g	h	i	j	k	l	m

Fig. 2. Configurations of common 8-neighbours of two 8-adjacent pixels p and q , '*' stands for any pixel value

Figure 2 shows generic configurations of intersecting neighbourhood of two pixels. To get all possible configurations, one should add configurations that are symmetrical to b, e, f, g, h, k , and l with respect to the pq axis, as well as rotations through $90^\circ, 180^\circ$, and 270° . Configurations of independent pixels from Fig. 2 according to propositions 4 and 5 are given in table 1. Note that some configurations do not exist for $\mathcal{G} = 4$ and consequently for $\mathcal{G} = \{4, 8\}$.

Table 1. Independent configurations from Fig. 2

	$\mathcal{G} = 8$	$\mathcal{G} = 4$	$\mathcal{G} = \{4, 8\}$
independent simple	b, c, e, f, g	a, b, g, k, l	b, g
independent b-simple	a, b, c, d, e, f, g	a, b, d, g, k, l	a, b, d, g

In order to check whether a pixel p is independent simple, we must test its pairwise independence from all its simple neighbours. If it is independent from all of its simple neighbours, the second condition of Definition 6 is met and only the third condition has to be verified. However, according to Definition 7, this condition does not have to be tested for b-simple pixels.

One of the configurations (marked as c in Fig. 2) requires more attention when $\mathcal{G} = 4$ and $\mathcal{G}' = 8$. This is due to the fact that for some configurations of '*'-neighbours of p this pixel can be removed, even if it is not independent. This case is described in detail in the next section.

3.3 Multiple Simple Neighbours

There are two cases where multiple configurations of simple or b-simple pixels have to be analysed. The first one refers to the third condition of Definition 6 and concerns the isolated set of pixels for $\mathcal{G} = 8$. The second one refers to the configuration c from Fig. 2 for $\mathcal{G} = 4$.

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0000 000 0000 0000 1000 0000 ****
0010 010 0110 0110 0110 0p11 *rq*
0100 010 0100 0110 0110 01q1 0ps*
0000 000 0000 0000 0001 0000 *0**
(a) (b) (c) (d) (e) (f) (g)

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Fig. 3. Multiple simple neighbours. (a,b) isolated pair. (c) isolated triple. (d) isolated quadruple (for $\mathcal{G} = 4$ and $\mathcal{G} = 8$). (e) isolated quadruple (for $\mathcal{G} = 4$ only). (f) example of configuration where p is dependent of q , but can be removed ($\mathcal{G} = 4$). (g) generic 'edge' configuration of neighbourhood ($\mathcal{G} = 4$, '*' stands for any pixel values, but such that r,q,s remain simple).

As far as the first case is concerned, for skeletonisation with $\mathcal{G} = 8$, isolated triple and quadruple configurations of simple pixels cannot be removed even if they are independent. This situation is similar to the case of an isolated pair of pixels as shown in Figs. 3a and 3b, where depending on the order of processing we can get different thinning results. Fortunately, a pair of simple pixels does not have to be considered, because such a pair is not pairwise independent and thus it cannot be removed. However, using pairwise tests, there is no obstacle to remove triple and quadruple configurations shown in Figs. 3c and 3d because every pixel belonging to these isolated connected components is independent. But such a removal would be order dependent because different pixel processing orders of the triple/quadruple result in different isolated pairs of pixels. Consequently, an additional test is required to protect pixels belonging to isolated triple or quadruple of simple pixels from being removed. The test can either follow the external boundary of the investigated triples or quadruples of simple pixels and check whether there are only background pixels or it can check the number of neighbours of every pixel belonging to it. If every pixel has exactly three (resp. four) foreground 8-neighbours then it means that the triple (resp. quadruple) is isolated.

For the configuration c of Fig. 2, for $\mathcal{G} = 4$, a supplementary analysis must be performed. This is due to the fact that for some combinations of the neighbourhood of the quadruple of pixels (consisting of p,q and two other pixels of value

Fig. 4. (a–c) Order dependent thinning with scanning orders: (a) from top-left (TL) to bottom-right (BR), (b) from BL to TR, (c) from BR to TL. (d) shows the order independent thinning result.

1 in their common neighbourhood), pixel p is independent from q . An example of such a case is shown in Fig. 3f. Due to the dependence of p on q , p cannot be set to 0, based on the previously described conditions, whereas it should be, because the whole quadruple of simple (resp. b-simple) pixels can be removed without changing the homotopy of the image. The same result can be obtained for any other similar configuration when p is an 'edge' pixel. Therefore such an 'edge' configuration must be treated separately. The 'edge' pixel is characterised by a pixel having three adjacent simple (resp. b-simple) neighbours with a dependent diagonal neighbour and two independent non-diagonal ones. This produces the configuration shown in Fig. 3g. Since p is independent of r and s , the rest of its 4-adjacent neighbours must be background neighbours. Otherwise p would not be simple. In addition, in case of simple pixels (not b-simple), the quadruple being considered must be checked whether it is isolated. If so, p cannot be removed. Owing to that, a similar (to the first case described in this section) check of the third condition of Definition 6 must be executed. In this case however, to test whether a quadruple p, q, r, s is isolated must to check whether only 4-adjacent external neighbours belong to the background. The difference between non-deletable quadruples for $\mathcal{G} = 8$ and $\mathcal{G} = 4$ is illustrated with the two examples shown in Fig. 3d and 3e.

4 Results

The fast implementation of the algorithm according to the proposed method is queue-based. It consists of three phases. In the first one, all order independent simple (resp. b-simple) pixels are put into the queue. In the second phase, pixels are removed from the queue and set to 0. In the third phase, all the order-independent neighbours of the pixels processed in the second phase are put into the queue. Second and third phases are performed iteratively until there are new order independent simple pixels to process.

Fig. 5. Results of order independent thinning without anchor pixels. Left column: homotopic marking (skeleton): (a) $\mathcal{G} = 8$, (b) $\mathcal{G} = 4$, (c) $\mathcal{G} = \{4, 8\}$. Right column - local-SKIZ: (d) $\mathcal{G} = 8$, (e) $\mathcal{G} = 4$, (f) $\mathcal{G} = \{4, 8\}$.

The impact of the order of processing image pixels on the result of order-dependent thinning is shown in Fig. 4 and compared to the order independent result. Three scanning orders were applied: from top-left to bottom-right (Fig. 4a), from bottom-left to top-right (Fig. 4b) and finally from bottom-right to top-left (Fig. 4c). One can observe that every skeleton is different, although everyone of them was obtained using the successive ('layer' by 'layer') removal of simple pixels. The result of the order independent approach is shown in Fig. 4d. The same result was obtained for all tested scanning orders. All results have been obtained for $\mathcal{G} = 8$. The influence of the chosen connectivity on the thinning result is shown in Fig. 5. The test image was thinned in all three combinations of connectivity and produced two types of result: homotopic marking and local-SKIZ. Comparing homotopic marking for $\mathcal{G} = 8$ (Fig. 5a) and $\mathcal{G} = 4$ (Fig. 5b) we can observe that thin closed objects were not preserved in the latter case. This is due to the fact that closed paths on the original image are often one-pixel thick and sometimes two adjacent pixels are diagonal neighbours. In this case these pixels are not 4-connected; for $\mathcal{G} = 4$ they belong to different connected components. This is also a reason why local-SKIZ in case of $\mathcal{G} = 4$ (Fig. 5d) is an empty image - contrary to skeletonisation, the local-SKIZ preserves only the closed paths, which are not present in this image for $\mathcal{G} = 4$.

5 Conclusions

In this paper, the issue of order independence in thinning was discussed. The skeletonisation by thinning is based on the successive removal of simple pixels, which are defined as pixels which can be removed from the image without modifying its homotopy. We have introduced the notion of the b-simple pixel, which is defined as a pixel whose removal does not change the background homotopy of the image. Based on these types of simpleness, different skeletons can be obtained: homotopic marking (for simple pixels) and the skeleton by influence zones of the complement of an input image (local-SKIZ, for b-simple pixels). The order independence was also investigated in the paper. When this property is satisfied by an iterative thinning algorithm, the result does not depend on the order of processing image pixels. The conditions which are necessary to remove a pixel whatever is the scanning order was formulated. All kinds of thinning methods were analysed in the paper for three types of connectivity. Two of them were the classical 8-foreground with 4-background and 4-foreground with 8-background. The third type is the one where the removed pixel must be simultaneously simple (resp. b-simple) for both above mentioned connectivity combinations.

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Appendix - Proofs

The following lemmas will be used to prove the first two propositions.

Lemmas

1. $\mathcal{T}_{\mathcal{G}}(p) = 1 \Rightarrow \overline{\mathcal{T}_{\mathcal{G}'}}(p) \leq 1$, 2. $\overline{\mathcal{T}_{\mathcal{G}'}}(p) = 1 \Rightarrow \mathcal{T}_{\mathcal{G}}(p) \leq 1$,
3. $\mathcal{T}_{\mathcal{G}}(p) = 0 \Rightarrow \overline{\mathcal{T}_{\mathcal{G}'}}(p) = 1$, 4. $\mathcal{T}_4(p) = 1 \Rightarrow \mathcal{T}_8(p) \geq 1$; $\overline{\mathcal{T}_4}(p) = 1 \Rightarrow \overline{\mathcal{T}_8}(p) \geq 1$.

Proof of Lemma 1. Let consider first the particular case where $\mathcal{G} = 4$ and therefore $\mathcal{G}' = 8$: $\mathcal{T}_4(p) = 1 \Rightarrow \overline{\mathcal{T}_8}(p) \leq 1$. Left side indicates that there can be one or more 4-conn. components of the foreground, but only one is 4-adjacent to p , so that all the other ones must be single diagonal neighbours of p . If $\mathcal{N}_8^0(p) \neq \emptyset$ then all its pixels must be 8-conn., so $\overline{\mathcal{T}_8}(p) = 1$. Otherwise (if $\mathcal{N}_8^0(p) = \emptyset$) $\overline{\mathcal{T}_8}(p) = 0$. Combining both we get $\overline{\mathcal{T}_8}(p) \leq 1$. In case of $\mathcal{T}_8(p) = 1 \Rightarrow \overline{\mathcal{T}_4}(p) \leq 1$ and $\mathcal{N}_4^0(p) = \emptyset$, we get an isolated pixel and $\mathcal{T}_4(p) = 0$. In case when $\mathcal{N}_4^0(p) \neq \emptyset$, it must contain only one 4-conn. component 4-adjacent to p (i.e., $\overline{\mathcal{T}_4} = 1$) otherwise there would be more than one 8-conn. component of p which are 8-adjacent to p .

Proof of Lemma 2. The proof is similar to that developed for Lemma 1.

Proof of Lemma 3. Left-hand side means that there are no foreground pixels in $N_{\mathcal{G}}(p)$, i.e., p is $\mathcal{G}$ -isolated pixel. Consequently there exists exactly one $\mathcal{G}'$ -conn. component of the background.

Proof of Lemma 4. Any 4-conn. component consists of one or more 8-conn. ones.

Proof of Proposition 1. A $\{4,8\}$ -simple pixel must be both 4-simple and 8-simple. According to Definition 3 for both combinations of connectivity we

have four conditions which should be met, and therefore Proposition 1 can be re-written as follows:

$$\overline{\mathcal{T}}_4(p) = \mathcal{T}_4(p) = 1 \Leftrightarrow \overline{\mathcal{T}}_4(p) = \mathcal{T}_8(p) = \mathcal{T}_4(p) = \overline{\mathcal{T}}_8(p) = 1.$$

The implication ' $\Leftarrow$ ' is obvious. For ' $\Rightarrow$ ' we have: $\overline{\mathcal{T}}_4(p) = 1 \wedge \mathcal{T}_4(p) = 1 \Leftrightarrow$

$$\Leftrightarrow [\text{Lemmas 1, 2}] \overline{\mathcal{T}}_4(p) = 1 \wedge \mathcal{T}_8(p) \leq 1 \wedge \mathcal{T}_4(p) = 1 \wedge \overline{\mathcal{T}}_8(p) \leq 1 \Rightarrow$$

$$\Rightarrow [\text{Lemma 4}] \overline{\mathcal{T}}_4(p) = 1 \wedge \mathcal{T}_8(p) = 1 \wedge \mathcal{T}_4(p) = 1 \wedge \overline{\mathcal{T}}_8(p) = 1.$$

Proof of Proposition 2. Proposition 2 can be rewritten using topological numbers as follows: $\overline{\mathcal{T}}_{\mathcal{G}'}(p) = 1 \Leftrightarrow (\overline{\mathcal{T}}_{\mathcal{G}'}(p) = 1 \wedge \mathcal{T}_{\mathcal{G}}(p) = 1) \vee \mathcal{T}_{\mathcal{G}}(p) = 0$. Let then perform the following transformations: $\overline{\mathcal{T}}_{\mathcal{G}'}(p) = 1 \Leftrightarrow [\text{Lemma 2}] \overline{\mathcal{T}}_{\mathcal{G}'}(p) = 1 \wedge \mathcal{T}_{\mathcal{G}}(p) \leq 1 \Leftrightarrow \overline{\mathcal{T}}_{\mathcal{G}'}(p) = 1 \wedge (\mathcal{T}_{\mathcal{G}}(p) = 1 \vee \mathcal{T}_{\mathcal{G}}(p) = 0) \Leftrightarrow (\overline{\mathcal{T}}_{\mathcal{G}'}(p) = 1 \wedge \mathcal{T}_{\mathcal{G}}(p) = 1) \vee (\overline{\mathcal{T}}_{\mathcal{G}'}(p) = 1 \wedge \mathcal{T}_{\mathcal{G}}(p) = 0) \Leftrightarrow [\text{Lemma 3}] (\overline{\mathcal{T}}_{\mathcal{G}'}(p) = 1 \wedge \mathcal{T}_{\mathcal{G}}(p) = 1) \vee \mathcal{T}_{\mathcal{G}}(p) = 0$.

Proof of Proposition 4. To analyse precisely dependencies in the neighbourhood of such a pair of pixels, we must consider all possible cases. In order to guarantee that p is order independent of q , we must show, that when q is set to 0, p remains simple. Let us first consider the case where $\mathcal{G} = 8$ and $\mathcal{G}' = 4$ (this case was described in [2]). According to Proposition 4, the following configurations of pixels from Fig. 2 are order-independent: b, c and e, f, g . All the other are dependent, which means that once q is modified, p will not be simple any more. In the proof we will check if the conditions from Definition 4 (referred to as the 1st, the 2nd and the 3rd conditions) are met for pixel p when q is modified. Let us consider now all particular cases from Fig. 2. **Conf. a, d:** Since p is simple, then according to the 3rd cond. $\mathcal{N}_8^1(p) = \{q\}$ (otherwise it would not be conn.). If so, when q is modified, then due to the 1st cond., p will become non simple and p is dependent on q . **Conf. b, e, f and g:** When q is removed, there will still be pixels belonging to $\mathcal{N}_4^0(p)$ and $\mathcal{N}_8^1(p)$. The 1st and the 2nd cond. will then be met. The 3rd cond. as well since $\mathcal{N}_8^1(p)$ will still be connected when q is removed. Consequently p is independent of q . **Conf. c:** According to the 3rd cond., $\mathcal{N}_4^0(p)$ must consists of at least one pixel. In this configuration, there must exist a 4-adjacent neighbour of p marked by '*'. When q is modified, there will still be some pixels in $\mathcal{N}_4^0(p)$ and $\mathcal{N}_8^1(p)$ and consequently the 1st and 2nd conditions will be met. Also, $\mathcal{N}_8^1(p)$ will not become disconnected when q is modified, so that the 3rd cond. is met as well. Pixel p is therefore independent of q . **Conf. h, i, k:** When q is modified, $\mathcal{N}_8^1(p)$ will become non-connected and the 3rd cond. will not be met so that p is dependent on q . **Conf. j, l, m:** According to the 3rd cond., $\mathcal{N}_4^0(p)$ must consists of at least one pixel. In these configurations it must be a 4-adjacent neighbour of p marked as '*'. The only path connecting between 'upper-*' and 'lower-*' pixels in $\mathcal{N}_8^1(p)$ goes through q . If q is modified then $\mathcal{N}_8^1(p)$ will become non-connected and the 3rd cond. will not be met any more so that p is dependent on q .

A similar analysis is performed for the second case when $\mathcal{G} = 4$ and $\mathcal{G}' = 8$. According to Proposition 4, configurations of pixels a, b, g, k, l from Fig. 2 are order-independent. In the remaining configurations, when q is modified, p become non-simple. Considering particular cases from Fig. 2 and referring to the

conditions of Definition 4 we get: **Conf.a:** Since p is simple, $N_4^1(p)$ consists of at least one pixel, which must be one (or both) of 4-adjacent '*'-pixels. $N_4^1(p)$ is then located within *-pixels, and q anyway does not belong to it. When q is modified, the 3rd cond. will still be met, as well as the 1st and 2nd conditions. Thus, p is independent of q . **Conf.b:** Pixel q is a 8-, but not 4-adjacent neighbour of p , so when q is removed the $N_4^1(p)$ does not change. Also, q does not belong to 4-path connecting pixels from $N_4^1(p)$. Consequently, when q is removed the 3rd cond. will still be met. Since the 1st and 2nd cond. are met as well, p is independent from q . **Conf.c,m:** Since p is simple there must exist at least one pixel in $N_8^0(p)$ and it has to be one of those from $N_8(p)$ which are marked as '*'. The only 4-path connecting pixels from $N_4^1(p)$ goes then through q . If q is modified, the 3rd cond. will not be met and therefore p is dependent on q . **Conf.d:** Since p is simple, then according to the 3rd cond. $N_8^1(p) = \{q\}$ (otherwise 3rd cond. would not be met). If so, when q is removed, then due to the 1st cond., p is becoming non simple, so p is dependent on q . **Conf.e,f,h-j:** these configuration cannot appear on an actual image, since either p (in f,j) or q (in e,i) or both (in h) cannot be simple according to the 3rd cond. **Conf.g,k,l:** The 1st and 2nd cond. are met no matter the value of q . When q is removed, also the 3rd cond. will be met, since it does not split the 4-conn. component of $N_8^1(p)$ due to the fact that q is the last pixel 4-adjacent to p in the 4-path in $N_8^1(p)$.

Proof of Proposition 5. Comparing the current case with the case of simple independent pixels, the difference in pixels' configuration which are independent is in configurations a and d (from Fig. 2). Contrary to the case of simple pixels, in the current one both configurations are independent. This is due to the fact that when q is modified, p becomes isolated. In case of b-simpleness, this is allowed (thanks to Proposition 2). In all other configurations, the isolated pixel cannot appear when q is modified, so for those configurations the proof is the same as the former one.