Spherical Harmonic Transforms Using Quadratures and Least Squares

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Abstract. Spherical Harmonic Transforms (SHTs) which are essentially Fourier transforms on the sphere are critical in global geopotential and related applications. For analysis purposes, discrete SHTs are difficult to formulate for an optimal discretization of the sphere, especially for applications with requirements in terms of near-isometric grids and special considerations in the polar regions. With the enormous global datasets becoming available from satellite systems, very high degrees and orders are required and the implied computational efforts are very challenging. Among the best known strategies for discrete SHTs are quadratures and least squares. The computational aspects of SHTs and their inverses using both quadrature and least-squares estimation methods are discussed with special emphasis on information conservation and numerical stability. Parallel and grid computations are imperative for a number of geodetic, geophysical and related applications, and these are currently under investigation.

1 Introduction

Domains with spherical topology are very common in astronomy, cosmology, geophysics, geodesy and related disciplines. On the spherical Earth as on the celestial sphere, computations can be done for regional and global domains using planar and spherical formulations. Spherical quadratures and least-squares estimation are used to convert continuous integral formulations into summations over data lattices. Spherical topologies are quite different from planar ones and these have important implications in the computational aspects of data processing.

Spherical geocomputations for regional domains of even continental extents can be reduced to planar computations and under assumptions of stationarity or shift invariance, discrete regular array computations can be optimized using Fast Fourier Transforms (FFTs). Specifically, convolution operations for filtering and other data processing applications thereby require only O(NlogN) instead of $O(N^2)$ operations for N data in one dimension, $O(N^2 \log N)$ instead of $O(N^4)$ operations for NxN data in two dimensions, and so on.

For global appplications, Gaussian, equiangular and other similar rectangular grids can be used for spherical quadratures and discrete convolutions. Various quadrature

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strategies are available in the literature going back to Gauss and Neumann, in addition to least-squares estimation techniques (e.g. [14]). Other approaches have also been used for discretization and analysis of functions on the sphere using icosahedral, triangular and curvilinear tesselations. Depending on the applications, these strategies may be preferable to the rectangular ones which will be discussed in the following.

2 Continuous and Discrete SHTs

The orthogonal or Fourier expansion of a function f(θ, λ) on the sphere $\mathbf{S}^2$ is given by

$$f(\theta,\lambda)=\sum_{n=0}^{\infty}\sum_{|m|\le n} f_{n,m}\; Y_n^m(\theta,\lambda) \tag{1}$$

using colatitude θ and longitude λ, where the basis functions $Y_n^m(\theta,\lambda)$ are called the spherical harmonics satisfying the (spherical) Laplace equation $\Delta_{S^2} Y_n^m(\theta,\lambda)=0$, for all $|m| \le n$ and n = 0, 1, 2, …. This is an orthogonal decomposition in the Hilbert space $\boldsymbol{L}^2(\mathbf{S}^2)$ of functions square integrable with respect to the standard rotation invariant measure dσ = sin θ dθ dλ on $\mathbf{S}^2$. In particular, the Fourier or spherical harmonic coefficients appearing in the preceding expansion are obtained as inner products

$$f_{n,m}=\int_{S^2} f(\theta,\lambda)\; \overline{Y}_n^m(\theta,\lambda)\; d\sigma \tag{2}$$

with the overbar denoting the complex conjugate. In most practical applications, the functions f(θ,λ) are band-limited in the sense that only a finite number of those coefficients are nonzero, i.e. $f_{n,m} \equiv 0$ for all $n \ge N$ and orders $|m| \le n$. Hence, using the regular grid $\theta_j = j\pi/2N$ and $\lambda_k = k\pi/N$, j,k = 0,…,2N-1, spherical harmonic synthesis can be formulated as

$$f(\theta_j,\lambda_k)=\sum_{n=0}^{N-1}\sum_{|m|\le n} f_{n,m}\; Y_n^m(\theta_j,\lambda_k) \tag{3}$$

and using some appropriate spherical quadrature, the corresponding spherical harmonic analysis can be formulated as

$$f_{n,m}=\sum_{j=0}^{2N-1}\sum_{k=0}^{2N-1} q_j\; f(\theta_j,\lambda_k)\; \overline{Y}_n^m(\theta_j,\lambda_k) \tag{4}$$

for quadrature weights q_j as discussed by various authors e.g. [2], [3], [4] and [6].

The usual geodetic spherical harmonic formulation is given as

$$f(\theta,\lambda)=\sum_{n=0}^{\infty}\sum_{m=0}^{n} [\ddot{C}_{nm}\cos m\lambda+\ddot{S}_{nm}\sin m\lambda]\; \ddot{P}_{nm}(\cos\theta) \tag{5}$$

where

$$\begin{Bmatrix}\ddot{C}_{nm}\\ \ddot{S}_{nm}\end{Bmatrix}=\frac{1}{4\pi}\int_{S^2} f(\theta,\lambda)\begin{Bmatrix}\cos m\lambda\\ \sin m\lambda\end{Bmatrix}\ddot{P}_{nm}(\cos\theta)\; d\sigma \tag{6}$$

and

$$\ddot{P}_{nm}(\cos\theta) = \sqrt{\frac{2(2n+1)(n-m)!}{(n+m)!}}\, P_{nm}(\cos\theta)$$
$$\ddot{P}_{n}(\cos\theta) = \sqrt{2n+1}\, P_{n}(\cos\theta) \tag{7}$$

are the normalized associated Legendre functions. These associated Legendre functions are related to the preceding spherical harmonic functions $Y_n^m(\theta,\lambda)$ as follows

$$Y_n^m(\theta,\lambda) = (-1)^m \sqrt{\frac{(2n+1)(n-m)!}{4\pi(n+m)!}}\, P_{nm}(\cos\theta)\, e^{-im\lambda} \tag{8}$$

and geodetic normalization is slightly different from the usual mathematical normalization (see e.g. [5] and [8]).

Explicitly, using the geodetic formulation and convention, one has for synthesis, given normalized spherical harmonic coefficients a_{nm} and b_{nm} for $m \le n$, $n = 0, 1, 2, \ldots, N-1$,

$$\begin{aligned} f(\theta,\lambda) &= \sum_{m=0}^{N-1}\sum_{n=m}^{N-1} (a_{nm}\cos m\lambda + b_{nm}\sin m\lambda)\, \ddot{P}_{nm}(\cos\theta) \\ &= \sum_{m=0}^{N-1}\left\{\left(\sum_{n=m}^{N-1} a_{nm}\ddot{P}_{nm}(\cos\theta)\right)\cos m\lambda + \left(\sum_{n=m}^{N-1} b_{nm}\ddot{P}_{nm}(\cos\theta)\right)\sin m\lambda\right\} \end{aligned} \tag{9}$$

and defining

$$\begin{Bmatrix} A_m(\theta) \\ B_m(\theta) \end{Bmatrix} = \sum_{n=m}^{N-1} \begin{Bmatrix} a_{nm} \\ b_{nm} \end{Bmatrix} \ddot{P}_{nm}(\cos\theta) \tag{10}$$

one has

$$\begin{aligned} f(\theta,\lambda) &= \sum_{m=0}^{N-1} \{A_m(\theta)\cos m\lambda + B_m(\theta)\sin m\lambda\} \\ &= \mathrm{Re\,IDFT}\,[A_m(\theta) + i\,B_m(\theta)] \end{aligned} \tag{11}$$

assuming discrete longitudes $\lambda_k = k\pi/N$, $k=0, 1, 2, \ldots, 2N-1$, for unspecified discrete colatitudes θ. Writing $C_m(\theta) = A_m(\theta) + iB_m(\theta)$, and correspondingly $c_{nm} = a_{nm} + ib_{nm}$, one then has

$$f(\theta,\lambda_k) = \mathrm{Re\,IDFT}_k\,[C_m(\theta)] \tag{12}$$

where

$$C_m(\theta) = \sum_{n=m}^{N-1} (a_{nm} + ib_{nm})\ddot{P}_{nm}(\cos\theta) = \sum_{n=m}^{N-1} c_{nm}\ddot{P}_{nm}(\cos\theta) \tag{13}$$

in which IDFT stands for inverse discrete Fourier transform. For analysis, given data $f(\theta_j, \lambda_k)$ at $\theta_j = j\pi/2N$ and $\lambda_k = k\pi/N$, $j, k = 0, 1, 2, \ldots, 2N-1$, the normalized spherical harmonic coefficients a_{nm} and b_{nm} for $m \le n$, $n = 0, 1, 2, \ldots, N-1$, can be evaluated as follows:

$$\begin{Bmatrix} a_{nm} \\ b_{nm} \end{Bmatrix} = \frac{\pi(-1)^m}{N} \sum_{j=0}^{2N-1}\sum_{k=0}^{2N-1} q_j f(\theta_j,\lambda_k) \begin{Bmatrix} \cos m\lambda_k \\ \sin m\lambda_k \end{Bmatrix} \ddot{P}_{nm}(\cos\theta_j) \tag{14}$$

or, using complex coefficients,

$$\begin{aligned} c_{nm} &= \frac{\pi(-1)^m}{N} \sum_{j=0}^{2N-1} \sum_{k=0}^{2N-1} q_j f(\theta_j, \lambda_k) e^{+im\lambda_k} \ddot{P}_{nm}(\cos\theta_j) \\ &= \pi \sum_{j=0}^{2N-1} (-1)^m q_j \ddot{P}_{nm}(\cos\theta_j)\ \mathrm{DFT}_k[f(\theta_j, \lambda_k)] \end{aligned} \tag{15}$$

with the preceding quadrature weights q_j. Notice that in practice, depending on conventions, DFT and IDFT could be interchanged in the preceding derivation and for computational efficiency, direct and inverse FFTs would be substituted.

The preceding derivation can easily be modified for data grids with $\Delta\theta = \Delta\lambda$ such as

$$\{(\theta_j, \lambda_k) \mid \theta_j = j\pi/2N, \lambda_k = k\pi/2N; j = 0, 1, \ldots, 2N\text{-}1,\ k = 0, 1, \ldots, 4N\text{-}1\}$$

with the possible exclusion of the poles as

$$\{(\theta_j, \lambda_k) \mid \theta_j = (j+{}^1\!/_2)\pi/2N, \lambda_k = k\pi/2N; j = 0, 1, \ldots, 2N\text{-}1,\ k = 0, 1, \ldots, 4N\text{-}1\}$$

which also permit the use of hemispherical symmetries in the associated Legendre functions

$$P_{nm}(\cos(\pi - \theta)) = (-1)^{n+m} P_{nm}(\cos\theta)\,. \tag{16}$$

More details can be found in [4]. Notice that these data grids have 2Nx4N or $8N^2$ quantities for N^2 spectral coefficients for band limitedness N.

3 Spectral Analysis Using Least Squares

Let us consider the regular data grid

$$\{(\theta_j, \lambda_k) \mid \theta_j = j\pi/N, \lambda_k = k\pi/2N; j = 0, 1, \ldots, N\text{-}1,\ k = 0, 1, \ldots, 2N\text{-}1\}$$

or with the exclusion of the poles,

$$\{(\theta_j, \lambda_k) \mid \theta_j = (j+{}^1\!/_2)\pi/N, \lambda_k = k\pi/2N; j = 0, 1, \ldots, N\text{-}1,\ k = 0, 1, \ldots, 2N\text{-}1\}$$

which implies $\Delta\theta = \Delta\lambda$. This is the grid normally considered in geodesy to model the geopotential field with least data for a spectral expansion to degree and order $N - 1$.

From the preceding discussion, one can write

$$f(\theta_j, \lambda_k) = \mathrm{Re}\ \mathrm{IDFT}_k\Big[\sum_{n=m}^{N-1} (a_{nm} + ib_{nm}) \ddot{P}_{nm}(\cos\theta_j)\Big] \tag{17}$$

for the unknown a_{nm} and b_{nm}. Hence

$$\mathrm{DFT}_k[f(\theta_j, \lambda_k) + i0] = \sum_{n=m}^{N-1} (a_{nm} + ib_{nm}) \ddot{P}_{nm}(\cos\theta_j) \tag{18}$$

which implies that for each $m = 0, 1, \ldots, N\text{-}1$, least squares can be used to estimate the spectral coefficients $a_{mm}, a_{m+1,m}, \ldots, a_{N-1,m}$ and $b_{mm}, b_{m+1,m}, \ldots, b_{N-1,m}$.

It is very important to notice that a maximum degree of $N - 1$ is critical for the least-squares solution for the spectral coefficients $a_{mm}, a_{m+1,m}, \ldots, a_{N-1,m}$ and $b_{mm}, b_{m+1,m}, \ldots, b_{N-1,m}$ using N partitions in latitude, i.e. $\theta_j = j\pi/N$, or $\theta_j = (j+{}^1\!/_2)\pi/N$, j =

0, 1, …, N- 1. In other words, the use of N parallels for degree N would lead to an underdetermined system for a_{mm}, $a_{m+1,m}$, …, $a_{N,m}$ and b_{mm}, $b_{m+1,m}$, …, $b_{N,m}$, as apparently suggested in [14].

In most practical implementations, the partition in latitude is shifted by half a grid interval to avoid any numerical complication at the pole for $\theta = 0$. Furthermore, the partition in latitude is not required to be equispaced for the least-squares estimation of the spectral coefficients. However, in longitude, equispacing is required for the usual implementation of the DFTs as FFTs.

4 Numerical Experimentation

As indicated earlier, there are several formulations for employing spherical harmonics as an analysis tool. One popular code that is readily available is Spherepack [1] of which a new version has been released recently. Other experimental codes are those of Driscoll and Healy [6] and the follow-ons, such as [7] and [12], plus the example algorithm described by Mohlenkamp [9][10] and offered as a partial sample implementation in [11]. Experimentation with these codes has shown scaling differences from that which is expected in a geodetic context [13].

Using the Driscoll and Healy [6] formulation modified as described in Section 2, extensive experimentation using different grids on several computer platforms in double precision (i.e. REAL*8) and quadruple precision (i.e. REAL*16) has been carried out. However, only double precision results are reported in this paper. The first synthesis started with unit coefficients, $a_{nm} = b_{nm} = 1$, except for $b_{n0} = 0$, for all degrees n and orders m, which corresponds to white noise. Then, following analysis of the generated spatial grid values, the coefficients are recomputed and root-mean-square (RMS) values are given for this synthesis/analysis. Then after another synthesis using these recomputed coefficients, RMS values of recomputed grid residuals are given for the second synthesis. Hence, starting with arbitrary coefficients $\{c_{nm}\}$, the procedure can be summarized as follows:

$$SHT[SHT^{-1}[\{c_{nm}\}]] - [\{c_{nm}\}] \quad \rightarrow \quad \text{RMS of first synthesis/analysis,}$$

and

$$SHT^{-1}[SHT[SHT^{-1}[\{c_{nm}\}]]] - SHT^{-1}[\{c_{nm}\}] \quad \rightarrow \quad \text{RMS of second synthesis.}$$

Notice that the first RMS is in the spectral domain while the second is in the spatial domain. The SHTs and SHT^{-1}s are evaluated and re-evaluated explicitly to study their numerical stability and the computational efficiency. Table 1(top) lists the RMS values and the computation times with grids $\Delta\theta = ½\Delta\lambda$ (i.e., 2Nx2N) and $\Delta\theta = \Delta\lambda$ (i.e., Nx2N and 2Nx4N). The above procedure is repeated for coefficients corresponding to 1/degree^2, i.e. explicitly, $a_{nm} = 1/(n+1)^2$, $b_{nm} = 0$ for m=0, and $1/(n+1)^2$, otherwise, for all degrees n and orders m, which simulate a physically realizable situation. The corresponding RMS values and computation times for various grids are listed in the bottom part of Table 1. It should be noted that the results for the grid Nx2N are based on least squares whereas for 2Nx2N and 2Nx4N the quadrature scheme discussed in Section 2 is used, and the computations were done on an AMD 64 Athlon FX-51 PC in REAL*8 precision.

Table 1. SHT RMS values and computation times for Nx2N, 2Nx2N and 2Nx4N grids, where N-1 is the maximum degree of expansion. Results for the Nx2N grid are obtained using the least-squares technique and for 2Nx2N and 2Nx4N, the quadrature scheme. **Top:** Input: unit spectral coefficients. **Bottom:** Input: 1/degree^2 spectral coefficients.

	Degrees	Grid	Synthesis/Analysis		Synthesis	
			RMS(coef.)	Time(sec.)	RMS(data)	Time(sec.)
N x 2N	0-63	64x128	2.632e-15	0.03	1.123e-13	0.02
	0-127	128x256	5.354e-15	0.44	4.368e-13	0.06
	0-255	256x512	1.062e-14	6.66	1.743e-12	0.53
	0-511	512x1024	2.126e-14	226.06	7.045e-12	4.20
	0-1023	1024x2048	4.245e-14	10290.73	2.796e-11	32.94
	0-1499	1500x3000	6.046e-14	15552.55	5.808e-11	102.23
2N x 2N	0-63	128x128	5.966e-15	0.01	7.763e-15	0.01
	0-127	256x256	1.693e-14	0.08	4.315e-14	0.07
	0-255	512x512	5.000e-14	0.69	1.906e-13	0.59
	0-511	1024x1024	6.879e-14	6.19	2.943e-13	5.12
	0-1023	2048x2048	2.817e-13	46.30	2.819e-12	37.99
	0-1499	3000x3000	7.121e-13	142.29	9.497e-12	117.34
2N x 4N	0-63	128x256	5.978e-15	0.01	7.844e-15	0.01
	0-127	256x512	1.686e-14	0.08	3.605e-14	0.07
	0-255	512x1024	4.994e-14	0.74	1.906e-13	0.63
	0-511	1024x2048	6.874e-14	6.32	2.787e-13	5.22
	0-1023	2048x4096	2.817e-13	47.54	2.818e-12	38.60
	0-1499	3000x6000	7.121e-13	146.22	9.497e-12	120.03
N x 2N	0-63	64x128	1.718e-17	0.05	7.765e-16	0.00
	0-127	128x256	2.253e-17	0.47	1.797e-15	0.08
	0-255	256x512	1.675e-17	6.95	2.804e-15	0.55
	0-511	512x1024	1.078e-17	247.97	3.583e-15	4.66
	0-1023	1024x2048	7.151e-18	11051.94	4.780e-15	35.67
	0-1499	1500x3000	6.177e-18	16728.81	6.167e-15	112.25
2N x 2N	0-63	128x128	4.142e-17	0.01	3.844e-17	0.01
	0-127	256x256	5.806e-17	0.08	9.084e-17	0.07
	0-255	512x512	5.201e-17	0.74	1.212e-16	0.59
	0-511	1024x1024	2.877e-17	6.45	5.578e-17	5.12
	0-1023	2048x2048	3.437e-17	48.90	2.097e-16	38.02
	0-1499	3000x3000	5.057e-17	151.04	5.208e-16	117.30
2N x 4N	0-63	128x256	4.120e-17	0.01	3.413e-17	0.01
	0-127	256x512	5.774e-17	0.09	6.358e-17	0.07
	0-255	512x1024	5.175e-17	0.79	1.185e-16	0.64
	0-511	1024x2048	2.876e-17	6.66	7.386e-17	5.19
	0-1023	2048x4096	3.438e-17	50.32	2.041e-16	38.53
	0-1499	3000x6000	5.057e-17	155.86	5.058e-16	119.97

The RMS of errors involved in the least squares (Nx2N) and quadratures (2Nx2N and 2Nx4N) schemes are comparable (Table 1) and varying in the range 10^{-12} to 10^{-15} for unit coefficients and 10^{-15} to 10^{-17} for 1/degree^2 coefficients. The computation times for both the approaches, however, vary significantly. Preliminary investigations indicate inefficiency in the least-squares approach resulting in extra efforts needed in the normal equations corresponding to each order m to estimate the related spectral coefficients. Work is in progress to optimize this part of the code to make the computation times comparable to the quadrature scheme.

Spectral analysis of the synthesis/analysis results can be done degree by degree to study the characteristics of the estimated spectral harmonic coefficients. The results of the second synthesis also enable a study of the spatial results parallel by parallel, especially for the polar regions. Such investigations are currently underway to better characterize the numerical stability and reliability of the SHT and SHT^{-1}. Ongoing experimentation is attempting to carry out the computations to higher degrees and orders.

5 Concluding Remarks

Considerable work has been done on solving the computational complexities, and enhancing the speed of calculation of spherical harmonics transforms for different equiangular grids. The approach of Driscoll and Healy [6] is exact for exact arithmetic, and with a number of modifications, different implementations have been experimented with, leading to RMS errors of orders 10^{-12} to 10^{-15} with unit coefficients of degrees and orders up to 1500. Comparable RMS have been obtained with the least-squares estimation approach. However the computational efforts with least squares are quite significantly larger than with the quadrature approach. When starting with spherical harmonic coefficients corresponding to 1/degree^2, the previously mentioned analysis and synthesis results are improved to 10^{-15}-10^{-18} with both approaches. The latter simulations are perhaps more indicative of the expected numerical accuracies in practice.

All these computational experiments started with unit and 1/degree^2 coefficients of degrees 0, 1, 2, …, N-1 for spherical N x 2N grids with least squares, and 2N x 2N and 2N x 4N grids with quadratures, offset in latitude by half a grid unit from the pole. It remains to be seen how these spatial lattices differ for the same spectrum of degree and order N – 1 for a specified N. This comparative spatial analysis is currently underway.

Computations for higher degrees and orders are under consideration assuming the availability of parallel FFT code in quadruple precision. As enormous quantities of data are involved the intended gravity field applications, parallel and grid computations are imperative for these applications. Preliminary experimentation with parallel processing has already been done [15].

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References

1. Adams, J.C. and P.N. Swarztrauber [1997]: SPHEREPACK 2.0: A Model Development Facility. http://www.scd.ucar.edu/softlib/SPHERE.html
2. Blais, J.A.R. and D.A. Provins [2002]: Spherical Harmonic Analysis and Synthesis for Global Multiresolution Applications. Journal of Geodesy, vol.76, no.1, pp.29-35.
3. Blais, J.A.R. and D.A. Provins [2003]: Optimization of Computations in Global Geopotential Field Applications. Computational Science – ICCS 2003, Part II, edited by P.M.A. Sloot, D. Abramson, A.V. Bogdanov, J.J. Dongarra, A.Y. Zomaya and Y.E. Gorbachev. Lecture Notes in Computer Science, vol.2658, pp.610-618. Springer-Verlag.
4. Blais, J.A.R., D.A. Provins and M.A. Soofi [2005]: Optimization of Spherical Harmonic Transform Computations, in ICCS 2005, V.S. Sunderam et al. (eds.), Lecture Notes in Computer Science (LNCS), vol. 3514, p. 74-81.
5. Colombo, O. [1981]: Numerical Methods for Harmonic Analysis on the Sphere. Report no. 310, Department of Geodetic Science and Surveying, The Ohio State University
6. Driscoll, J.R. and D.M. Healy, Jr. [1994]: Computing Fourier Transforms and Convolutions on the 2-Sphere. Advances in Applied Mathematics, 15, pp. 202-250.
7. Healy, D., Jr., D. Rockmore, P. Kostelec and S. Moore [1998]: FFTs for the 2-Sphere - Improvements and Variations, To appear in Advances in Applied Mathematics, Preprint from http://www.cs.dartmouth.edu/~geelong/publications (June 1998).
8. Heiskanen, W.A. and H. Moritz [1967]: Physical Geodesy, W.H. Freeman and Company, San Francisco, 363pp.
9. Mohlenkamp, M.J. [1997]: A Fast Transform for Spherical Harmonics. PhD thesis, Yale University.
10. Mohlenkamp, M.J. [1999]: A Fast Transform for Spherical Harmonics. The Journal of Fourier Analysis and Applications, 5, 2/3, pp. 159-184, Preprint from http://amath.colorado.edu/faculty/mjm.
11. Mohlenkamp, M.J. [2000]: Fast spherical harmonic analysis: sample code. http://amath.colorado.edu/faculty/mjm.
12. Moore, S., D. Healy, Jr., D. Rockmore and P. Kostelec [1998]: SpharmonKit25: Spherical Harmonic Transform Kit 2.5, http://www.cs.dartmouth.edu/~geelong/sphere/.
13. Provins, D.A. [2003]: Earth Synthesis: Determining Earth's Structure from Geopotential Fields, Unpublished PhD thesis, University of Calgary, Calgary.
14. Sneeuw, N. [1994]: Global Spherical Harmonic Analysis by Least-Squares and Numerical Quadrature Methods in Historical Perspective. Geophys. J. Int. 118, 707-716.
15. Soofi, M.A. and J.A.R. Blais [2005]: Parallel Computations of Spherical Harmonic Transforms, Oral presentation at the Annual Meeting of the Canadian Geophysical Union, Banff, Alberta, Canada.