A Picture for Complex Stochastic Boolean Systems: The Intrinsic Order Graph*

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Abstract. Complex stochastic Boolean systems, depending on a large number n of statistically independent random Boolean variables, appear in many different scientific, technical or social areas. Each one of the 2^n binary states associated to such systems is denoted by its corresponding binary n -tuple of 0s and 1s, $(u_1, \dots, u_n)$, and it has a certain occurrence probability $\Pr\{(u_1, \dots, u_n)\}$. The ordering between the 2^n binary n -tuple probabilities, $\Pr\{(u_1, \dots, u_n)\}$, can be illustrated by a directed graph which “scales” them by decreasing order, the so-called intrinsic order graph. In this context, this paper provides a simple algorithm for iteratively drawing the intrinsic order graph, for any complex stochastic Boolean system and for any number n of independent random Boolean variables. The presentation is self-contained.

1 Introduction

This paper deals with the modelling of complex stochastic Boolean systems, that is, those complex systems which depend on a large number n of random Boolean variables $x_1, \dots, x_n$. These systems, with very simple components (the Boolean variables x_i only take two possible values: 0, 1) but complex overall behavior, can be found in many different knowledge areas: Wherever a stochastic Boolean phenomenon with n basic variables appears (Biology, Biochemistry, Climatology, Computer Science, Networks, Engineering, Economics, Sociology, etc.).

According to the usual terminology in Statistics, a stochastic Boolean system can be modeled by the n -dimensional Bernoulli distribution (see, e.g., [6]). This distribution consists on n random variables $x_1, \dots, x_n$, which only take two possible values, 0 or 1, with probabilities

$$\Pr\{x_i = 1\} = p_i, \quad \Pr\{x_i = 0\} = 1 - p_i \quad (1 \leq i \leq n),$$

so that the sample space is the set $\{0, 1\}^n$ of the 2^n binary n -tuples (binary strings), $u = (u_1, \dots, u_n)$, of 0s and 1s. Throughout this work, we assume that

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the random variables x_i are mutually independent, so that the occurrence probability of each binary n -tuple can be easily computed as the product

$$\Pr \{(u_1, \dots, u_n)\} = \prod_{i=1}^n \Pr \{x_i = u_i\} = \prod_{i=1}^n p_i^{u_i} (1 - p_i)^{1-u_i}, \quad (1)$$

that is, $\Pr \{(u_1, \dots, u_n)\}$ is the product of factors p_i if $u_i = 1$, $1-p_i$ if $u_i = 0$.

The following natural question immediately arises in the study of stochastic Boolean systems: How can we order the 2^n binary strings $(u_1, \dots, u_n)$ by decreasing/increasing order of their occurrence probabilities? Obviously, this question has a relevant, theoretical and practical, interest. However, in spite of the simplicity of Equation (1) to compute the probabilities $\Pr \{(u_1, \dots, u_n)\}$, this is not a simple question due to its exponential nature. To avoid this obstacle, in [3, 4] the authors have established a simple positional criterion, the so-called *Intrinsic Order Criterion* (IOC), that *a priori* assures us that for certain pairs of binary n -tuples $u, v \in \{0, 1\}^n$ (exactly for those pairs whose bits satisfy IOC), the inequality $\Pr \{u\} \geq \Pr \{v\}$ *intrinsically* holds, that is, $\Pr \{u\} \geq \Pr \{v\}$ for any values of the parameters p_i satisfying certain non restrictive hypothesis. Moreover, in [1, 2] the author has “roughly” constructed the intrinsic order graph for the first values of n , just by direct application of IOC. However, a general method for easily constructing this graph for all $n \in \mathbb{N}$, has not yet been provided.

In this context, *the main goal of this paper is to provide a simple algorithm for recursively drawing the intrinsic order graph, for any number n of Boolean variables. This directed graph is a very useful picture for all complex stochastic Boolean systems (related to any knowledge area), because it “scales” the binary n -tuples of 0s and 1s, $(u_1, \dots, u_n) \in \{0, 1\}^n$, by decreasing order of their occurrence probabilities.* In Sect. 2, we describe all previous results required for making this paper self-contained. The rest of the paper is devoted to our new approach. In Sect. 3, we present a simple characterization of the covering relation of the intrinsic order. Finally, in Sect. 4, from this characterization we derive the algorithm for drawing the intrinsic order graph.

2 The Intrinsic Order

In [3, 4], we have established the following characterization theorem that allows us to compare two given binary string probabilities, $\Pr \{u\}$ and $\Pr \{v\}$, without computing them.

Theorem 1 (The intrinsic order theorem). *Let $x_1, \dots, x_n$ be n independent Bernoulli variables, with parameters $p_i = \Pr \{x_i = 1\}$ ($1 \leq i \leq n$) satisfying:*

$$0 < p_1 \leq \dots \leq p_n \leq \frac{1}{2}. \quad (2)$$

Then, the probability of the n -tuple $(u_1, \dots, u_n) \in \{0, 1\}^n$ is intrinsically greater than or equal to the probability of the n -tuple $(v_1, \dots, v_n) \in \{0, 1\}^n$ (that is, for

all set of parameters $\{p_i\}_{i=1}^n$ such that (2)) if, and only if, the matrix

$$M_v^u := \begin{pmatrix} u_1 & \cdots & u_n \\ v_1 & \cdots & v_n \end{pmatrix}$$

either has no $\binom{1}{0}$ columns, or for each $\binom{1}{0}$ column there exists (at least) one corresponding preceding $\binom{0}{1}$ column (IOC).

Remark 1. In the following, we assume that the parameters p_i always satisfy condition (2). Note that this hypothesis is not restrictive for practical applications because, if for some $i : p_i > \frac{1}{2}$, then we only need to consider the variable $\bar{x}_i = 1 - x_i$, instead of x_i . Next, we order the n Bernoulli variables by increasing order of their probabilities.

Remark 2. The $\binom{0}{1}$ column preceding to each $\binom{1}{0}$ column is not required to be necessarily placed at the immediately previous position, but just at previous position.

Remark 3. The term *corresponding*, used in Theorem 1, has the following meaning: For each two $\binom{1}{0}$ columns in matrix M_v^u , there must exist (at least) two different $\binom{0}{1}$ columns preceding to each other. In other words, IOC requires that for each $\binom{1}{0}$ column C in matrix M_v^u , if we denote by $n_1^0(C)$ ($n_0^1(C)$, resp.) the number of $\binom{0}{1}$ columns ($\binom{1}{0}$ columns, resp.) preceding C , then $n_1^0(C) > n_0^1(C)$.

Theorem 1 naturally leads to the following partial order relation on the set $\{0, 1\}^n$. The so-called intrinsic order (because it only depends of the relative positions of 0s and 1s in the binary n -tuples u, v , but not on the basic probabilities p_i), will be denoted by “ $\preceq$ ”, and we shall denote by P_n the partially ordered set (poset, for short) $(\{0, 1\}^n, \preceq)$. See [5] for more details about posets.

Definition 1. For all $u, v \in \{0, 1\}^n$

$u \succeq v$ iff $\Pr\{u\} \geq \Pr\{v\}$ for all set $\{p_i\}_{i=1}^n$ s.t. (2) iff M_v^u satisfies IOC.

Example 1. $(0, 0, 1, 1) \succeq (1, 1, 0, 0)$ because matrix

$$\begin{pmatrix} 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 \end{pmatrix}$$

satisfies IOC (Remark 2).

Example 2. $(0, 1, 1, 0) \not\succeq (1, 0, 0, 0)$ and $(1, 0, 0, 0) \not\succeq (0, 1, 1, 0)$ because

$$\text{neither } \begin{pmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}, \text{ nor } \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \end{pmatrix}$$

satisfies IOC (Remark 3).

From now on, we shall indistinctly denote any n -tuple $u \in \{0, 1\}^n$ by its binary representation $(u_1, \dots, u_n)$ or by its decimal representation $u_{(10)}$. Also, we shall denote by “ $\leq_{lex}$ ” the usual lexicographic order (truth-table order) between the binary n -tuples, with the convention that $0 < 1$, i.e.,

$$(u_1, \dots, u_n) \equiv u_{(10)} := \sum_{i=1}^n 2^{n-i} u_i, \quad u \leq_{lex} v \Leftrightarrow u_{(10)} \leq v_{(10)}.$$

The next corollary (see [2] for the proof) states the relation between the (partial) intrinsic order “ $\preceq$ ” and the (total) lexicographic order “ $\leq_{lex}$ ”.

Corollary 1. *For all $u, v \in \{0, 1\}^n$*

$$u \succeq v \Rightarrow u \leq_{lex} v, \text{ i.e., } u \succeq v \Rightarrow u_{(10)} \leq v_{(10)}.$$

3 Covering Relation in P_n

As is well-known [5], every finite poset is completely determined by its cover relations. Specifically, with respect to our poset P_n , we say that v is covered by u (or u covers v), denoted by $v \triangleleft u$ (or $u \triangleright v$) if $v \prec u$ and there is no $w \in \{0, 1\}^n$ such that $v \prec w \prec u$. For instance, in $P_2 = (\{0, 1\}^2, \preceq)$, we have $(1, 1) \triangleleft (1, 0) \triangleleft (0, 1) \triangleleft (0, 0)$ because $(1, 1) \prec (1, 0) \prec (0, 1) \prec (0, 0)$ (Definition 1, Theorem 1) with no other elements between them. However, $(1, 1) \prec (0, 1)$ but $(1, 1) \not\triangleleft (0, 1)$, because $(1, 0)$ is between them. In this section, we provide a simple matrix characterization of the covering relation associated to the intrinsic order. This condition for “ $\triangleleft$ ”, required for drawing the intrinsic order graph in the next section, will be obviously a particular case of (more restrictive than) the IOC condition for “ $\preceq$ ”. First, we need the following lemma.

Lemma 1. *Let $u, v \in \{0, 1\}^n$ and $1 \leq i \leq n - 1$.*

(i) *If $v \preceq u$ then $(v_1, \dots, v_i) \preceq (u_1, \dots, u_i)$.*

(ii) *If $v \preceq u$ and $\begin{pmatrix} u_1 \dots u_i \\ v_1 \dots v_i \end{pmatrix}$ has the same number of $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ columns as $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$ columns then $(v_{i+1}, \dots, v_n) \preceq (u_{i+1}, \dots, u_n)$.*

(iii) *If $v \preceq u$ and the columns $\begin{pmatrix} u_i \\ v_i \end{pmatrix}$ and $\begin{pmatrix} u_{i+1} \\ v_{i+1} \end{pmatrix}$ are both different from $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$, then the matrix obtained by permuting these two columns of M_v^u , keeping its other columns, satisfies IOC.*

(iv) *If $v \preceq u$, $\begin{pmatrix} u_i \\ v_i \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$, $\begin{pmatrix} u_j \\ v_j \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ for some $j > i$, and there is no k such that $i < k < j$ and $\begin{pmatrix} u_k \\ v_k \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, then the matrix obtained by replacing the i -th and j -th columns of M_v^u by $\begin{pmatrix} 0 \\ 0 \end{pmatrix}$ or $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$, keeping its other columns, satisfies IOC.*

Proof. Using the matrix description IOC of the intrinsic order (Theorem 1), the proof is straightforward. $\square$

Theorem 2 (Covering relation in P_n). *Let $n \geq 1$ and $u, v \in \{0, 1\}^n$. Then $v \triangleleft u$ if and only if either*

$$M_v^u = \begin{pmatrix} u_1 \dots u_{n-1} & 0 \\ u_1 \dots u_{n-1} & 1 \end{pmatrix}, \text{ or} \quad (3)$$

$$M_v^u = \begin{pmatrix} u_1 \dots u_{i-1} & 0 & 1 & u_{i+2} \dots u_n \\ u_1 \dots u_{i-1} & 1 & 0 & u_{i+2} \dots u_n \end{pmatrix} \quad (1 \leq i \leq n - 1), \quad (4)$$

where we assume that in case (3) the $n - 1$ first columns of matrix M_v^u are deleted if $n = 1$; while in case (4) the $i - 1$ first (the $n - i - 1$ last, resp.) columns of matrix M_v^u are deleted if $i = 1$ (if $i = n - 1$, resp.).

Proof. Sufficient condition. First, note that, for both cases (3) and (4), $v \prec u$ since the two corresponding matrices M_v^u obviously satisfy IOC. So, for both cases, we only need to prove that if $v \preceq w \preceq u$ then $w = u$ or $w = v$.

If (3) holds then using Corollary 1, we have $v \preceq w \preceq u \Rightarrow u \leq_{lex} w \leq_{lex} v$ and then $w = u$ or $w = v$.

If (4) holds then using Corollary 1, we have $v \preceq w \preceq u \Rightarrow u \leq_{lex} w \leq_{lex} v$ and then two subcases are possible:

$$(w_1, \dots, w_{i+1}) = \begin{cases} (u_1, \dots, u_{i-1}, 0, 1) & (4)\text{-a,} \\ (u_1, \dots, u_{i-1}, 1, 0) & (4)\text{-b.} \end{cases}$$

- If $i = n - 1$, then (4)-a is equivalent to say $w = u$ while (4)-b is equivalent to say $w = v$. Then, $w = u$ or $w = v$.
- Otherwise, if $1 \leq i < n - 1$, then using Lemma 1-(ii), we have: On one hand, since $v \preceq w$ and matrix $\begin{pmatrix} w_1 & \dots & w_{i-1} & w_i & w_{i+1} \\ u_1 & \dots & u_{i-1} & 1 & 0 \end{pmatrix}$ has the same number of $\binom{1}{0}$ columns as $\binom{0}{1}$ columns (namely, 1 if (4)-a holds and 0 if (4)-b holds) then we get $(u_{i+2}, \dots, u_n) \preceq (w_{i+2}, \dots, w_n)$. On the other hand, since $w \preceq u$ and matrix $\begin{pmatrix} u_1 & \dots & u_{i-1} & 0 & 1 \\ w_1 & \dots & w_{i-1} & w_i & w_{i+1} \end{pmatrix}$ has the same number of $\binom{1}{0}$ columns as $\binom{0}{1}$ columns (namely, 0 if (4)-a holds and 1 if (4)-b holds) then we get $(w_{i+2}, \dots, w_n) \preceq (u_{i+2}, \dots, u_n)$. So, due to the antisymmetry of the intrinsic order we obtain $(w_{i+2}, \dots, w_n) = (u_{i+2}, \dots, u_n)$, and then $w = u$ (if (4)-a holds) or $w = v$ (if (4)-b holds).

Necessary condition. We provide a constructive proof by finding $w \in \{0, 1\}^n$ yielding $v \not\preceq w \not\preceq u$, for all $v \prec u$ such that the pattern of matrix M_v^u is different from (3) and (4). First, note that since $v \prec u$ we assure (Corollary 1) that $u <_{lex} v$, i.e., the first column of M_v^u different from $\binom{0}{1}$ and $\binom{1}{1}$ is $\binom{0}{1}$. Let $\binom{u_i}{v_i}$ be this column. Then

$$M_v^u = \begin{pmatrix} u_1 & \dots & u_{i-1} & 0 & u_{i+1} & u_{i+2} & \dots & u_n \\ u_1 & \dots & u_{i-1} & 1 & v_{i+1} & v_{i+2} & \dots & v_n \end{pmatrix}, \quad (5)$$

where we can assure that $1 \leq i < n$, because if $i = n$ then M_v^u in (5) would be like (3), and we assume that the $i - 1$ first (the $n - i - 1$ last, resp.) columns of matrix (5) are deleted if $i = 1$ (if $i = n - 1$, resp.). The same assumption is established for the $i - 1$ first (the $n - i - 1$ last, resp.) components of the chosen vectors w . We distinguish the following four cases.

(1) $\binom{u_{i+1}}{v_{i+1}} = \binom{0}{0}$. In this case we choose

$$w = (u_1, \dots, u_{i-1}, 0, 1, v_{i+2}, \dots, v_n), \text{ so that } v \neq w \neq u \text{ and}$$

$$M_v^w = \begin{pmatrix} u_1 & \dots & u_{i-1} & 0 & 1 & v_{i+2} & \dots & v_n \\ u_1 & \dots & u_{i-1} & 1 & 0 & v_{i+2} & \dots & v_n \end{pmatrix} \text{ and } M_w^u = \begin{pmatrix} u_1 & \dots & u_{i-1} & 0 & 0 & u_{i+2} & \dots & u_n \\ u_1 & \dots & u_{i-1} & 0 & 1 & v_{i+2} & \dots & v_n \end{pmatrix}$$

satisfy IOC: the first one obviously; the second one due to Lemma 1-(iii).

(2) $\binom{u_{i+1}}{v_{i+1}} = \binom{1}{1}$. In this case we choose

$$w = (u_1, \dots, u_{i-1}, 1, 0, u_{i+2}, \dots, u_n), \text{ so that } v \neq w \neq u \text{ and}$$

$$M_v^w = \begin{pmatrix} u_1 & \dots & u_{i-1} & 1 & 0 & u_{i+2} & \dots & u_n \\ u_1 & \dots & u_{i-1} & 1 & 1 & v_{i+2} & \dots & v_n \end{pmatrix} \text{ and } M_w^u = \begin{pmatrix} u_1 & \dots & u_{i-1} & 0 & 1 & u_{i+2} & \dots & u_n \\ u_1 & \dots & u_{i-1} & 1 & 0 & u_{i+2} & \dots & u_n \end{pmatrix}$$

satisfy IOC: the first one due to Lemma 1-(iii); the second one obviously.

(3) $\binom{u_{i+1}}{v_{i+1}} = \binom{0}{1}$. Here we distinguish two subcases:

– (3.1) M_v^u has no $\binom{1}{0}$ columns. In this subcase we choose

$w = (u_1, \dots, u_{i-1}, 0, 1, v_{i+2}, \dots, v_n)$, so that $v \neq w \neq u$ and

$$M_v^w = \begin{pmatrix} u_1 & \dots & u_{i-1} & 0 & 1 & v_{i+2} & \dots & v_n \\ u_1 & \dots & u_{i-1} & 1 & 1 & v_{i+2} & \dots & v_n \end{pmatrix} \text{ and } M_w^u = \begin{pmatrix} u_1 & \dots & u_{i-1} & 0 & 0 & u_{i+2} & \dots & u_n \\ u_1 & \dots & u_{i-1} & 0 & 1 & v_{i+2} & \dots & v_n \end{pmatrix}$$

satisfy IOC, since they have no $\binom{1}{0}$ columns.

– (3.2) M_v^u has at least one $\binom{1}{0}$ column. Let $\binom{u_j}{v_j}$, $i+2 \leq j \leq n$, be the left-most $\binom{1}{0}$ column of M_v^u . In this subcase we choose

$w = (u_1, \dots, u_{i-1}, 0, 1, v_{i+2}, \dots, v_{j-1}, 1, v_{j+1}, \dots, v_n)$, so that $v \neq w \neq u$

and with the assumption that the substring $(v_{i+2}, \dots, v_{j-1})$ of w is deleted if $j = i+2$. Then

$$M_v^w = \begin{pmatrix} u_1 & \dots & u_{i-1} & 0 & 1 & v_{i+2} & \dots & v_{j-1} & 1 & v_{j+1} & \dots & v_n \\ u_1 & \dots & u_{i-1} & 1 & 1 & v_{i+2} & \dots & v_{j-1} & 0 & v_{j+1} & \dots & v_n \end{pmatrix},$$

$$M_w^u = \begin{pmatrix} u_1 & \dots & u_{i-1} & 0 & 0 & u_{i+2} & \dots & u_{j-1} & 1 & u_{j+1} & \dots & u_n \\ u_1 & \dots & u_{i-1} & 0 & 1 & v_{i+2} & \dots & v_{j-1} & 1 & v_{j+1} & \dots & v_n \end{pmatrix}$$

satisfy IOC: the first one obviously; the second one due to Lemma 1-(iv).

(4) $\binom{u_{i+1}}{v_{i+1}} = \binom{1}{0}$. Recall that for all possible cases (1)-(4) we assured that $1 \leq i < n$. Moreover, in this fourth case we can assure that $1 \leq i < n-1$ and $(v_{i+2}, \dots, v_n) \neq (u_{i+2}, \dots, u_n)$, because if $i = n-1$ or if $i < n-1$ with $(v_{i+2}, \dots, v_n) = (u_{i+2}, \dots, u_n)$ then, in both cases, M_v^u in (5) would have the pattern (4). In this last case we choose

$w = (u_1, \dots, u_{i-1}, 0, 1, v_{i+2}, \dots, v_n)$, so that $v \neq w \neq u$ and

$$M_v^w = \begin{pmatrix} u_1 & \dots & u_{i-1} & 0 & 1 & v_{i+2} & \dots & v_n \\ u_1 & \dots & u_{i-1} & 1 & 0 & v_{i+2} & \dots & v_n \end{pmatrix} \text{ and } M_w^u = \begin{pmatrix} u_1 & \dots & u_{i-1} & 0 & 1 & u_{i+2} & \dots & u_n \\ u_1 & \dots & u_{i-1} & 0 & 1 & v_{i+2} & \dots & v_n \end{pmatrix}$$

satisfy IOC: the first one obviously; the second one due to Lemma 1-(iv). $\square$

4 The Intrinsic Order Graph

The classical picture of a poset P is its Hasse diagram. This is a directed graph (digraph, for short) whose vertices are the elements of P , whose edges are the cover relations, and with the usual convention that if $v \prec u$ then u is drawn

“above” v [5]. For instance, the digraph of P_1 is $\begin{smallmatrix} 0 \\ 1 \end{smallmatrix}$, since $1 \triangleleft 0$. The next theorem provides us with a fast algorithm for iteratively building up the Hasse diagram of P_n , for all $n > 1$, from the Hasse diagram of P_1 . The binary n -tuples are represented by their decimal numbering.

Theorem 3 (Iterative construction of P_n from P_1). *For all $n > 1$, the digraph of $P_n = \{0, \dots, 2^n - 1\}$ can be drawn simply by adding to the digraph of $P_{n-1} = \{0, \dots, 2^{n-1} - 1\}$ its isomorphic copy $2^{n-1} + P_{n-1} = \{2^{n-1}, \dots, 2^n - 1\}$. This addition must be performed placing the powers of 2 at consecutive levels of the Hasse diagram of P_n . Finally, the edges $(u \triangleright v)$ connecting one vertex u of P_{n-1} with one vertex v of $2^{n-1} + P_{n-1}$ is given by the set of vertex pairs*

$$\{(u, v) \mid u \triangleright v\} = \{(u_{(10)}, 2^{n-2} + u_{(10)}) \mid 2^{n-2} \leq u_{(10)} \leq 2^{n-1} - 1\}.$$

Proof. For all $n > 1$, consider the following partition of $\{0, 1\}^n$

$$\begin{aligned} \{0, 1\}^n &= (\{0\} \times \{0, 1\}^{n-1}) \cup (\{1\} \times \{0, 1\}^{n-1}), \text{ i.e.,} \\ \{0, \dots, 2^n - 1\} &= \{0, \dots, 2^{n-1} - 1\} \cup \{2^{n-1}, \dots, 2^n - 1\}. \end{aligned}$$

On one hand, note that according to IOC (Theorem 1), the subposets of P_n , $(\{0\} \times \{0, 1\}^{n-1}, \preceq)$ and $(\{1\} \times \{0, 1\}^{n-1}, \preceq)$, are both isomorphic to P_{n-1} in the following sense. For all $(u_1, \dots, u_{n-1}), (v_1, \dots, v_{n-1}) \in \{0, 1\}^{n-1}$

$$\begin{aligned} (u_1, \dots, u_{n-1}) \preceq (v_1, \dots, v_{n-1}) &\Leftrightarrow (0, u_1, \dots, u_{n-1}) \preceq (0, v_1, \dots, v_{n-1}) \\ &\Leftrightarrow (1, u_1, \dots, u_{n-1}) \preceq (1, v_1, \dots, v_{n-1}). \end{aligned} \quad (6)$$

On the other hand, note that according to Theorem 2, we have for all $n > 1$

$$(0, \dots, 0, 1) \triangleright (0, \dots, 1, 0) \triangleright \dots \triangleright (1, 0, \dots, 0), \text{ i.e., } 2^0 \triangleright 2^1 \triangleright \dots \triangleright 2^{n-1}. \quad (7)$$

Equations (6) and (7) prove that P_n can be iteratively built up by adding to the digraph of P_{n-1} its isomorphic copy $2^{n-1} + P_{n-1}$, placing the powers of 2 at consecutive levels of the Hasse diagram of P_n . Finally, we prove the last assertion about the edges connecting the two mentioned subposets of P_n . Let $u \in \{0\} \times \{0, 1\}^{n-1} \equiv P_{n-1}$, $v \in \{1\} \times \{0, 1\}^{n-1} \equiv 2^{n-1} + P_{n-1}$ and put $u = (0, u_1, \dots, u_{n-1})$, $v = (1, v_1, \dots, v_{n-1})$. Using Theorem 2 and taking into account that the pattern (3) is not possible for matrix M_v^u because $n \geq 2$, we have

$$\begin{aligned} u \triangleright v &\Leftrightarrow M_v^u = \begin{pmatrix} 0 & u_1 & \dots & u_{n-1} \\ 1 & v_1 & \dots & v_{n-1} \end{pmatrix} \text{ has the pattern (4)} \\ &\Leftrightarrow \begin{pmatrix} u_1 \\ v_1 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \text{ and, if } n > 2 \text{ then } v_i = u_i \text{ for all } i = 2, \dots, n-1 \\ &\Leftrightarrow u = \begin{cases} (0, 1) & \text{if } n = 2 \\ (0, 1, u_2, \dots, u_{n-1}) & \text{if } n > 2 \end{cases}, \quad v = \begin{cases} (1, 0) & \text{if } n = 2 \\ (1, 0, u_2, \dots, u_{n-1}) & \text{if } n > 2 \end{cases} \end{aligned}$$