Polarizable Theta-Stable Parabolic Subalgebras and $K_{\mathbb{C}}$ -Saturation in the Non-compact Real Forms of G_2 and F_4

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Abstract. A general method for finding theta-stable parabolic subalgebras was given in 2004. In this paper, we develop *LiE* subroutines to find representatives of conjugacy classes of polarizable theta-stable parabolic subalgebras. Using a theorem of Tauvel, we implement algorithms for computing the $K_{\mathbb{C}}$ -saturation of the nilradical of such parabolic subalgebras. Furthermore, we provide a tool for testing a long standing conjecture that relates the wave front set of a representation of a real or p-adic group to special nilpotent orbits. Incidentally, the conjecture is true for classical groups.

1 Introduction

Let $\mathfrak{g}$ be a real semisimple Lie algebra with adjoint group G and Cartan decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ relative to a Cartan involution θ . We will denote by $\mathfrak{g}_{\mathbb{C}}$ the complexification of $\mathfrak{g}$. Let σ be the conjugation of $\mathfrak{g}_{\mathbb{C}}$ with respect to $\mathfrak{g}$. Then $\mathfrak{g}_{\mathbb{C}} = \mathfrak{k}_{\mathbb{C}} \oplus \mathfrak{p}_{\mathbb{C}}$ where $\mathfrak{k}_{\mathbb{C}}$ and $\mathfrak{p}_{\mathbb{C}}$ are obtained by complexifying $\mathfrak{k}$ and $\mathfrak{p}$ respectively. K will be a maximal compact Lie subgroup of G with Lie algebra $\mathfrak{k}$ and $K_{\mathbb{C}}$ will be the connected subgroup of the adjoint group $G_{\mathbb{C}}$ of $\mathfrak{g}_{\mathbb{C}}$, with Lie algebra $\mathfrak{k}_{\mathbb{C}}$. It is well known that $K_{\mathbb{C}}$ acts on $\mathfrak{p}_{\mathbb{C}}$ and the number of nilpotent orbits of $K_{\mathbb{C}}$ in $\mathfrak{p}_{\mathbb{C}}$ is finite. Furthermore, for a nilpotent $e \in \mathfrak{p}_{\mathbb{C}}$, $K_{\mathbb{C}} \cdot e$ is a connected component of $G_{\mathbb{C}} \cdot e \cap \mathfrak{p}_{\mathbb{C}}$ and $\dim K_{\mathbb{C}} \cdot e = 1/2 \dim G_{\mathbb{C}} \cdot e$.

Let $\mathfrak{q}$ be a parabolic subalgebra of $\mathfrak{g}_{\mathbb{C}}$ with Levi decomposition $\mathfrak{q} = \mathfrak{l} \oplus \mathfrak{u}$. Denote by Q the connected Lie subgroup of $G_{\mathbb{C}}$ with Lie algebra $\mathfrak{q}$. Then there is a unique nilpotent orbit $\mathcal{O}_{\mathfrak{g}_{\mathbb{C}}}$ of $G_{\mathbb{C}}$ on $\mathfrak{g}_{\mathbb{C}}$ meeting $\mathfrak{u}$ in an open dense set. The intersection $\mathcal{O}_{\mathfrak{g}_{\mathbb{C}}} \cap \mathfrak{u}$ consists of a single Q -orbit under the adjoint action of Q on $\mathfrak{u}$. These facts were first proved by Richardson [9].

The Kostant-Sekiguchi correspondence is a bijection between nilpotent orbits of G in $\mathfrak{g}$ and nilpotent orbits of $K_{\mathbb{C}}$ on $\mathfrak{p}_{\mathbb{C}}$. Thus, the correspondence allows us to study certain questions about real nilpotent orbits by looking at nilpotent orbits of $K_{\mathbb{C}}$ on the symmetric space $\mathfrak{p}_{\mathbb{C}}$. Therefore the following is a natural

question: Maintaining the above notations and assuming that $\mathfrak{q}$ is θ -stable, *when does $Q \cap K_{\mathbb{C}}$ admit an open dense orbit on $\mathfrak{u} \cap \mathfrak{p}_{\mathbb{C}}$?* This would be the equivalent of the Richardson theorem for the real case. It turns out that this statement is not true in general. Patrice Tauvel gave a counter example in [12] on page 652 for $\mathfrak{g}_{\mathbb{C}} = D_4$. However, he was able to prove a version of Richardson's theorem for polarizable parabolic subalgebras.

Definition 1. Let $\mathfrak{q} = \mathfrak{q} \cap \mathfrak{k}_{\mathbb{C}} \oplus \mathfrak{q} \cap \mathfrak{p}_{\mathbb{C}}$ be a θ stable parabolic subalgebra of $\mathfrak{g}_{\mathbb{C}}$ with Levi decomposition $\mathfrak{q} = \mathfrak{l} \oplus \mathfrak{u}$. Let e be a nilpotent element of $\mathfrak{u} \cap \mathfrak{p}_{\mathbb{C}}$. We say that $\mathfrak{q}$ is a polarization of $\mathfrak{g}_{\mathbb{C}}$ at e if

$$2\dim \mathfrak{q} = \dim \mathfrak{g}_{\mathbb{C}}^e + \dim \mathfrak{g}_{\mathbb{C}}$$

where $\mathfrak{g}_{\mathbb{C}}^e$ is the centralizer of e in $\mathfrak{g}_{\mathbb{C}}$.

A theta-stable parabolic $\mathfrak{q} = \mathfrak{l} \oplus \mathfrak{u}$ is said to be polarizable if there exists $e \in \mathfrak{u} \cap \mathfrak{p}_{\mathbb{C}}$ such that $\mathfrak{q}$ is a polarization of $\mathfrak{g}_{\mathbb{C}}$ at e .

Observe that since $\dim \mathfrak{g}_{\mathbb{C}}^e = \dim \mathfrak{g}_{\mathbb{C}} - \dim G_{\mathbb{C}} \cdot e$ and $\dim K_{\mathbb{C}} \cdot e = 1/2 \dim G_{\mathbb{C}} \cdot e$ the above formula becomes:

$$\dim \mathfrak{q} = \dim \mathfrak{g}_{\mathbb{C}} - \dim K_{\mathbb{C}} \cdot e.$$

This is the criterion that we will use to test for polarization.

The next proposition could be seen as a version of Richardson's theorem for complex symmetric spaces.

Proposition 1. (*P. Tauvel*). Maintaining the above notations, suppose that there exists z in $\mathfrak{p}_{\mathbb{C}}$ such that $\mathfrak{q}$ is a polarization of $\mathfrak{g}_{\mathbb{C}}$ at z . Then

- i. There exists a unique $K_{\mathbb{C}}$ -nilpotent orbit, $\mathcal{O}_k$, in $\mathfrak{p}_{\mathbb{C}}$ such that $\mathcal{S} = \mathfrak{u} \cap \mathfrak{p}_{\mathbb{C}} \cap \mathcal{O}_k$ is an open and dense in $\mathfrak{u} \cap \mathfrak{p}_{\mathbb{C}}$.
- ii. $\mathcal{S}$ is a $Q \cap K_{\mathbb{C}}$ -orbit.
- iii. if $x \in \mathcal{S}$ then $[x, \mathfrak{k}_{\mathbb{C}} \cap \mathfrak{q}] = \mathfrak{u} \cap \mathfrak{p}_{\mathbb{C}}$, $[x, \mathfrak{p}_{\mathbb{C}} \cap \mathfrak{q}] = \mathfrak{u} \cap \mathfrak{k}_{\mathbb{C}}$ and $\mathfrak{q}$ is a polarization of $\mathfrak{g}_{\mathbb{C}}$ at x .

Proof. See [12] proposition 4.6. □

The orbit $\mathcal{O}_k$ is said to be the $K_{\mathbb{C}}$ -saturation of $\mathfrak{u} \cap \mathfrak{p}_{\mathbb{C}}$. We use the notation $\mathcal{O}_k = K_{\mathbb{C}}(\mathfrak{u} \cap \mathfrak{p}_{\mathbb{C}})$. A study of the proof of the above proposition reveals that if e is in $\mathfrak{u} \cap \mathfrak{p}_{\mathbb{C}}$ and $\mathfrak{q}$ is a polarization of $\mathfrak{g}_{\mathbb{C}}$ at e then $\mathcal{O}_k = K_{\mathbb{C}} \cdot e$. This observation allows us to design an algorithm for computing the $K_{\mathbb{C}}$ -saturation of $\mathfrak{u} \cap \mathfrak{p}_{\mathbb{C}}$ when $\mathfrak{q}$ is polarizable.

2 Algorithms

2.1 Root Decomposition

Let $\mathfrak{h}$ be a fundamental Cartan subalgebra of $\mathfrak{g}$. Then $\mathfrak{h} = \mathfrak{t} \oplus \mathfrak{v}$, where $\mathfrak{t}$ is a Cartan subalgebra of $\mathfrak{k}$ and $\mathfrak{v} \subseteq \mathfrak{p}$. We have the following decompositions in the root spaces of $\mathfrak{g}_{\mathbb{C}}$ generated by the roots of $\mathfrak{h}_{\mathbb{C}}$:

$$\begin{aligned} \mathfrak{k}_{\mathbb{C}} &= \mathfrak{t}_{\mathbb{C}} \oplus \sum_{\alpha \text{ compact imaginary}} \mathbb{C}X_{\alpha} \oplus \sum_{(\alpha, \theta\alpha) \text{ complex pairs}} \mathbb{C}(X_{\alpha} + \theta(X_{\alpha})), \\ \mathfrak{p}_{\mathbb{C}} &= \mathfrak{v}_{\mathbb{C}} \oplus \sum_{\alpha \text{ non compact imaginary}} \mathbb{C}X_{\alpha} \oplus \sum_{(\alpha, \theta\alpha) \text{ complex pairs}} \mathbb{C}(X_{\alpha} - \theta(X_{\alpha})). \end{aligned}$$

Here X_{α} is a nonzero vector of the root space $\mathfrak{g}_{\mathbb{C}}^{\alpha}$. An imaginary root α is compact (noncompact) if its root space $\mathfrak{g}_{\mathbb{C}}^{\alpha}$ lies in $\mathfrak{k}_{\mathbb{C}}$ ($\mathfrak{p}_{\mathbb{C}}$). See ([7]) for more details.

2.2 Algorithms

Algorithm I. The following algorithm generates all the θ -stable parabolic subalgebras associated with a system of simple roots Δ .

Input:

- Δ : set of simple roots
- Υ : set of compact imaginary roots
- Γ : set of non-compact imaginary roots
- Λ : set of complex roots
- Φ^{+} : set of positive roots

Computation:

1. $\mathfrak{h}_{\mathbb{C}} = \bigoplus_{\alpha \in \Delta} \mathbb{C}H_{\alpha}$
2. $\mathfrak{u} \leftarrow \sum_{\alpha \in \Phi^{+}} \mathbb{C}X_{\alpha}$
3. $\mathfrak{q} \leftarrow \mathfrak{h}_{\mathbb{C}} \oplus \mathfrak{u}$ [This is the Borel subalgebra defined by Φ^{+}]
4. **For** each subset S of Δ **Do**
5. **For** each $\alpha \in S$ **Do**
 - If** $\alpha \in \Lambda$ **Then** $S \leftarrow S \cup \{\theta(\alpha)\}$ **EndIf-EndFor**
6. $\Phi \leftarrow \langle S \rangle$ [The root subsystem generated by the positive roots in S]
7. $\mathfrak{l} \leftarrow \mathfrak{h}_{\mathbb{C}} \oplus \sum_{\alpha \in \Phi \cap \Upsilon} \mathbb{C}X_{\alpha} \oplus \sum_{\alpha \in \Phi \cap \Gamma} \mathbb{C}X_{\alpha} \oplus \sum_{\alpha \in \Phi \cap \Lambda} \mathbb{C}(X_{\alpha} + \theta(X_{\alpha}))$
 $\oplus \sum_{\alpha \in \Phi \cap \Lambda} \mathbb{C}(X_{\alpha} - \theta(X_{\alpha}))$
8. $\hat{\Phi} = \Phi^{+} \setminus (\Phi \cap \Phi^{+})$
9. $\mathfrak{u}' \leftarrow \bigoplus_{\mu \in \hat{\Phi}} \mathbb{C}X_{\mu}$
10. $Next(\mathfrak{q}) \leftarrow \mathfrak{l} \oplus \mathfrak{u}'$ [build the next parabolic subalgebra in the list]

EndFor

End.

Algorithm II. The next algorithm determines whether or not a given parabolic is polarizable. If it is polarizable then the $K_{\mathbb{C}}$ -saturation of its nilradical in $\mathfrak{p}_{\mathbb{C}}$ is returned. Although we use the algorithm on exceptional Lie groups it does handle the classical cases also. The dimensions of the nilpotent orbits of the

exceptional complex Lie groups are given in [1] and such data are used in order to check the polarization condition. In [5] we gave algorithms for computing representatives of $K_{\mathbb{C}}$ -nilpotent orbits for exceptional Lie groups. Djoković [2] has computed such representatives also. Each representative of a nilpotent orbit is given as a linear combination of non zero vectors of root spaces.

We write $e = c_{\mu_1}X_{\mu_1} + \dots c_{\mu_m}X_{\mu_m}$ where the μ_i 's are roots to denote a representative. Furthermore, $\mu_1, \dots, \mu_m$ are simple roots of a reductive Lie subalgebra of $\mathfrak{g}_{\mathbb{C}}$. For example in $F_{4(4)}$, $e = \sqrt{2}(X_{\alpha_1+2\alpha_2+3\alpha_3+\alpha_4} + X_{-\alpha_1-\alpha_2-\alpha_3})$ is a representative of orbit #9 in Djoković's tables and these two short roots generate a subalgebra of type A_2 . We write $\langle e \rangle$ to denote the subalgebra of rank m generated by $\mathcal{R}_e = \{\mu_1, \dots, \mu_m\}$.

Input:

- L : List of lists of representatives of each $K_{\mathbb{C}}$ orbits.
- q : A theta-stable parabolic subalgebra $\mathfrak{q} = \mathfrak{l} \oplus \mathfrak{u}$ of $\mathfrak{g}_{\mathbb{C}}$.
- $\mathcal{R}_e$: defined above

Computation:

0. $\mathcal{R} \leftarrow \{\alpha | X_{\alpha} \in \mathfrak{u} \cap \mathfrak{p}_{\mathbb{C}}\}$
1. **For** each $K_{\mathbb{C}}$ -orbit $\mathcal{O}_k$ **Do**
2. **If** $\dim \mathfrak{q} = \dim \mathfrak{g}_{\mathbb{C}} - \dim \mathcal{O}_k$ **Then** [$\mathfrak{q}$ may be polarizable]
3. **For** each representative e of $\mathcal{O}_k$ **Do**
4. $r \leftarrow \text{rank } \langle e \rangle$ [$\langle e \rangle$ is the subalgebra generated by e]
5. **For** each subset $S \subseteq \mathcal{R}$
such that $|S| = r$ and $Type(\langle S \rangle) = Type(\langle e \rangle)$ **Do**
6. **If** S is $K_{\mathbb{C}}$ -conjugate to $\mathcal{R}_e$ **Then**
 q is polarizable and $\mathcal{O}_k = K_{\mathbb{C}}(\mathfrak{u} \cap \mathfrak{p}_{\mathbb{C}})$
7. **Else** go back to step 3.

EndIf-EndFor-EndFor

EndIf-EndFor

End.

Comments. We implement both algorithms in *LiE*. The first algorithm is an extension of the procedure discussed in [6]. A proof of correctness was given also in that paper. A careful examination of the second algorithm should reveal that step 6 is indeed the most costly condition on average because it may require traversing the Weyl group of $K_{\mathbb{C}}$.

The number of $K_{\mathbb{C}}$ -conjugacy classes of systems of simple roots is given by $\frac{|W(G_{\mathbb{C}})|}{|W(K_{\mathbb{C}})|}$ where $W(G_{\mathbb{C}})$ and $W(K_{\mathbb{C}})$ are the Weyl groups of $G_{\mathbb{C}}$ and $K_{\mathbb{C}}$ respectively. One can enumerate all such classes by successive reflections through non compact imaginary roots. Thus for each class the algorithms will give a list of theta-stable polarizable parabolic subalgebras. There could be redundancies as some parabolic subalgebras could lie in different classes of simple roots. However, this problem can be easily resolve by keeping a list of distinct parabolic subalgebras. A more subtle problem is to decide when two theta parabolic subalgebras

are conjugate under $K_{\mathbb{C}}$ which may require traversing $W(K_{\mathbb{C}})$. Keep in mind that in general $W(K_{\mathbb{C}})$, although finite, could be large. This situation is under investigation.

Finally, we shall discuss an important conjecture in Representation theory. Certain complex nilpotent orbits are said to be *special*. For a definition of the term special in this context see [1], [10]. Special nilpotent orbits are the building blocks of the celebrated Springer's correspondence which relate nilpotent orbits to Representation theory. The Wave Front theory (See [4]) suggests that the $K_{\mathbb{C}}$ -saturation is always a connected component of a special nilpotent complex orbit. Moëglin [4] proved that the conjecture is true for classical p-adic groups. Trapa [13] seems to have a proof for classical real Lie groups. The results below support the conjecture for polarizable parabolic subalgebras; all the $K_{\mathbb{C}}$ -saturation obtained come from special complex nilpotent orbits of $G_{\mathbb{C}}$ on $\mathfrak{g}_{\mathbb{C}}$.

3 Results

Because of space constraints we will only give a small subset of our results.

3.1 G_2

$\mathfrak{g} = G_{2(2)}$ and $\dim G_2 = 14$.

$\Delta = \{\alpha_1, \alpha_2\}$, $\mathfrak{h}_{\mathbb{C}} = \mathbb{C}H_{\alpha_1} \oplus \mathbb{C}H_{\alpha_2}$

Table I

1. $\mathfrak{q} = \mathfrak{h}_{\mathbb{C}} \oplus \mathbb{C}X_{\alpha_1+\alpha_2} \oplus \mathbb{C}X_{-\alpha_2} \oplus \mathbb{C}X_{\alpha_1} \oplus \mathbb{C}X_{2\alpha_1+\alpha_2} \oplus \mathbb{C}X_{3\alpha_1+\alpha_2} \oplus \mathbb{C}X_{3\alpha_1+2\alpha_2}$ $\dim \mathfrak{q} = 8$ $\mathcal{O}_k = K_{\mathbb{C}}(\sqrt{6}X_{\alpha_1+\alpha_2} + \sqrt{10}X_{-\alpha_2})$ $\dim \mathcal{O}_k = 6$
2. $\mathfrak{q} = (\mathfrak{h}_{\mathbb{C}} \oplus \mathbb{C}X_{\alpha_1} \oplus \mathbb{C}X_{-\alpha_1}) \oplus \mathbb{C}X_{\alpha_1+\alpha_2} \oplus \mathbb{C}X_{\alpha_2} \oplus \mathbb{C}X_{2\alpha_1+\alpha_2} \oplus \mathbb{C}X_{3\alpha_1+\alpha_2} \oplus \mathbb{C}X_{3\alpha_1+2\alpha_2}$ $\dim \mathfrak{q} = 9$ $\mathcal{O}_k = K_{\mathbb{C}}(X_{\alpha_1+\alpha_2} + X_{3\alpha_1+\alpha_2})$ $\dim \mathcal{O}_k = 5$
3. $\mathfrak{q} = (\mathfrak{h}_{\mathbb{C}} \oplus \mathbb{C}X_{\alpha_1+\alpha_2} \oplus \mathbb{C}X_{-\alpha_1-\alpha_2}) \oplus \mathbb{C}X_{\alpha_1} \oplus \mathbb{C}X_{-\alpha_2} \oplus \mathbb{C}X_{2\alpha_1+\alpha_2} \oplus \mathbb{C}X_{3\alpha_1+\alpha_2} \oplus \mathbb{C}X_{3\alpha_1+2\alpha_2}$ $\dim \mathfrak{q} = 9$ $\mathcal{O}_k = K_{\mathbb{C}}(X_{2\alpha_1+\alpha_2} + X_{-\alpha_2})$ $\dim \mathcal{O}_k = 5$

3.2 $\mathfrak{g}_{\mathbb{C}} = F_4$

We order the roots of F_4 as in the table below and we use the Bourbaki system of simple roots $\Delta = \{\alpha_1, \alpha_2, \alpha_3, \alpha_4\}$. The Cartan involution θ with $+1$ -eigenspace $\mathfrak{k}$ and -1 -eigenspace $\mathfrak{p}$ depends on the real forms. If $\mathfrak{g} = F_{4(4)}$ then $\mathfrak{k}_{\mathbb{C}} = sp_3(\mathbb{C}) \oplus sl_2(\mathbb{C})$ and the vector space $\mathfrak{p}_{\mathbb{C}}$ is the complex span of non-zero root vectors X_{β} where $\beta = c_1\alpha_1 + c_2\alpha_2 + c_3\alpha_3 + c_4\alpha_4$ with $c_1 = \pm 1$. If $\mathfrak{g} = F_{4(-20)}$ then

$\mathfrak{k}_{\mathbb{C}} = \mathfrak{so}_9(\mathbb{C})$ and the vector space $\mathfrak{p}_{\mathbb{C}}$ is the complex span of non-zero root vectors X_{β} where $\beta = c_1\alpha_1 + c_2\alpha_2 + c_3\alpha_3 + c_4\alpha_4$ with $c_4 = \pm 1$.

Positive roots of F_4

1. [1,0,0,0]	9. [0,1,2,0]	17. [1,2,2,1]
2. [0,1,0,0]	10. [0,1,1,1]	18. [1,1,2,2]
3. [0,0,1,0]	11. [1,1,2,0]	19. [1,2,3,1]
4. [0,0,0,1]	12. [1,1,1,1]	20. [1,2,2,2]
5. [1,1,0,0]	13. [0,1,2,1]	21. [1,2,3,2]
6. [0,1,1,0]	14. [1,2,2,0]	22. [1,2,4,2]
7. [0,0,1,1]	15. [1,1,2,1]	23. [1,3,4,2]
8. [1,1,1,0]	16. [0,1,2,2]	24. [2,3,4,2]

In the preceding table each vector indexed by i represents the coefficients of the i^{th} positive root in the Bourbaki base Δ . We use X_i or X_{-i} to denote a non zero root vector in the root spaces of i and $-i$ respectively. The maximum Cartan subalgebra is $\mathfrak{h}_{\mathbb{C}} = \mathbb{C}H_{\alpha_1} \oplus \mathbb{C}H_{\alpha_2} \oplus \mathbb{C}H_{\alpha_3} \oplus \mathbb{C}H_{\alpha_4}$.

$\mathfrak{g} = F_{4(4)}$ and $\dim F_4 = 52$.

Table II

1. $\mathfrak{q} = \mathfrak{h}_{\mathbb{C}} \oplus (2, 3, 4, 6, 7, 9, 10, 13, 16, 24, 14, 15, 17, 18,$ $19, 20, 21, 22, 23, -1, -5, -8, -11, -12)$ $\dim \mathfrak{q} = 28 \quad \mathcal{O}_k = K_{\mathbb{C}}(\sqrt{22}X_{16} + \sqrt{30}X_{15} + \sqrt{42}X_{-11} + 4X_{-12}) \quad \dim \mathcal{O}_k = 24$
2. $\mathfrak{q} = (\mathfrak{h}_{\mathbb{C}} \oplus \pm 3) \oplus (2, 4, 6, 7, 9, 10, 13, 16, 24, 12, 14, 15, 17, 18,$ $19, 20, 21, 22, 23, -1, -5, -8, -11)$ $\dim \mathfrak{q} = 29 \quad \mathcal{O}_k = K_{\mathbb{C}}(\sqrt{12}X_{12} + \sqrt{14}X_{14} + \sqrt{10}X_{-11} + \sqrt{20}X_{-5}) \quad \dim \mathcal{O}_k = 23$
3. $\mathfrak{q} = (\mathfrak{h}_{\mathbb{C}} \oplus \pm 14) \oplus (2, 3, 4, 6, 7, 9, 10, 13, 16, 24, 15, 17, 18, 19, 20,$ $21, 22, 23, -1, -5, -8, -11, -12)$ $\dim \mathfrak{q} = 29 \quad \mathcal{O}_k = K_{\mathbb{C}}(\sqrt{14}X_{14} + \sqrt{18}X_{18} + \sqrt{8}X_{-11} + \sqrt{10}X_{-12}) \quad \dim \mathcal{O}_k = 23$
4. $\mathfrak{q} = (\mathfrak{h}_{\mathbb{C}} \oplus \pm(14, 15)) \oplus (2, 3, 4, 6, 7, 9, 10, 13, 16, 24, 17, 18, 19, 20,$ $21, 22, 23, -1, -5, -8, -11, -12)$ $\dim \mathfrak{q} = 30 \quad \mathcal{O}_k = K_{\mathbb{C}}(\sqrt{5}X_{17} + 3X_{18} + X_{-11} + \sqrt{8}X_{-12}) \quad \dim \mathcal{O}_k = 22$
5. $\mathfrak{q} = (\mathfrak{h}_{\mathbb{C}} \oplus \pm(2, 4)) \oplus (3, 6, 7, 9, 10, 13, 16, 24, 11, 14, 15, 17,$ $18, 19, 20, 21, 22, 23, -1, -5, -8, -11)$ $\dim \mathfrak{q} = 30 \quad \mathcal{O}_k = K_{\mathbb{C}}(\sqrt{5}X_{17} + X_{18} + 3X_{11} + \sqrt{8}X_{-8}) \quad \dim \mathcal{O}_k = 22$

6. $\mathfrak{q} = (\mathfrak{h}_{\mathbb{C}} \oplus \pm(2, 11, 14)) \oplus (4, -12, 15, -8, 17, 18, 3, 20, 6, 7, 24, 10, -5, 19$ $, -1, 21, 9, 13, 16, 22, 23)$ $\dim \mathfrak{q} = 31 \quad \mathcal{O}_k = K_{\mathbb{C}}(\sqrt{5}X_{17} + 3X_{18} + \sqrt{8}X_{-12}) \quad \dim \mathcal{O}_k = 21$
7. $\mathfrak{q} = (\mathfrak{h}_{\mathbb{C}} \oplus \pm(3, 12, 15)) \oplus (2, 4, 6, 7, 9, 10, 13, 16, 24, -3, 14, 17, 18$ $, 19, 20, 21, 22, 23, -1, -5, -8, -11)$ $\dim \mathfrak{q} = 31 \quad \mathcal{O}_k = K_{\mathbb{C}}(\sqrt{10}X_{14} + \sqrt{6}(X_{18} + X_{-8})) \quad \dim \mathcal{O}_k = 21$
8. $\mathfrak{q} = (\mathfrak{h}_{\mathbb{C}} \oplus \pm(3, 4, 7)) \oplus (2, 6, 9, 10, 13, 16, 24, 5, 8, 11,$ $12, 14, 15, 17, 18, 19, 20, 21, 22, 23, -1)$ $\dim \mathfrak{q} = 31 \quad \mathcal{O}_k = K_{\mathbb{C}}(\sqrt{10}X_{-1} + \sqrt{6}(X_{18} + X_8)) \quad \dim \mathcal{O}_k = 21$
9. $\mathfrak{q} = (\mathfrak{h}_{\mathbb{C}} \oplus \pm(3, 12, 14, 15)) \oplus (2, 4, 6, 7, 9, 10, 13, 16, 24, 17, 18$ $, 19, 20, 21, 22, 23, -1, -5, -8, -11)$ $\dim \mathfrak{q} = 32 \quad \mathcal{O}_k = K_{\mathbb{C}}(\sqrt{3}X_{17} + X_{18} + 2X_{-5} + X_{-11}) \quad \dim \mathcal{O}_k = 20$
10. $\mathfrak{q} = (\mathfrak{h}_{\mathbb{C}} \oplus \pm(2, 4, 8, 12)) \oplus (3, 6, 7, 9, 10, 13, 16, 24, 11, 14$ $15, 17, 18, 19, 20, 21, 22, 23, -1, -5)$ $\dim \mathfrak{q} = 32 \quad \mathcal{O}_k = K_{\mathbb{C}}(\sqrt{3}X_{17} + X_{18} + 2X_{-5} + X_{11}) \quad \dim \mathcal{O}_k = 20$
11. $\mathfrak{q} = (\mathfrak{h}_{\mathbb{C}} \oplus \pm(3, 4, 7, 24)) \oplus (2, 6, 9, 10, 13, 16, 14, 17,$ $19, 20, 21, 22, 23, -1, -5, -8, -11, -12, -15, -18)$ $\dim \mathfrak{q} = 32 \quad \mathcal{O}_k = K_{\mathbb{C}}(\sqrt{2}X_{14} + X_{20} + \sqrt{3}X_{22} + X_{-8}) \quad \dim \mathcal{O}_k = 20$
12. $\mathfrak{q} = (\mathfrak{h}_{\mathbb{C}} \oplus \pm(2, 4, 24, 11, 14, 15, 17, 18, 20)) \oplus (3, 6, 7, 10, 19, 21,$ $-8, -12, 9, 13, 16, 22, 23, -1, -5)$ $\dim \mathfrak{q} = 37 \quad \mathcal{O}_k = K_{\mathbb{C}}(\sqrt{2}(X_{19} + X_{-8})) \quad \dim \mathcal{O}_k = 15$
13. $\mathfrak{q} = (\mathfrak{h}_{\mathbb{C}} \oplus \pm(3, 4, 7, 5, 8, 11, 12, 15, 18)) \oplus (2, 6, 9, 10, 13,$ $16, 24, 14, 17, 19, 20, 21, 22, -1, 23)$ $\dim \mathfrak{q} = 37 \quad \mathcal{O}_k = K_{\mathbb{C}}(X_{14} + X_{21} + X_{-1}) \quad \dim \mathcal{O}_k = 15$
14. $\mathfrak{q} = (\mathfrak{h}_{\mathbb{C}} \oplus \pm(2, 3, 6, 9, 24, 12, 15, 17, 19)) \oplus (4, 7, 10, 13,$ $18, 20, 21, 22, 23, -1, -5, -8, -11, -14, 16)$ $\dim \mathfrak{q} = 37 \quad \mathcal{O}_k = K_{\mathbb{C}}(\sqrt{2}(X_{18} + X_{-1})) \quad \dim \mathcal{O}_k = 15$
15. $\mathfrak{q} = (\mathfrak{h}_{\mathbb{C}} \oplus \pm(2, 3, 4, 6, 7, 9, 10, 13, 16)) \oplus (1, 5, 8, 11, 12, 14,$ $15, 17, 18, 19, 20, 21, 22, 23, 24)$ $\dim \mathfrak{q} = 37 \quad \mathcal{O}_k = K_{\mathbb{C}}(\sqrt{2}(X_{23} + X_1)) \quad \dim \mathcal{O}_k = 15$

$$\mathfrak{g} = F_{4(-20)}.$$

Table III

1. $\mathfrak{q} = (\mathfrak{h}_{\mathbb{C}} \oplus \pm(1, 2, 3, 5, 6, 8, 9, 11, 14)) \oplus (16, 18, 20, 21, 22, ,$ $23, 24, -4, -7, -10, -12, -13, -15, -17, -19)$ $\dim \mathfrak{q} = 37 \quad \mathcal{O}_k = K_{\mathbb{C}}(\sqrt{2}(X_{19} + X_4)) \quad \dim \mathcal{O}_k = 15$

4 Conclusion

The results presented in this paper are the output of an implementation of the above algorithms in the language *LiE* [14]. We choose this language because it contains many basic functions that can be used in such investigations. We are also quite familiar with the accuracy and the speed of its underlying algorithms. We have not implemented our algorithms in other languages. It would be an interesting exercise to produce implementations in GAP, Mathematica or Maple. One of the reviewers wanted to know whether or not our program can be used to produce counter-examples similar to the one given by Tauvel. Although this is a clever suggestion, we did not pursue it. However, we think that it is worth pointing out that Richardson's theorem holds for $sl_n(\mathbb{R})$.

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