Extending Maple Capabilities for Solving and Displaying Inequalities

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Abstract. Solving inequalities is a very important topic in computational algebra. This paper presents a new *Maple* package, **IneqGraphics**, for displaying the two-dimensional solution sets of several inequalities of real variables. The package also deals with inequalities involving complex variables by displaying the corresponding solutions on the complex plane. The package provides graphical solutions to many inequalities (such as those involving polynomial, rational, logarithmic, exponential and trigonometric functions) that cannot be solved by using the standard *Maple* commands, which are basically reduced to linear inequalities. In addition, our outputs are consistent with *Maple*'s notation and results in the sense that the package provides a similar output for those cases that can also be solved with *Maple*. To show the performance of the package, several illustrative and interesting examples are described.

1 Introduction

Solving inequalities is a very important topic in Mathematics, with outstanding applications in many problems of theoretical and applied science [1, 2, 4, 6, 7]. However, since there is not a general methodology for solving inequalities, their symbolic computation is still a challenging problem in computational algebra [3]. For example, *Maple* includes some commands for dealing with inequalities, but they are drastically limited to the linear case [5]. This drawback motivated us to create a *Maple* package, **IneqGraphics**, that is introduced in this paper. The package displays the two-dimensional solution sets of several inequalities. Typically, in our package the input might be comprised of any sequence of inequalities (connected by either the **and** or **or** operators) involving nonlinear expressions - such as polynomial, rational, trigonometric, exponential and logarithmic functions - that cannot be solved by using the standard *Maple* commands. The

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package also deals with inequalities involving complex variables by displaying the corresponding solutions on the complex plane.

The structure of this paper is as follows: Section 2 describes the main standard *Maple* tools for solving inequalities. Then, Section 3 introduces the new *Maple* package, **IneqGraphics**, and describes the commands implemented within. The performance of the package is also discussed in this section by using some illustrative examples. Finally, Section 4 closes with the main conclusions of this paper and some further remarks.

2 Standard *Maple* Tools for Solving Inequalities

Maple incorporates only a few commands for solving inequalities. The command `solve(ineq,var)`, solves one inequality in one variable, `var`, for all the cases where the equality points of `ineq` can be properly ordered. While it is not required for the inequality to be linear (for instance, you can solve $e^x > x + 1$) it only accepts one inequality, thus strongly limiting its applicability.

In addition, the *Maple* 10 package `plots` provides the command `inequal` that plots regions defined by logical combinations of linear inequalities in the $\mathbb{R}^2$ space. Although its input allows us to enter more than one equality or inequality, they must necessarily be linear. For example, the package fails to display the solution sets of each of the inequalities $x^2 + y < 1$ and $x^2 + y^2 \leq 16$ on the set $[-5, 5] \times [-5, 5]$:

```
>with(plots):
```

```
Warning, the name changecoords has been redefined
```

```
> inequal(x^2+y<1,x=-5..5,y=-5..5);
```

```
Error, (in inequal) the inequalities should be linear in 'x' and 'y'
```

```
> inequal(x^2+y^2<16,x=-5..5,y=-5..5);
```

```
Error, (in inequal) the inequalities should be linear in 'x' and 'y'
```

This problem has been partially overcome in [8] where the author presents an interesting extension of the command `inequal` to the case of nonlinear expressions. In particular, his new command `inequalities` admits systems of nonlinear inequalities, including the case of expressions given in polar coordinates. However, the package does not solve the problem in all its generality as the inequalities can only be connected by the `and` operator, not all nonlinear expressions can be considered and the polar form fails in presence of loops, multivalued curves and self-intersection scenarios, to mention some of the limitations we noticed.

The package **IneqGraphics**, described in the next section, overcomes many of those limitations and allows the user to solve a large family of real and complex inequality systems and equations and display their two-dimensional solution sets accordingly.

3 The Package IneqGraphics: Some Illustrative Examples

As shown in the previous section, inequalities involving nonlinear functions cannot be visualized by means of the `inequal` command. They can be solved, however, by loading the package developed by the authors:

```
> libname:="C:\\Program files\\Maple 10\\mypackages", libname:
> with(IneqGraphics);
[complexineqplot, ineqplot]
```

which includes the command

```
ineqplot(ineqs, x=xmin..xmax, y=ymin..ymax, ops)
```

for displaying the two-dimensional region of the set of points satisfying the inequalities `ineqs` of real numbers inside the square $[xmin, xmax] \times [ymin, ymax]$. For example, the inequalities indicated in the previous section can be solved as follows:

```
> ineqplot(x^2+y<1,x=-5..5,y=-5..5);
```

See Figure 1(left)

```
> ineqplot(x^2+y^2<=16,x=-5..5,y=-5..5);
```

See Figure 1(right)

Fig. 1. Some examples of inequality solutions on the square $[-5, 5] \times [-5, 5]$: (left) $x^2 + y < 1$; (right) $x^2 + y^2 \leq 16$

Similarly, Fig. 2 displays the solution sets of the inequalities $1 < x^2 + y^2 < 4$ and $\sin(x^2 + y^2) \geq \frac{1}{2}$ on the squares $[-2.5, 2.5] \times [-2.5, 2.5]$ and $[-4, 4] \times [-4, 4]$, respectively. Note that the *Maple* command `inequal` does not provide any solution for these inequalities either.

```
> ineqplot({1<x^2+y^2,x^2+y^2<4},x=-2.5..2.5,y=-2.5..2.5);
```

See Figure 2(left)

```
> ineqplot(sin(x^2+y^2)>=1/2,x=-4..4,y=-4..4,
feasiblepoints=50,linespoints=2500);
```

See Figure 2(right)

Fig. 2. Some examples of inequality solutions: (left) $1 < x^2 + y^2 < 4$ on the square $[-2.5, 2.5] \times [-2.5, 2.5]$; (right) $\sin(x^2 + y^2) \geq \frac{1}{2}$ on the square $[-4, 4] \times [-4, 4]$

As mentioned above, our input can be comprised of several inequalities combined by using the **and** and **or** operators simultaneously. The next two examples are aimed to show this fact:

```
> ineqplot(1<abs(x)+abs(y) and abs(x)+abs(y)<=2 or abs(x+y)<1/2,
x=-3..3,y=-3..3,linespoints=400,feasiblepoints=35);
```

See Figure 3(left)

```
> ineqplot((x+2)^2+y^2<=1 or (x-2)^2+y^2<=1 or ((x+2)^2+y^2>=3
and (x-2)^2+y^2>=3),x=-4..4,y=-2..2,feasiblepoints=30);
```

See Figure 3(right)

The previous command, **ineqplot**, can be generalized to inequalities involving complex numbers. The new command:

```
complexineqplot(ineqs,z=(Rezmin..Rezmax,Imzmin..Imzmax),opts)
```

displays the solution sets of the inequalities **ineqs** of complex numbers inside the square in the complex plane given by $[\text{Rezmin}, \text{Rezmax}] \times [\text{Imzmin}, \text{Imzmax}]$. In this case, the functions appearing within the inequalities need to be real-valued functions of a complex argument, e.g. *Abs*, *Re* and *Im*. For example:

```
> complexineqplot({1<abs(z^2-z+1),abs(z^2-z+1)<4},
z=(-2...3.,-3...3.),feasiblepoints=25);
```

Fig. 3. Some examples of inequality solutions connected with the **and** and **or** operators: (left) $1 < |x| + |y| \leq 2$ or $|x+y| < \frac{1}{2}$ on the square $[-3, 3] \times [-3, 3]$; (right) $(x+2)^2 + y^2 \leq 1$ or $(x-2)^2 + y^2 \leq 1$ or $(x+2)^2 + y^2 \geq 3$ or $(x-2)^2 + y^2 \geq 3$ on the square $[-4, 4] \times [-2, 2]$

See Figure 4(left)

```
> complexineqplot({1<abs(sin(z-2)+z),abs(sin(z-2)+z)<4},
z=(-5...8.,-4...4.),feasiblepoints=35,linespoints=550);
```

See Figure 4(right)

We would like to point out that the output obtained from the package is consistent with *Maple*'s notation and results in the sense that the package provides a similar output for those cases already solved by *Maple*. Figure 5 illustrates this fact: on the left, the solutions provided by the standard *Maple* command

Fig. 4. Some examples of inequality solutions for $z \in \mathbb{C}$: (left) $1 < \|z^2 - z + 1\| < 4$, $Re(z) \in [-2, 3]$, $Im(z) \in [-3, 3]$; (right) $1 < \|\sin(z-2) + z\| < 4$, $Re(z) \in [-5, 8]$, $Im(z) \in [-4, 4]$

Fig. 5. Solution sets for the inequality systems: (top) $x + y \leq 2$ and $x - y \geq 3$; (bottom) $\frac{x}{2} + 7y > 1$ and $y < 1$ and $x < 10$. The solutions have been obtained by applying: (left) the standard *Maple* command `inequal`; (right) the new command `ineqplot`.

`inequal` to the inequalities $x + y \leq 2$ and $x - y \geq 3$ (top) and $\frac{x}{2} + 7y > 1$ and $y < 1$ and $x < 10$ (bottom) on $[-1, 6] \times [-7, 2]$ and $[-15, 15] \times [-1, 1.5]$ respectively are displayed. On the right, the solutions obtained by using the new command `ineqplot` are shown. As the reader can see, the new solution sets match perfectly those obtained from the standard *Maple* commands.

The last examples aim to show how complicated the inequality systems can be: in addition to include polynomial, rational, exponential, logarithmic and trigonometric functions, combinations and even compositions of these (and other) functions can also be considered. In Figures 6 and 7 the solutions sets of the inequality systems:

$$x^2 + y^2 < 5 \text{ and } |\log(x^2 + 1) + y| > 1 \quad (1)$$

$$x^2 - y < 2 \text{ and } e^x + e^y > 1 \text{ and } e^x + e^y < 4 \quad (2)$$

Fig. 6. (left, right) Solution sets for the inequality systems given by (1) and (2) respectively

Fig. 7. (left, right) Solution sets for the inequality systems given by (3) and (4) respectively

$$4 - \left(\sqrt{\left(\sqrt{x^2 + y^2} - 4 \right)^2 + y^2} - 4 \right)^2 > 0 \text{ and } x + y > 0 \quad (3)$$

$$\sqrt{\sin^2(x) + (\sin(y) - 2)^2} \sqrt{(\sin(x) + 1)^2 + \sin^2(y)} \sqrt{(\sin(x) - 1)^2 + \sin^2(y)} \leq \frac{1}{8} \quad (4)$$

on $[-3, 3] \times [-3, 3]$, $[-2, 2] \times [-2.5, 1.6]$, $[-10, 10] \times [-6, 6]$ and $[-6, 6] \times [-7, 4]$ respectively, are displayed.

4 Conclusions and Further Remarks

In this paper a new *Maple* package, *IneqGraphics*, to solve real and complex inequalities and display their associated two-dimensional solution sets is

introduced. The package extends *Maple*'s capabilities by providing graphical solutions to many inequalities (such as those involving polynomial, rational, trigonometric, exponential and logarithmic functions) that cannot be solved by using the standard *Maple* commands and packages. The performance of the package has been illustrated by means of several interesting examples. In all the cases, the output obtained is consistent with *Maple*'s notation and results in the sense that the package provides a similar output for those cases already solved by *Maple*. We are currently working on the extension of these results to the three-dimensional case and the implementation of new routines for other kind of inequalities by using quantifier elimination (QE) and cylindrical algebraic decomposition (CAD) algorithms. The obtained results will be reported elsewhere.

References

1. Beckenbach, E.F., Bellman, R.E.: An Introduction to Inequalities. Random House, New York (1961)
2. Hardy, G.H., Littlewood, J.E., Pólya, G.: Inequalities (Second Edition). Cambridge University Press, Cambridge (1952)
3. Ipanaque, R., Iglesias, A.: A Mathematica Package for Solving and Displaying Inequalities. Lecture Notes in Computer Science, **3039** (2003) 303-310
4. Kazarinoff, N.D.: Geometric Inequalities. Random House, New York (1961)
5. Maple User Manual. Maple Waterloo Inc., Canada (2005)
6. Mitrinovic, D.S.: Analytic Inequalities. Springer-Verlag, New York (1970)
7. Mitrinovic, D.S., Pecaric, J.E., Volenec, V.: Recent Advances in Geometric Inequalities. Kluwer, Dordrecht (1989)
8. Web page: <http://www.math.ubc.ca/~israel/advisor/advisor6/h9r1.htm>