Newton's Method for the Ellipsoidal l_p Norm Facility Location Problem

Yu Xia*

The Institute of Statistical Mathematics,
4-6-7 Minami-Azabu, Minato-ku,
Tokyo 106-8569, Japan
yuxia@ism.ac.jp

Abstract. We give the minisum ellipsoidal l_p norm facility location model. Our model includes both $p \geq 2$ and $p < 2$ cases. We derive the optimality conditions for our model and transform the optimality conditions into a system of equations. We then solve the system by perturbed Newton's method. Some numerical examples are presented.

1 Introduction

We want to locate a new facility. The objective is to minimize the weighted sum of distances from it to some existing facilities. In addition, the new facility is required to be within some region. The distances from the new facility to some resources, such as transportation centers, should not exceed some upperbounds.

Because it usually takes a long time and a large amount of capital to build a facility, after its establishment, the facility is supposed to operate for a long period. Since traffic between the existing locations and the new one occurs on a regular basis during the facility's life circle, it is important to calculate the location of the new facility as accurate as possible. The approximation of road distances by weighted l_p distance measure has been studied in a series of papers ([3, 4], etc.), where it is argued with empirical study that l_p distances weighted by an inflation factor tailored to given regions can better describe the irregularity in the transportation networks such as hills, bends, and are therefore superior to the weighted rectangular and Euclidean norms. Denote the l_p norm (Minkowski distance of order p , $p > 1$) of a d -dimensional Euclidean space $\mathbb{R}^d$ as

$$l_p(\mathbf{z}) \stackrel{\text{def}}{=} \left[\sum_{i=1}^d |z_i|^p \right]^{1/p}.$$

In this paper, we measure distances by the ellipsoidal l_p norm distance:

$$l_{pM}(\mathbf{z}) \stackrel{\text{def}}{=} l_p(M\mathbf{z}),$$

* Research supported by a JSPS postdoctoral fellowship for foreign researchers. I thank suggestions and comments of an anonymous referee.

where M is a linear transformation. It is not hard to see that l_{pM} is a norm when M is nonsingular. Note that the l_{pM} distance is the l_p norm when M is the identity. The Euclidean distance (2-norm distance), the Manhattan distance (l_1 norm distance), and the Chebyshev distance (l_∞ norm distance) are special cases of l_{pM} . The l_{pM} distance also includes the l_{pb} norm (see, for instance [2]):

$$l_{pb}(\mathbf{z}) \stackrel{\text{def}}{=} \left[\sum_{i=1}^n b_i |x_i|^p \right]^{1/p}, \quad b_i > 0 \ (i = 1, \dots, n).$$

The l_{pM} distance measure can better describe the actual transportation networks than the weighted l_p distance or the l_{pb} distance can. In real word, the road connecting two points is not likely a straight line, but a curve, due to existing landmarks or geographic conditions. The curvature of the road can be better described via the rotation of the axes of the coordinates for the facilities and a proper choice of p . In addition, the transportation costs on different segments of a road may be different due to the physical conditions and traffic flows of different segments. For instance, some segments may consist of bridges or uphill. The difference can be modeled by the scaling of the axes. The model is more accurate with different p 's and different transformation matrices for different locations instead of one p and one M for all locations.

The rest of the paper is organized as follows. In §2, we give the ellipsoidal l_p norm facility location model. In §3, we derive optimality conditions for the model and reformulate them into equations. We then give our perturbed Newton's method for the equations. In §4, we present some numerical examples.

2 The Mathematical Model

Let vectors $\mathbf{f}_1, \dots, \mathbf{f}_n$ represent the n existing facilities, or demand points. Let vectors $\mathbf{f}_j$ ($j = n+1, \dots, n+s$) represent the resources. It is possible that some resources $\mathbf{f}_j$ ($n < j \leq n+s$) are just some existing facilities $\mathbf{f}_i$ ($1 \leq i \leq n$). Let vector $\mathbf{x}$ denote the new facility to be located. Below is our model:

$$\min_{\mathbf{x}} \quad \sum_{i=1}^n w_i \|M_i(\mathbf{x} - \mathbf{f}_i)\|_{p_i} \quad (1a)$$

$$\text{s.t.} \quad A\mathbf{x} \leq \mathbf{b} \quad (1b)$$

$$\|M_{n+j}(\mathbf{x} - \mathbf{f}_{n+j})\|_{p_{n+j}} \leq r_j \quad (j = 1, \dots, s). \quad (1c)$$

In the above model, the weight $w_i > 0$ ($i = 1, \dots, n$) is a combination of demands and inflation factor which counts for the condition of transportation equipments, cost of manpower, etc., for the existing facility $\mathbf{f}_i$. And M_i ($i = 1, \dots, n+s$) are d_i -by- d matrices, which are not necessarily nonsingular. A is a given m -by- d matrix. And $\mathbf{b}$ is a given vector of dimension m . r_i are constants. The constraints (1b) give the region in which the new facility can be located. The constraints (1c) specify that the new facility should be within certain distances from some resources.

Model (1) is reduced to the classic l_p norm location problem if all M_i ($i = 1, \dots, n+s$) are the identity matrix and the constraints (1b) and (1c) are empty.

3 The Algorithm

In this section, we derive optimality conditions for (1) and reformulate them into a system of equations. Then we present our global algorithm for the equations.

We first show that (1) is a convex program in order to give necessary and sufficient conditions for optimality.

For any $0 \leq \lambda \leq 1$, and $\mathbf{x}_1, \mathbf{x}_2 \in \mathbb{R}^d$, by Minkowski's inequality, we have

$$\begin{aligned} \sum_{i=1}^n w_i \|M_i [\lambda \mathbf{x}_1 + (1-\lambda) \mathbf{x}_2 - \mathbf{f}_i]\|_{p_i} &= \sum_{i=1}^n w_i \|\lambda M_i(\mathbf{x}_1 - \mathbf{f}_i) \\ &+ (1-\lambda) M_i(\mathbf{x}_2 - \mathbf{f}_i)\|_{p_i} \leq \lambda \sum_{i=1}^n w_i \|M_i(\mathbf{x}_1 - \mathbf{f}_i)\|_{p_i} + (1-\lambda) \sum_{i=1}^n w_i \|M_i(\mathbf{x}_2 - \mathbf{f}_i)\|_{p_i}. \end{aligned}$$

Therefore, the objective function (1a) is convex in $\mathbf{x}$.

For $i = 1, \dots, n+s$, define q_i as the scalar satisfying $\frac{1}{q_i} + \frac{1}{p_i} = 1$.

Next, we consider the Lagrangian dual to (1).

We use z_{il} to denote the l th element of the vector $\mathbf{z}_i$, and M_{il} to denote the l th row of the matrix M_i .

By Hölder's inequality, when $M_i(\mathbf{x} - \mathbf{f}_i) \neq \mathbf{0}$,

$$w_i \|M_i(\mathbf{x} - \mathbf{f}_i)\|_{p_i} \geq (\|\mathbf{z}_i\|_{q_i} \leq w_i) \mathbf{z}_i^T M_i(\mathbf{x} - \mathbf{f}_i),$$

where the equality holds iff

$$\|\mathbf{z}_i\|_{q_i} = w_i, \|M_i(\mathbf{x} - \mathbf{f}_i)\|_{p_i}^{p_i} |z_{il}|^{q_i} = w_i^{q_i} |M_{il}(\mathbf{x} - \mathbf{f}_i)|^{p_i}, \text{sign}(\mathbf{z}_{il}) = \text{sign}[M_{il}(\mathbf{x} - \mathbf{f}_i)].$$

The subdifferential of $\|M_i(\mathbf{x} - \mathbf{f}_i)\|$ at any point where $M_i(\mathbf{x} - \mathbf{f}_i) = \mathbf{0}$ is $\{M_i^T \mathbf{z}_i : \|\mathbf{z}_i\|_{q_i} \leq 1\}$.

Let $\boldsymbol{\eta} \stackrel{\text{def}}{=} (\eta_1, \dots, \eta_s)^T$, $\boldsymbol{\lambda} \stackrel{\text{def}}{=} (\lambda_1, \dots, \lambda_m)^T$ denote the dual variables. Then the dual to (1) is

$$\begin{aligned} \max_{\boldsymbol{\lambda} \geq \mathbf{0}, \boldsymbol{\eta} \geq \mathbf{0}} \min_{\mathbf{x}} \sum_{i=1}^n w_i \|M_i(\mathbf{x} - \mathbf{f}_i)\|_{p_i} + \boldsymbol{\lambda}^T (A\mathbf{x} - \mathbf{b}) + \sum_{j=1}^s \eta_j \left(\|M_{n+j}(\mathbf{x} - \mathbf{f}_{n+j})\|_{p_{n+j}} \right. \\ \left. - r_j \right) = \max_{\boldsymbol{\lambda} \geq \mathbf{0}, \boldsymbol{\eta} \geq \mathbf{0}} \min_{\mathbf{x}} \max_{\substack{\|\mathbf{z}_i\|_{q_i} \leq w_i \\ (i=1, \dots, n) \\ \|\mathbf{z}_j\|_{q_j} \leq \eta_j \\ (j=n+1, \dots, n+s)}} \left[\sum_{i=1}^n \mathbf{z}_i^T M_i(\mathbf{x} - \mathbf{f}_i) + \boldsymbol{\lambda}^T (A\mathbf{x} - \mathbf{b}) \right. \\ \left. + \sum_{j=1}^s \mathbf{z}_{j+n}^T M_{j+n}(\mathbf{x} - \mathbf{f}_{j+n}) - \sum_{j=1}^s \eta_j r_j \right] = \max_{\boldsymbol{\lambda} \geq \mathbf{0}, \boldsymbol{\eta} \geq \mathbf{0}} \min_{\mathbf{x}} \max_{\substack{\|\mathbf{z}_i\|_{q_i} \leq w_i \ (i=1, \dots, n) \\ \|\mathbf{z}_j\|_{q_j} \leq \eta_j \ (j=n+1, \dots, n+s)}} \\ \left[\left(\sum_{i=1}^{n+s} \mathbf{z}_i^T M_i + \boldsymbol{\lambda}^T A \right) \mathbf{x} - \sum_{i=1}^{n+s} \mathbf{z}_i^T M_i \mathbf{f}_i - \boldsymbol{\lambda}^T \mathbf{b} - \sum_{j=1}^s \eta_j r_j \right]. \end{aligned}$$

Let $\mathbf{z}(\mathbf{x})$ denote the optimal solution of $\mathbf{z}$ to the inner maximization problem for a fixed $\mathbf{x}$. The first term in the above expression implies $\sum_{i=1}^{n+s} \mathbf{z}(\mathbf{x})_i^T M_i + \boldsymbol{\lambda}^T A = \mathbf{0}$; otherwise, the value of the second minimization is unbounded.

Therefore, the Lagrangian dual to (1) is:

$$\begin{aligned} \min_{\boldsymbol{\lambda}, \boldsymbol{\eta}, \mathbf{z}_i} \quad & \sum_{i=1}^{n+s} \mathbf{z}_i^T M_i \mathbf{f}_i + \boldsymbol{\lambda}^T \mathbf{b} + \sum_{j=1}^s \eta_j r_j \\ \text{s.t.} \quad & \sum_{i=1}^{n+s} M_i^T \mathbf{z}_i + A^T \boldsymbol{\lambda} = \mathbf{0} \\ & \|\mathbf{z}_i\|_{q_i} \leq w_i \quad (i = 1, \dots, n) \\ & \|\mathbf{z}_j\|_{q_j} \leq \eta_j \quad (j = n+1, \dots, n+s) \\ & \boldsymbol{\lambda} \geq \mathbf{0} \\ & \boldsymbol{\eta} \geq \mathbf{0} . \end{aligned} \quad (2)$$

The dual of the linearly constrained l_p norm facility location problem has been studied in [5]; however, our model includes the resource constraints (1c).

Assume there is a strict interior solution, i.e., $\exists \tilde{\mathbf{x}} \in \mathbb{R}^d$ such that $A\tilde{\mathbf{x}} \leq \mathbf{b}$ and $\|M_j(\mathbf{x} - \mathbf{f}_j)\|_{p_j} < r_j$ ($j = n+1, \dots, n+s$). Then there is no duality gap between (2) and (1) (see for instance [6]), since the objective and the constraints are convex. In the course of deriving (2), we also obtain that at an optimum, the variables $\mathbf{x}, \mathbf{z}, \boldsymbol{\lambda}, \boldsymbol{\eta}$ satisfy the following conditions:

$$\begin{aligned} & \sum_{i=1}^{n+s} M_i^T \mathbf{z}_i + A^T \boldsymbol{\lambda} = \mathbf{0} , \\ & \eta_{j-n} \left(r_{j-n}^{p_j} - \|M_j(\mathbf{x} - \mathbf{f}_j)\|_{p_j}^{p_j} \right) = 0 \quad (j = n+1, \dots, n+s) , \\ & \eta_j \geq 0 \quad (j = 1, \dots, s) , \\ & \|M_{n+j}(\mathbf{x} - \mathbf{f}_{n+j})\|_{p_{n+j}} \leq r_j \quad (j = 1, \dots, s) , \\ & \lambda_i (b_i - A_i \mathbf{x}) = 0 \quad (i = 1, \dots, m) , \\ & \lambda_i \geq 0 \quad (i = 1, \dots, m) , \\ & A_i \mathbf{x} \leq b_i \quad (i = 1, \dots, m) , \\ & \alpha_i |z_{il}|^{q_i} = |M_{il}(\mathbf{x} - \mathbf{f}_i)|^{p_i} \quad (i = 1, \dots, n; l = 1, \dots, d_i) , \\ & \alpha_j |z_{jl}|^{q_j} = |M_{jl}(\mathbf{x} - \mathbf{f}_j)|^{p_j} \quad (j = 1+n, \dots, n+s; l = 1, \dots, d_j) , \\ & \text{sign}(z_{il}) = \text{sign}[M_{il}(\mathbf{x} - \mathbf{f}_i)] \quad (i = 1, \dots, n+s; l = 1, \dots, d_i) , \\ & \alpha_i (w_i - \|\mathbf{z}_i\|_{p_i}) = 0 \quad (i = 1, \dots, n) , \\ & \|\mathbf{z}_i\|_{q_i} \leq w_i \quad (i = 1, \dots, n) , \\ & \alpha_{n+j} (\eta_j - \|\mathbf{z}_{n+j}\|_{p_{n+j}}) = 0 \quad (j = 1, \dots, s) , \\ & \|\mathbf{z}_{n+j}\|_{q_{n+j}} \leq \eta_j \quad (j = 1, \dots, s) , \end{aligned}$$

where we define both $\text{sign}(0) = 1$ and $\text{sign}(0) = -1$ to be true and use A_i to denote the i th row of A .

Usually the new facility is not located on the existing facility or resource sites. Therefore, we assume the linear constraints $A\mathbf{x} \leq \mathbf{b}$ excluding $\mathbf{f}_i$ ($i = 1, \dots, n+s$). Then, we can omit the variables α_i ($i = 1, \dots, n+s$), since

$$\|z_i\|_{q_i} = w_i \quad (i = 1, \dots, n) , \quad \|z_j\|_{q_j} = \eta_j \quad (j = n+1, \dots, n+s) .$$

The optimality conditions are reduced to the following:

$$\begin{aligned}
& \sum_{i=1}^{n+s} M_i^T \mathbf{z}_i + A^T \boldsymbol{\lambda} = \mathbf{0} , \\
& \eta_j \left(r_j^{p_{n+j}} - \|M_{n+j}(\mathbf{x} - \mathbf{f}_{n+j})\|_{p_{n+j}}^{p_{n+j}} \right) = 0 \quad (j = 1, \dots, s) , \\
& \eta_j \geq 0 \quad (j = 1, \dots, s) , \\
& \|M_{n+j}(\mathbf{x} - \mathbf{f}_{n+j})\|_{p_{n+j}} \leq r_j \quad (j = 1, \dots, s) , \\
& \lambda_i (b_i - A_i \mathbf{x}) = 0 \quad (i = 1, \dots, m) , \\
& \lambda_i \geq 0 \quad (i = 1, \dots, m) , \\
& A\mathbf{x} \leq \mathbf{b} , \\
& \left(\sum_{l=1}^{d_i} |M_{il}(\mathbf{x} - \mathbf{f}_i)|^{p_i} \right)^{\frac{1}{q_i}} |z_{il}| = w_i |M_{il}(\mathbf{x} - \mathbf{f}_i)|^{\frac{p_i}{q_i}} \quad (i = 1, \dots, n; l = 1, \dots, d_i) , \\
& \left(\sum_{l=1}^{d_j} |M_{jl}(\mathbf{x} - \mathbf{f}_j)|^{p_j} \right)^{\frac{1}{q_j}} |z_{jl}| = \eta_{j-n} |M_{jl}(\mathbf{x} - \mathbf{f}_j)|^{\frac{p_j}{q_j}} \\
& \quad (j = n+1, \dots, n+s; l = 1, \dots, d_j) , \\
& \text{sign}(z_{il}) = \text{sign}[M_{il}(\mathbf{x} - \mathbf{f}_i)] \quad (i = 1, \dots, n+s; l = 1, \dots, d_i) .
\end{aligned}$$

However, the above system is not easy to solve, because it includes both equalities and inequalities. Next, we use the nonlinear complementarity function $\min$ to reformulate the complementarity conditions in order to transform the above system into a system of equations (3), which can be solved by Newton's method. Note that $\min(a, b) = 0$ is equivalent to $a \geq 0, b \geq 0$, and at least one of a, b is 0. To avoid some nonsmooth points, we distinguish between $p_i \geq 2$ and $p_i < 2$ in our formulation. Notice that $p_i \geq 2 \Rightarrow q_i \leq 2$ and $p_i \leq 2 \Rightarrow q_i \geq 2$.

From $\frac{1}{p_i} + \frac{1}{q_i} = 1$, ($i = 1, \dots, n+s$), we have

$$\frac{p_i}{q_i} = p_i - 1 = \frac{1}{q_i - 1}, \quad \frac{q_i}{p_i} = q_i - 1 = \frac{1}{p_i - 1} .$$

The system of equations (3) is given on page 13.

Let F represent the left-hand-side of (3). Let $\Psi \stackrel{\text{def}}{=} \frac{F'F}{2}$. Any global optimization method that locates a global minimal solution to Ψ solves (1). We use the gradient decent method with perturbed nonmonotone line search for almost everywhere differentiable minimization [7] to find a global minimum of Ψ .

The Algorithm

Denote $\mathbf{u} \stackrel{\text{def}}{=} (\mathbf{x}; \boldsymbol{\lambda}; \boldsymbol{\eta}; \mathbf{z})$. Let $\Delta \mathbf{u}$ represent the Newton's direction to (3).

Initialization. Set constants $s > 0, 0 < \sigma < 1, \beta \in (0, 1), \gamma \in (\beta, 1), nml \geq 1$.

For each $k \geq 0$, assume Ψ is differentiable at $\mathbf{u}^k$. Set $k = 0$.

Do while. $\|F\|_\infty \geq \text{opt}$, $\|\mathbf{u}^{k+1} - \mathbf{u}^k\|_\infty \geq \text{steptol}$, and $k \leq \text{itlimit}$.

1. Find the Newton's direction for (3): $\Delta \mathbf{u}^k$.
2. (a) Set $\alpha^{k,0} = s$, $i = 0$.
- (b) Find the smallest nonnegative integer l for which

$$\Psi(\mathbf{u}^k) - \Psi(\mathbf{u}^k + \beta^l \alpha^{k,i} \Delta \mathbf{u}^k) \geq_{0 \leq j \leq m(k)} -\sigma \beta^l \alpha^{k,i} \nabla \Psi(\mathbf{u}^k)^T \Delta \mathbf{u}^k.$$

where $m(0) = 0$ and $0 \leq m(k) \leq \min[m(k-1) + 1, nml]$.

- (c) If Ψ is nondifferentiable at $(\mathbf{u}^k + \beta^l \alpha^{k,i} \Delta \mathbf{u}^k)$, find $t \in [\gamma, 1)$ so that Ψ is differentiable at $(\mathbf{u}^k + t\beta^l \alpha^{k,i} \Delta \mathbf{u}^k)$, set $\alpha^{k,i+1} = t\beta^l \alpha^{k,i}$, $i + 1 \rightarrow i$, go to step 2b.

Otherwise, set $\alpha^k = \beta^l \alpha^{k,i}$, $\mathbf{u}^{k+1} = \mathbf{u}^k + \alpha^k \Delta \mathbf{u}$, $k + 1 \rightarrow k$.

The Equations $F(\mathbf{u}) = 0$

$$\sum_{i=1}^{n+s} M_i^T \mathbf{z}_i + A^T \boldsymbol{\lambda} = \mathbf{0} \quad (3a)$$

$$\min [\eta_{j-n}, r_{j-n}^{p_j} - \|M_j(\mathbf{x} - \mathbf{f}_j)\|_{p_j}^{p_j}] = 0 \quad (j = n+1, \dots, n+s), \quad (3b)$$

$$\min (\lambda_i, b_i - A_i \mathbf{x}) = 0 \quad (i = 1, \dots, m), \quad (3c)$$

$$\left(\sum_{l=1}^{d_i} |M_{il}(\mathbf{x} - \mathbf{f}_i)|^{p_i} \right)^{\frac{1}{q_i}} z_{il} - w_i |M_{il}(\mathbf{x} - \mathbf{f}_i)|^{\frac{p_i}{q_i}} \text{sign}(M_{il}(\mathbf{x} - \mathbf{f}_i)) = 0 \quad (3d)$$

$$(i \in \{1, \dots, n\}, p_i \geq 2; l = 1, \dots, d_i)$$

$$\left(\sum_{l=1}^{d_i} |M_{il}(\mathbf{x} - \mathbf{f}_i)|^{p_i} \right)^{\frac{1}{p_i}} |z_{il}|^{\frac{q_i}{p_i}} \text{sign}(z_{il}) - w_i^{\frac{q_i}{p_i}} M_{il}(\mathbf{x} - \mathbf{f}_i) = 0 \quad (3e)$$

$$(i \in \{1, \dots, n\}, p_i < 2; l = 1, \dots, d_i)$$

$$\left(\sum_{l=1}^{d_j} |M_{jl}(\mathbf{x} - \mathbf{f}_j)|^{p_j} \right)^{\frac{1}{q_j}} z_{jl} - \eta_{j-n} |M_{jl}(\mathbf{x} - \mathbf{f}_j)|^{\frac{p_j}{q_j}} \text{sign}(M_{jl}(\mathbf{x} - \mathbf{f}_j)) = 0 \quad (3f)$$

$$(j \in \{n+1, \dots, n+s\}, p_j \geq 2; l = 1, \dots, d_j)$$

$$\left(\sum_{l=1}^{d_j} |M_{jl}(\mathbf{x} - \mathbf{f}_j)|^{p_j} \right)^{\frac{1}{p_j}} |z_{jl}|^{\frac{q_j}{p_j}} \text{sign}(z_{jl}) - \eta_{j-n}^{\frac{q_j}{p_j}} M_{jl}(\mathbf{x} - \mathbf{f}_j) = 0 \quad (3g)$$

$$(j \in \{n+1, \dots, n+s\}, p_j < 2; l = 1, \dots, d_j).$$

4 Numerical Experiments

We have implemented the above algorithm in MATLAB. We adopt the suggested parameters in [1]. The machine accuracy of the computer running the code is $\epsilon = 2.2204e - 16$. Our computer program stops either

$\|F\|_\infty < opt = \epsilon^{1/3} = 6.0555e - 5$, or the infinity norm of the Newton's direction is less than $septomol = \epsilon^{2/3}$; or the number of iterations exceeds $itlimit = 100$. We set $s = 1$, $\beta = 0.5$, $\sigma = 1.0e - 4$, $nml = 10$.

Below is a numerical example with 10 existing facilities and 2 resources:

$$\begin{aligned} f_1 &= (0.8175669, 0.9090309, 0.2491902), f_2 = (0.3332802, 0.4928804, 0.0171481), \\ f_3 &= (0.5420494, 0.8212501, 0.3767346), f_4 = (0.5911344, 0.9217661, 0.6447813), \\ f_5 &= (0.2692597, 0.2433543, 0.6320366), f_6 = (0.9949809, 0.8636484, 0.3300618), \\ f_7 &= (0.5093652, 0.9573201, 0.1044826), f_8 = (0.2043801, 0.8580418, 0.2739731), \\ f_9 &= (0.1839796, 0.2708408, 0.7940208), f_{10} = (0.4267598, 0.0453742, 0.1054062), \\ f_{11} &= (0.2745187, 0.5993611, 0.1221155), f_{12} = (0.2181376, 0.9671702, 0.5442553). \end{aligned}$$

The weights for $\mathbf{f}_1, \dots, \mathbf{f}_{10}$ are: $\mathbf{w} = (0.7781196, 0.2842939, 0.0304183, 0.4431445, 0.2179517, 0.5138524, 0.3852747, 0.4875440, 0.4408099, 0.6238188)$.

The p for $\mathbf{f}_1, \dots, \mathbf{f}_{12}$ are $\mathbf{p} = (1.7, 1.5, 2.5, 2.1, 2.5, 2.1, 2.3, 2.5, 2.5, 2.1, 2.9, 1.9)$.

The transformation matrices for the distance measure between the new facility and the existing facilities and the resources are:

$$\begin{aligned} M_1 &= \begin{bmatrix} 0.4043783 & 0.2878919 & 0.5358692 \\ 0.4152851 & 0.5412418 & 0.2719560 \\ 0.3998512 & 0.3457063 & 0.3102344 \end{bmatrix} & M_2 &= \begin{bmatrix} 0.6493642 & 0.4302950 & 0.8370728 \\ 0.9980712 & 0.2561977 & 0.7632986 \\ 0.2469849 & 0.1924696 & 0.8870779 \end{bmatrix} \\ M_3 &= \begin{bmatrix} 0.3580290 & 0.1878973 & 0.5333254 \\ 0.5734145 & 0.0221892 & 0.1189078 \\ 0.8408229 & 0.0575776 & 0.8342093 \end{bmatrix} & M_4 &= \begin{bmatrix} 0.3855365 & 0.5720651 & 0.8868140 \\ 0.0013690 & 0.1720985 & 0.6703435 \\ 0.5097000 & 0.4759156 & 0.7431721 \end{bmatrix} \\ M_5 &= \begin{bmatrix} 0.1805674 & 0.2649339 & 0.6593017 \\ 0.9350963 & 0.9854972 & 0.6605832 \\ 0.9342599 & 0.8032064 & 0.4336484 \end{bmatrix} & M_6 &= \begin{bmatrix} 0.6954798 & 0.2586548 & 0.0355097 \\ 0.7623554 & 0.3843390 & 0.4729695 \\ 0.4053601 & 0.0080771 & 0.2474898 \end{bmatrix} \\ M_7 &= \begin{bmatrix} 0.0132654 & 0.9779720 & 0.0447804 \\ 0.6306576 & 0.7188327 & 0.8326453 \\ 0.5479869 & 0.5564485 & 0.2328634 \end{bmatrix} & M_8 &= \begin{bmatrix} 0.1321284 & 0.6354911 & 0.6550676 \\ 0.4827887 & 0.5106806 & 0.1679761 \\ 0.5706509 & 0.5329994 & 0.1684392 \end{bmatrix} \\ M_9 &= \begin{bmatrix} 0.5942561 & 0.9369910 & 0.3738004 \\ 0.8647487 & 0.7414568 & 0.2842745 \\ 0.1066821 & 0.7481810 & 0.2772064 \end{bmatrix} & M_{10} &= \begin{bmatrix} 0.7579091 & 0.6877772 & 0.4022239 \\ 0.1689570 & 0.2506596 & 0.7048818 \\ 0.1025312 & 0.9542088 & 0.6352403 \end{bmatrix} \\ M_{11} &= \begin{bmatrix} 0.0136439 & 0.2677678 & 0.4533213 \\ 0.8115766 & 0.2669812 & 0.5141040 \\ 0.2874519 & 0.7550024 & 0.1576553 \end{bmatrix} & M_{12} &= \begin{bmatrix} 0.8747555 & 0.7809548 & 0.8933810 \\ 0.8366846 & 0.5110550 & 0.1887794 \\ 0.8529809 & 0.5310465 & 0.3032273 \end{bmatrix} \end{aligned}$$

The data for the feasible region are

$$\begin{aligned} A &= \begin{pmatrix} -0.3725763 & -0.6852143 & -0.2041526 \\ -0.9844071 & -0.2049625 & -0.8364794 \\ 0.4568025 & 0.3087572 & 0.0137238 \end{pmatrix}, \\ \mathbf{b} &= (-0.8759974, -1.2869180, 0.5790744)^T. \end{aligned}$$

The upper bounds for the distances are $\mathbf{r} = (0.6361848, 0.5080279)^T$.

We start from $\mathbf{x} = \mathbf{0}$, $\mathbf{z} = \mathbf{0}$, $\boldsymbol{\lambda} = \mathbf{0}$, $\boldsymbol{\eta} = \mathbf{0}$. Iterations are depicted in the figure “Example 1”. We then perturb each element of $\mathbf{b}$, A , $\mathbf{f}$, randomly by a real number in $(-0.5, 0.5)$ respectively. The starting point for each perturbed problem is the solution to “Example 1”. The iterations are described in the figures: “Perturbation of A”, “Perturbation of b”, and “Perturbation of f”.

References

1. John E. Dennis, Jr. and Robert B. Schnabel. *Numerical methods for unconstrained optimization and nonlinear equations*. Prentice Hall Series in Computational Mathematics. Prentice Hall Inc., Englewood Cliffs, NJ, 1983.
2. J. Fernández, P. Fernández, and B. Pelegrin. Estimating actual distances by norm functions: a comparison between the $l_{k,p,\theta}$ -norm and the $l_{b_1,b_2,\theta}$ -norm and a study about the selection of the data set. *Comput. Oper. Res.*, 29(6):609–623, 2002.
3. R.F. Love and JG Morris. Modelling inter-city road distances by mathematical functions. *Operational Research Quarterly*, 23:61–71, 1972.
4. R.F. Love and JG Morris. Mathematical models of road travel distances. *Management Science*, 25:130–139, 1979.
5. Robert F. Love. The dual of a hyperbolic approximation to the generalized constrained multi-facility location problem with l_p distances. *Management Sci.*, 21(1):22–33, 1974/75.
6. R. Tyrrell Rockafellar. *Convex analysis*. Princeton University Press, Princeton, N.J., 1970. Princeton Mathematical Series, No. 28.
7. Yu Xia. An algorithm for perturbed second-order cone programs. Technical Report AdvOI-Report No. 2004/17, McMaster University, 2004.