Optimal Constant Weight Codes

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Abstract. A new class of binary constant weight codes is presented. We establish new lower bound and exact values on $A(n, 2k, k + 1)$, in particular, $A(30, 12, 7) = 9$, $A(48, 16, 9) = 11$, $A(51, 16, 9) = 12$, $A(58, 18, 10) = 12$.

An (n, d, w) constant weight binary code is a code of length n , code distance d in which all code words have the same number of “ones”. The number of “ones” is w . We will denote the maximal possible size (a number of code words) of an (n, d, w) constant weight code by $A(n, d, w)$.

The most important and interesting problem is finding the largest possible size $A(n, d, w)$ of an (n, d, w) constant weight code (hereafter called optimal code). The results of code searching used to be put in tables of optimal codes. The first lower bound appeared in 1977 in the book of MacWilliams and Sloane ([1], pp.684-691). A table of binary constant weight codes of length $n \leq 28$ with explicit constructions for most of the 600 codes was presented in the encyclopedic work of E E. Brouwer, J. B. Shearer, N. J. A. Sloane [2]. Today Neil J. A. Sloane presents his table of constant weight codes [3] online and performs continual updates.

Let us consider an $(n, 2k, k + 1)$ constant weight code with length n , code distance $2k$ and weight of all $k + 1$. Johnson’s upper bound (se, [2], p. 525) in this case is

$$A(n, 2k, k + 1) \leq \frac{kn}{(k + 1)^2 - n} \quad (\text{if denominator is positive}) \quad (1)$$

Theorem. $A(n = \frac{k^2 + 3k + 2}{2}, 2k, k + 1) = k + 2$ holds for all k .

The research was support by The Royal Swedish academy of Sciences.

Proof. The code is constructed from representation $\frac{k^2 + 3k + 2}{2}$ element set

$M = \{1, 2, 3, \dots, \frac{k^2 + 3k + 2}{2}\}$ as union of $k + 2$ subsets M_i with $k + 1$ elements

in every subset and $|M_i \cap M_j| = 1$ for $i \neq j$. We now give explicit construction of the code.

For $k = 2$ and $M = \{1, 2, 3, 4, 5, 6\}$ we can find 4 subsets $M_1 = \{1, 2, 4\}, M_2 = \{1, 3, 5\}, M_3 = \{2, 3, 6\}, M_4 = \{4, 5, 6\}$ where all vectors of constant weight code $(6, 4, 3)$ are:

$$\begin{aligned} v_1^{k=2} &= (1, 1, 0, 1, 0, 0) \\ v_2^{k=2} &= (1, 0, 1, 0, 1, 0) \\ v_3^{k=2} &= (0, 1, 1, 0, 0, 1) \\ v_4^{k=2} &= (0, 0, 0, 1, 1, 1) \end{aligned} \quad (2)$$

For $k = 3$ we obtain all code words of the code $(10, 6, 4)$ based on code words

$$v_1^{k=2}, v_2^{k=2}, v_3^{k=2}, v_4^{k=2}$$

$$\begin{aligned} v_1^{k=3} &= (v_1^{k=2}, 1, 0, 0, 0) = (1, 1, 0, 1, 0, 0, 1, 0, 0, 0) \\ v_2^{k=3} &= (v_2^{k=2}, 0, 1, 0, 0) = (1, 0, 1, 0, 1, 0, 0, 1, 0, 0) \\ v_3^{k=3} &= (v_3^{k=2}, 0, 0, 1, 0) = (0, 1, 1, 0, 0, 1, 0, 0, 1, 0) \\ v_4^{k=3} &= (v_4^{k=2}, 0, 0, 0, 1) = (0, 0, 0, 1, 1, 1, 0, 0, 0, 1) \\ v_5^{k=3} &= (0, 0, 0, 0, 0, 0, 1, 1, 1, 1) \end{aligned} \quad (3)$$

For $k = j$, we obtain all code words using the same construction the upper bound follows from (1)

Corollary 1. $A(n = \frac{k^2 l^2}{2} + \frac{kl^2}{2} + k + 1, 2lk, lk + 1) \geq kl + l$ for all k and l .

Proof. Using the same method as in the theorem we can construct the

$n = \frac{k^2 l^2}{2} + \frac{kl^2}{2} + k + l$ element set as union of $kl + l$ subsets with $kl + 1$ elements in every subset. The second Johnson bound (see, [5], p. 525) yields the upper bound as $A(n, d, w) \leq kl + l$ for $l = 3, 4, 5$.

Example 1. For $l = 3$ and $k = 2$, we have a nontrivial optimal code with parameters $(30, 12, 7)$. The optimal code consists of 9 vectors of weight 7. In this case the set $M = \{1, 2, \dots, 30\}$. We have following subsets and code words respectively.

$$\begin{aligned}
M_1 &= \{1,2,3,4,5,6,19\} \leftrightarrow v_1 = (1,1,1,1,1,0,0,0,0,0,0,0,0,0,0,1,0,0,0,0,0,0,0,0,0) \\
M_2 &= \{7,8,9,10,11,12,19\} \leftrightarrow v_2 = (0,0,0,0,0,0,1,1,1,1,0,0,0,0,0,1,0,0,0,0,0,0,0,0,0) \\
M_3 &= \{13,14,15,16,17,18,19\} \leftrightarrow v_3 = (0,0,0,0,0,0,0,0,0,0,0,0,1,1,1,1,1,0,0,0,0,0,0,0,0) \\
M_4 &= \{1,7,13,20,21,22,29\} \leftrightarrow v_4 = (1,0,0,0,0,0,1,0,0,0,0,0,1,0,0,0,0,0,1,1,0,0,0,0,0,1,0) \\
M_5 &= \{2,8,14,23,24,25,29\} \leftrightarrow v_5 = (0,1,0,0,0,0,0,1,0,0,0,0,0,1,0,0,0,0,0,0,0,0,1,1,0,0,0,1,0) \\
M_6 &= \{3,9,15,26,27,28,29\} \leftrightarrow v_6 = (0,0,1,0,0,0,0,0,1,0,0,0,0,0,1,0,0,0,0,0,0,0,0,0,0,1,1,1,0) \\
M_7 &= \{4,10,16,20,23,26,30\} \leftrightarrow v_7 = (0,0,0,1,0,0,0,0,0,1,0,0,0,0,0,1,0,0,0,1,0,0,1,0,0,1,0,0,0,1) \\
M_8 &= \{5,11,17,21,24,27,30\} \leftrightarrow v_8 = (0,0,0,0,1,0,0,0,0,0,1,0,0,0,0,0,1,0,0,0,1,0,0,1,0,0,1,0,0,1) \\
M_9 &= \{6,12,18,22,25,28,30\} \leftrightarrow v_9 = (0,0,0,0,0,1,0,0,0,0,0,1,0,0,0,0,0,1,0,0,0,1,0,0,1,0,0,1,0,1)
\end{aligned} \tag{4}$$

We can find that $A(30,12,7) = 9$, $A(58,18,10) = 12$ and $A(51,16,9) = 12$.

Corollary 2.

$$A(n = \frac{k^2 l^2}{2} + \frac{3kl^2}{2} + l^2 - kl - l + k + 1, 2l(k+1) - 2, l(k+1)) \geq kl + 2l - 1$$

for all k and l

Example 2. In the case $l = 3$ we obtain optimal codes and particular for $k = 2$ we have an optimal code $A(48,16,9) = 11$. The constant weight code with parameters $(48,16,9)$ we obtain using construction in Example 1. The set $M = \{1,2,...,48\}$ and subsets TM_i

$$\begin{aligned}
TM_i &= \{M_i, 30+2i-1, 30+2i\}, \text{ for } i=1,...,9 \\
TM_{10} &= \{31,33,35,37,39,41,43,45,47\} \\
TM_{11} &= \{32,34,36,38,40,42,44,46,48\}
\end{aligned} \tag{5}$$

M_i is subset from Example 1.

All code words and parameters of the code can be found using MATHEMATICA

In[3] := TM=

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{{1, 2, 3, 4, 5, 6, 19, 31, 32}, {7, 8, 9, 10, 11, 12, 19, 33, 34},
{13, 14, 15, 16, 17, 18, 19, 35, 36}, {1, 7, 13, 20, 21, 22, 29, 37, 38},
{2, 8, 14, 23, 24, 25, 29, 39, 40}, {3, 9, 15, 26, 27, 28, 29, 41, 42},
{4, 10, 16, 20, 23, 26, 30, 43, 44}, {5, 11, 17, 21, 24, 27, 30, 45, 46},
{6, 12, 18, 22, 25, 28, 30, 47, 48}, {31, 33, 35, 37, 39, 41, 43, 45, 47},
{32, 34, 36, 38, 40, 42, 44, 46, 48}}

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MM={};Do[If[Count[K[[i]],1]==9,AppendTo[MM,K[[i]]]],{i,
1,Length[K]}]
Union[Flatten[Table[Count[Mod[MM[[i]]+MM[[j]],2],1],{i,
1,Length[MM]},{j,i+1,Length[MM]}],1]]
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(* We calculate all code words and code distance*)
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Out[3]=

[illegible]

16

Matrix in out [3] gives all code vectors for optimal constant weight code with parameter (48, 16, 9) and we calculated what code distance between all pair code vectors is 16.

Conclusion

We establish new lower bound and exact values on $A(n, 2k, k + 1)$, in particular, $A(30, 12, 7) = 9$, $A(48, 16, 9) = 11$, $A(51, 16, 9) = 12$, $A(58, 18, 10) = 12$.

References

1. MacWilliams, F. J., and Sloane, N. J. A. (1977) The Theory of Error - Correcting Codes. North - Holland, Amsterdam.
2. E.E. Brouwer, J.B. Shearer, N.J.A. Sloane - A new table of Constant weight codes. IEEE Transactions of information theory, v 36, No 6 (1990)
3. Neil J. A. Sloane: Home Page <http://www.research.att.com/~njas/codes/Andw/>, Table of Constant Weight Binary Codes, online version