The Generic McMillan Degree: A New Method Using Integer Matrices

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Abstract. In this paper the problem of computing the generic McMillan degree of a *Structured Transfer Function (STF)* rational matrix is considered. The problem of determining the generic McMillan degree is tackled using genericity arguments and an optimisation procedure based on path properties of integer matrices is developed. This novel approach exploits the structure of integer matrices and leads to an efficient new algorithm for the computation of the generic value of the McMillan degree.

1 Introduction

The study of system properties based on ill-defined models is of great interest in the context of early design of large scale systems [5]. Our interest here is the study of the McMillan degree on special types of transfer functions referred to as *Structured Transfer Functions (STF)*. These are large dimension transfer functions with certain elements fixed to zero, some elements being constant, and other elements expressing the dominant dynamics of the system, which have been identified by some preliminary modelling effort. The McMillan degree indicates the complexity of the system, and can be calculated from the orders of the denominators in the Smith-McMillan form [4], [2]; this method is impractical for large dimension and uncertain models and the method suggested in [2], defining the pole polynomial as the least common multiple of the minors of all orders is used. Given an STF matrix $H(s)$, for example,

$$H(s) = \begin{bmatrix} A_1 A_2 A_3 & A_1^2 A_2 & A_2 \\ 0 & A_2^2 A_1 & A_2^3 \\ A_3 A_2 & A_1 & A_1 A_2 A_3 \end{bmatrix}, \quad (1)$$

where the A_i represent constant terms, first or second order dynamics. We can use partial fraction expansion and decompose $H(s)$ in the following manner:

$$H(s) = \underbrace{\begin{bmatrix} A_1 & A_1^2 & 0 \\ 0 & A_1 & 0 \\ 0 & A_1 & 0 \end{bmatrix}}_{H_1(s)} + \underbrace{\begin{bmatrix} A_2 & A_2 & A_2 \\ 0 & 0 & A_2^3 \\ A_2 & 0 & A_2 \end{bmatrix}}_{H_2(s)} + \underbrace{\begin{bmatrix} A_3 & 0 & 0 \\ 0 & A_2^3 & 0 \\ A_3 & 0 & A_3 \end{bmatrix}}_{H_3(s)} \quad (2)$$

where the matrices $H_i(s)$, $i = 1, 2, 3$, are called *simple* STF matrices. The generic McMillan degree, denoted by $\delta(H)$, can be found by the sum [5]:

$$\delta(H) = \delta(H_1) + \delta(H_2) + \delta(H_3(s)) . \quad (3)$$

It has been shown in [5], that for simple matrices $H_i(s)$ we can define a map

$$H_i(s) \in \mathbb{R}^{m \times p}(s) \rightarrow I_i \in \mathbb{N}^{m \times p} ,$$

such that the entries of the integer matrix I_i , correspond to the orders of the entries in $H_i(s)$, and thus reducing the problem of determining the generic McMillan degree, into an assignment problem of integer matrices [1].

2 Algorithm for Determining the Weight of an Integer Matrix

Given a matrix $A = [a_{ij}] \in \mathbb{N}^{m \times p}$, $m \geq p$, we define:

Definition 1. A k -length independent path $\{a_{i_1 j_1}, a_{i_2 j_2}, \dots, a_{i_k j_k}\}$, on the matrix A , is a set of elements from the matrix such that there is no common index in the sets $\{i_1, i_2, \dots, i_k\}$ and $\{j_1, j_2, \dots, j_k\}$. The weight of a path is defined as the sum: $\gamma = a_{i_1 j_1} + a_{i_2 j_2} + \dots + a_{i_k j_k}$.

The *maximal weight* of all the independent paths of a matrix is denoted by $\gamma(A)$, and it is simply referred to as the weight of the matrix. Concerning the relationship between the generic McMillan degree of a rational matrix and the weight of the paths of an integer matrix, we have the following result [5]:

Theorem 1. The generic McMillan degree of the simple structured matrix $H_i(s)$ is equal to the maximal weight of its corresponding integer matrix I_i :

$$\delta_{gm}(H_i) = \gamma(I_i) \quad (4)$$

A matrix is called *column irreducible*, if the *highest coefficient matrix* (the Boolean matrix indicating the location of elements in a column with maximal value) has full structural rank, otherwise the matrix is called *column reducible*. The algorithm for the computation of the maximal weight uses the following result, based on the properties of Boolean matrices.

Lemma 1. [4] If the highest coefficient matrix has full rank, then A is column irreducible and the maximal weight is given by the sum of the maximum element of each column. Otherwise A is column reducible and the sum of the maximum element of each column does not yield its maximal weight.

The objective here is to provide a combination of p elements, which form an *independent path* (i.e. they belong in a different row and a different column of A), with the maximum *weight* $\gamma(A)$, for any irreducible or reducible matrix A .

Assume that the columns of A are ordered according to descending weight and form the table of the column contents with a resulting matrix $\mathcal{C}(A) \equiv C$, i.e.

$$\begin{array}{cccc}
 \text{col}(1) & \text{col}(2) & \cdots & \text{col}(p) \\
 \hline
 \delta_{11} & \delta_{12} & \cdots & \delta_{1p} \\
 \delta_{21} & \delta_{22} & \cdots & \delta_{2p} \\
 \vdots & \vdots & \ddots & \vdots \\
 \delta_{\sigma_1(i)1} & \delta_{\sigma_2(i)2} & \cdots & \delta_{\sigma_p(i)p}
 \end{array} \quad (5)$$

where $\delta_{1j} \geq \delta_{2j} \geq \cdots \geq \delta_{\sigma_j(i)j}, \forall j = 1, \dots, p$ and the $\sigma_j(i), j = 1, \dots, p$ are arrangement functions, one for each column, which take the values $\{1, \dots, m\}$. We represent (5) by a matrix $B \in \mathbb{N}^{m \times p}$, and will refer to it as the *arrangement matrix* of A . We also define $C \in \mathbb{N}^{m \times p}$, as the *row index matrix* of A , given by:

$$C = [\gamma_{ij}] = [\sigma_j(i)j], \quad i = 1, 2, \dots, m \quad j = 1, 2, \dots, p \quad (6)$$

where $\sigma_j(i)$ is the row coordinate of the i -th maximal element of the j -th column of A . Let $D \in \mathbb{Z}^{m \times p}$ be the *loss allocation matrix* of A , given by:

$$D = [d_{ij}] = [b_{1j} - b_{ij}], \quad i = 1, 2, \dots, m \quad j = 1, 2, \dots, p \quad (7)$$

i.e. the amount d_{ij} is the difference of the element $a_{\gamma_{ij},j}$ from the maximum, which is $a_{\gamma_{1j},j}$. We will refer to an element of D as a *loss*. The search for the independent path with the maximum weight involves the following steps:

Algorithm

Preliminary Step: If the highest coefficient matrix has full rank, $\gamma(A) = \delta(A) = \sum_{i=1}^p \delta_{1i}$ and the independent path is given by the highest coefficient matrix. If A is reducible, then $\gamma(A) < \delta(A)$ and the procedure to determine $\gamma(A)$ continues as follows:

Searching Routine: In D , there will be a minimum element (*minimum loss*), denoted by m_1 . Using the row index and loss allocation matrices, we can find all elements in A , for which $a_{\gamma_{1j},j} - a_{\gamma_{ij},j} = m_1$. We substitute those elements with the maximum, i.e. set $a_{\gamma_{1j},j} = a_{\gamma_{ij},j}$ for each column, and if the new highest coefficient matrix, which is a Boolean matrix, has full rank, then there definitely exists at least one independent path, which contains exactly one non-zero element from each column. We choose the independent path with the minimum loss, and denote this loss by m_{l_1} . We now have a lower bound:

$$\sigma(A) = \delta(A) - m_{l_1} \quad (8)$$

Let m_i denote a minimum element of D , after the i -th step; we have:

- (i) If $m_{l_i} \geq \delta(A) - \sigma(A) = m_{l_1}$, then all independent paths, give a greater loss, we search in D for m_{i+1} :
- (ii) If $m_{l_i} < m_{l_1}$, then m_{l_i} becomes the new lower bound and the search continues for m_{i+1} similarly.

The search stops when we reach a loss $m_{i+1} \geq m_i$, and the weight of A , i.e. the generic McMillan degree of the corresponding rational matrix, will be

$$\gamma(A) = \delta(A) - m_i . \quad (9)$$

3 Conclusions

The computation of the McMillan degree of a structured transfer function matrix has been considered using properties of column irreducibility of integer matrices, which are similar to those determining the relationship between complexity and degree of polynomial matrices [4]. The proposed algorithm avoids the general searching methods and determines the optimal solution in a small number of steps. An alternative approach for the computation of the generic McMillan degree may be developed by using standard results from a class of integer optimisation problems, more known as *optimal assignment*.

The optimal assignment problem frequently appears in Operational Research as the problem of having to assign n workers to n jobs, where we assume that the performance of the i th person for the j th job can in some sense be determined [1], [7]. Amongst the most popular methods for solving optimal assignment problems, are the *Hungarian*, and the *Bradford* [7]. Such methods provide an alternative way of looking at the problem and their performance to the case of large dimension systems has to be evaluated. The comparison of this new, algebraically based algorithm to the standard methodologies, is under investigation. There are strong indications that exploring the structural criteria based on the reduceness properties, variants of the optimal assignment algorithms may be developed which explore the sparse structure of the matrices and thus lead to algorithms with reduced complexity.

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