Tree-Based Key Distribution Patterns

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Abstract. We revisit a key agreement scheme presented by Leighton and Micali [11], generalize the scheme, and present a new framework of tree-based key distribution pattern (TKDP). We presents a method of constructing TKDPs from cover-free families. We show the existence of TKDPs by probabilistic method. We can reduce the upper bounds on the minimum number of rows of $(t, w, \mathcal{T})$ -TKDPs, which are obtained from probabilistic methods, asymptotically by a factor of w as compared to Leighton and Micali's schemes by choosing optimal trees $\mathcal{T}$ instead of chains.

Keywords: key predistribution, cover-free family.

1 Introduction

In our model, the network consists of a *trusted authority* (TA) and a set of *users* $\mathcal{U} = \{U_1, \dots, U_n\}$. Let $2^{\mathcal{U}}$ denote the collection of all subsets of users. $\mathcal{P} \subseteq 2^{\mathcal{U}}$ and $\mathcal{F} \subseteq 2^{\mathcal{U}}$ will denote the collection of *privileged subsets* and the collection of *forbidden subsets*, respectively. $\Gamma = (\mathcal{P}, \mathcal{F})$ is called an *access structure*. Roughly speaking, a *key predistribution scheme* (KPS) with an access structure Γ is a method for the TA to distribute secret shares to each user in the network, so that any user in a privileged subset P can easily compute their group key K_P , while any coalition $F \in \mathcal{F}$ disjoint from P can compute K_P only with negligible probability. In this paper, we will consider an access structure $\Gamma_{(t,w)} = (\mathcal{P}, \mathcal{F})$ such that $\mathcal{P} = \{P \subseteq \mathcal{U} : |P| = t\}$ and $\mathcal{F} = \{F \subseteq \mathcal{U} : |F| = w\}$. Thus any t users can compute a common key while any coalition of at most w users can obtain no information on the key. (Such KPSs are called (t, w) -KPSs.) One method of constructing KPSs uses *key distribution patterns* (KDPs), due to Mitchell and Piper [13]. We will extend the study of KDPs by allowing hashed shares of information in the generation of secret keys.

1.1 Motivation: Leighton and Micali Scheme

This work is motivated by a key agreement scheme proposed by Leighton and Micali [11]. Here we briefly describe their scheme (with slight modification):

1. The TA publishes a (one-way) hash function

$$h : \{0, 1\}^l \longrightarrow \{0, 1\}^l,$$

where l is a secret key length used for symmetric encryption.

2. For each user U_j , a public sequence $A_j \in \{0, \dots, L-1\}^b$ is chosen in a uniformly random way, where b and L are positive integers. The public sequences act as public identities of users, and they comprise a $b \times n$ matrix $M = (A_1^T | \dots | A_n^T) = (\alpha_{i,j})$.
3. The TA chooses random seeds $s_i = s(i, 0) \in \{0, 1\}^l$ for $i = 1, \dots, b$ and computes hashed shares $s(i, \alpha) = h^\alpha(s_i)$ for $\alpha = 1, \dots, L-1$. ($h^\alpha(s)$ indicates applying the function h iteratively α times on the input s .)
4. User U_j receives a set of secret shares $s(i, \alpha_{i,j})$ for $i = 1, \dots, b$. Therefore, b is the number of secret shares assigned to each user.
5. A pairwise key between U_{j_1} and U_{j_2} is defined to be

$$K_{j_1, j_2} = s(1, \delta_1) + \dots + s(b, \delta_b),$$

where $\delta_i = \max(\alpha_{i, j_1}, \alpha_{i, j_2})$. Addition is defined in $(\mathbb{Z}_2^l, +)$.

We can easily derive a condition that a coalition of w users, $U_{j'_1}, \dots, U_{j'_w}$ obtain a hashed key between innocent users U_{j_1} and U_{j_2} as follows:

$$\min(\alpha_{i, j'_1}, \dots, \alpha_{i, j'_w}) \leq \max(\alpha_{i, j_1}, \alpha_{i, j_2}),$$

for all $i = 1, \dots, b$. The authors of [11] computed the probability that a set of randomly chosen sequences protects pairwise keys against adversarial coalitions of size at most w . They claim that the probability becomes large enough when w is about $(b/e \ln n)^{1/3}$.

Example 1.1. Suppose public sequences $A_1 = (5, 2, 1)$, $A_2 = (2, 3, 1)$ and $A_3 = (1, 3, 3)$ are assigned to users U_1 , U_2 and U_3 , respectively. Then U_1 , U_2 and U_3 receive key sets $\{h^5(s_1), h^2(s_2), h(s_3)\}$, $\{h^2(s_1), h^3(s_2), h(s_3)\}$ and $\{h(s_1), h^3(s_2), h^3(s_3)\}$, respectively. U_1 and U_2 can communicate with each other with key $K_{1,2} = h^5(s_1) + h^3(s_2) + h(s_3)$. An adversary U_3 can compute $h^5(s_1)$ and $h^3(s_2)$, but not $h(s_3)$; therefore the key $K_{1,2}$ is secure against U_3 .

1.2 Extension of Leighton and Micali Scheme

We extend Leighton and Micali's scheme in three directions.

Pairwise Key to Group Key. Any t users $U_{j_1}, \dots, U_{j_t}$ can define their group key to be

$$K_{j_1, \dots, j_t} = s(1, \delta_1) + \dots + s(b, \delta_b),$$

where $\delta_i = \max(\alpha_{i, j_1}, \dots, \alpha_{i, j_t})$. In order for a coalition of w users, $U_{j'_1}, \dots, U_{j'_w}$, to obtain the group key, the following should occur:

$$\min(\alpha_{i, j'_1}, \dots, \alpha_{i, j'_w}) \leq \max(\alpha_{i, j_1}, \dots, \alpha_{i, j_t}),$$

for all $i = 1, \dots, b$.

Linear Ordering to Tree Ordering. Suppose that a matrix $M = (\alpha_{i,j})$ of public sequences is determined. For each row of the matrix, a "seed" secret

Fig. 1. Orderings of symbols

share is generated and each entry represents the hash depth applied to the seed. Therefore, we observe a hierarchy or an ordering between entries (symbols) in the sense that one information is enough to compute the other, but not conversely. For two symbols α_1 and α_2 , we define $\alpha_1 \leq \alpha_2$ if the information(= $h^{\alpha_2}(s)$) associated with α_2 is easily computed from the information(= $h^{\alpha_1}(s)$) associated with α_1 . We can represent the ordering of symbols for each row as a chain structure (Fig. 1(a)), which coincides with the natural linear ordering of integers.

However we can also define nonlinear orderings based on rooted trees. For example, Fig. 1(b) shows an ordering of symbols based on a balanced binary tree of depth 2. Seven hashed shares are derived from a seed for each row, using (public) symbols of the tree in the hash computation. (The symbols are interpreted as binary sequences of a fixed length in the computation.)

Probabilistic Construction to Deterministic Construction. We can construct a matrix M of public sequences in a deterministic way, instead of choosing random columns. For example, cover-free families, orthogonal arrays or ordered designs could be used. We also introduce a symbol ∞ as an entry of M , which means “no information”. Based on a chain ordering, we can replace the largest symbol $L - 1$ by ∞ ; since the corresponding hashed share $h^{(L-1)}(s)$ (for some seed s) can be computed by any other single user, it makes no contribution to secure communication. Therefore we would rather assign no information to the symbol in order to save memory storage per user.

1.3 Other Related Works

Blom [3] presented key predistribution schemes that require each user store $w + 1$ keys to protect each pairwise key against any coalition attack of at most w users. Blundo *et al.* [4] generalized Blom’s scheme to allow for any group of size t to establish its group key. For a group key to be secure against a coalition of w users, each user has to store $\binom{t+w-1}{t-1}$ keys. They also proved the key storage is

information theoretically optimal. Mitchell and Piper [13] presented KPSs using key predistribution patterns, which involve very little arithmetic computation. The KDPs are equivalent to cover-free families, which are widely studied and generalized (for example, [17] and [18]). Such unconditionally secure key predistribution schemes are studied as special families of linear key predistribution schemes [14].

The efficiency of KPSs can be improved by choosing orthogonal arrays or perpendicular arrays as KDPs and then using resilient functions in the generation of keys [16]. Recently, Attrapadung *et al.* [2] studied key predistribution schemes using key chains from pseudo-random sequence generators (PRSGs). In a network of n users, each user stores $O\left(\frac{2^{n-1}}{n}\right)$ keys to establish group keys of any size, which are secure against any coalition attacks, based on the security of the PRSGs.

Since Eschenauer and Gligor's work [8], a series of papers on (pairwise) key predistribution for distributed sensor networks have been published ([5], [9], [10], [12], [15]). The main difference of these KPSs from conventional ones is that they allow certain pairs of nodes to share no common key. The attack model and the metric to evaluate the resiliency are also different, which are based on probabilistic analysis.

1.4 Our Contributions

By extending Leighton and Micali's scheme, we present a new framework of tree-based key distribution patterns (TKDPs). In section 3, we present a simple method of constructing TKDPs from cover-free families. In section 4, we show the existence of TKDPs by probabilistic methods, similar to [7]. Especially, the upper bounds on the minimum number of rows of (t, w, T) -TKDPs, which are obtained from probabilistic methods, are asymptotically w times smaller than Leighton and Micali's schemes (with the same parameters), when we choose optimal trees $\mathcal{T}$ instead of chains. One advantage of TKDP-based KPSs over Leighton and Micali's schemes is smaller hash depths are applied to each secret seed, which means less computation is required.

2 Tree-Based Key Distribution Pattern

We are now prepared to present a framework of a *tree-based key distribution pattern* (TKDP). First, we define a rooted tree $\mathcal{T}$ on L vertices, labeled from 0 to $L - 1$. Especially, the root vertex is labeled by 0. As we observed in the previous section, a rooted tree defines a partial ordering on L vertices. We say $j_1 < j_2$ if a vertex j_1 is an ancestor of j_2 . For example, we observe that $0 < 3$ and $2 < 5$ in Fig. 1(b). But there is no relation between 1 and 2. We also define $j < \infty$ for every $j \in \{0, \dots, L - 1\}$. From now on, we will identify the set of users with the integers from 1 to n .

Definition 2.1. Let $\mathcal{T}$ be a rooted tree labeled by $\{0, \dots, L - 1\}$ and let $\Gamma = (\mathcal{P}, \mathcal{F})$ be an access structure. Let $M = (\alpha_{i,j})$ be a $b \times n$ matrix with

$\alpha_{i,j} \in \{0, \dots, L-1\} \cup \{\infty\}$ for $1 \leq i \leq b$ and $1 \leq j \leq n$. We say that M is a tree-based key distribution pattern $(\Gamma, \mathcal{T})$ -TKDP(b, n), provided that for every disjoint pair of $P \in \mathcal{P}$ and $F \in \mathcal{F}$, there exist $1 \leq i^* = i_{(P,F)}^* \leq b$ and $j^* = j_{(P,F)}^* \in P$ such that

1. $\alpha_{i^*,j} \leq \alpha_{i^*,j^*}$ for all $j \in P$, and
2. $\alpha_{i^*,j} \not\leq \alpha_{i^*,j^*}$ for all $j \in F$.

The notation $(t, w, \mathcal{T})$ -TKDP(b, n) will denote a $(\Gamma_{(t,w)}, \mathcal{T})$ -TKDP(b, n).

KPS from TKDP. We construct a KPS with an access structure Γ from a TKDP in the following manner:

1. The TA publishes a $(\Gamma, \mathcal{T})$ -TKDP(b, n), denoted $M = (\alpha_{i,j})$, a hash function

$$h : \{0, 1\}^{l_1+l_2} \longrightarrow \{0, 1\}^{l_1},$$

and a tree $\mathcal{T}$ on a set $\{0, \dots, L-1\}$ of vertices.

2. For $1 \leq i \leq b$ and $1 \leq j \leq L-1$, the TA chooses random values $s_i = s(i, 0)$ of length l_1 and computes hashed shares $s(i, j)$ recursively based on the tree $\mathcal{T}$; if a vertex j_2 is a child of j_1 , then $s(i, j_2) = h(s(i, j_1) \parallel j_2)$, where the label of each vertex is represented as a binary sequence of length l_2 .
3. User j receives secret shares $s(i, \alpha_{i,j})$ for $1 \leq i \leq b$, where $s(i, \infty) = \emptyset$ by definition.
4. Let

$$I_P = \{1 \leq i \leq b : \exists \bar{j} \in P \text{ such that } \alpha_{i,j} \leq \alpha_{i,\bar{j}}, \forall j \in P\}.$$

The key K_P for a privileged set P is defined to be

$$K_P = \sum_{i \in I_P} s(i, \alpha_{i,\bar{j}}).$$

For a given $i \in I_P$, $\bar{j}$ is not necessarily unique, but $\alpha_{i,\bar{j}}$ is.

Any user $j \in P$ can compute every term in K_P by using the function h , while a disjoint coalition $F \in \mathcal{F}$ cannot compute at least one term

$$s(i_{(P,F)}^*, \alpha_{i_{(P,F)}^*, \bar{j}}),$$

since $\alpha_{i_{(P,F)}^*, j} \not\leq \alpha_{i_{(P,F)}^*, \bar{j}}$ for all $j \in F$ (by Def. 2.1). In this case, we say *row $i_{(P,F)}^*$ protects P against F* . If l_1 is the bit length of a secure key for a symmetric encryption, then we can say the KPS is secure with respect to the access structure Γ .

Remark 2.1. We assume a random oracle model for the function h in the sense that we cannot obtain the value of $h(s)$ without query to the random oracle.

Remark 2.2. Each user should store bl_1 bits, or equivalently, b secret keys. Therefore, we would like to have TKDPs that minimize b with the other parameters fixed.

Example 2.1. As we will see in Example 3.1,

$$M = \begin{pmatrix} 0 & 0 & 3 & 0 & 5 & 6 & 7 \\ 1 & 0 & 0 & 4 & 0 & 6 & 7 \\ 0 & 2 & 0 & 4 & 5 & 0 & 7 \\ 1 & 2 & 3 & 0 & 0 & 0 & 7 \\ 0 & 2 & 3 & 4 & 0 & 6 & 0 \\ 1 & 0 & 3 & 4 & 5 & 0 & 0 \\ 1 & 2 & 0 & 0 & 5 & 6 & 0 \end{pmatrix}$$

is a $(2, 2, \mathcal{T}_7)$ -TKDP $(7, 7)$, where $\mathcal{T}_7$ is a tree of depth 1 with 7 leaves (Fig. 2).

Fig. 2. Star-like tree: $\mathcal{T}_7$

The TA generates random secret seeds s_i for $i = 1, \dots, 7$. Two users U_1 and U_2 (corresponding to the first two columns) receive sets of keys

$$K_1 = \{s_1, h(s_2 \parallel 1), s_3, h(s_4 \parallel 1), s_5, h(s_6 \parallel 1), h(s_7 \parallel 1)\},$$

and

$$K_2 = \{s_1, s_2, h(s_3 \parallel 2), h(s_4 \parallel 2), h(s_5 \parallel 2), s_6, h(s_7 \parallel 2)\},$$

respectively. For $P = \{1, 2\}$, we have $I_P = \{1, 2, 3, 5, 6\}$. Both U_1 and U_2 can compute the group key

$$K_{1,2} = s_1 + h(s_2 \parallel 1) + h(s_3 \parallel 2) + h(s_5 \parallel 2) + h(s_6 \parallel 1).$$

An adversarial coalition of two users, say, U_3 and U_4 , cannot compute $h(s_5 \parallel 2)$ and $h(s_6 \parallel 1)$. In fact, $K_{1,2}$ is secure against any coalition of size 2.

3 Cover-Free Families and TKDPs

In this section, we will show some relations between *cover-free families* (or equivalently, conventional key distribution patterns) and TKDPs. A *set system* is a pair $(X, \mathcal{A})$, where $\mathcal{A}$ is a finite set of subsets of X , called *blocks*.

Definition 3.1. A set system $(X, \mathcal{A})$ is called a (t, w) -cover-free family provided that, for any t blocks $B_1, \dots, B_t \in \mathcal{A}$ and any w other blocks $A_1, \dots, A_w \in \mathcal{A}$, we have

$$\bigcap_{i=1}^t B_i \not\subseteq \bigcup_{j=1}^w A_j.$$

A (t, w) -cover-free family will be denoted as a (t, w) -CFF(b, n) if $|X| = b$ and $|\mathcal{A}| = n$. Let $(X, \mathcal{A})$ be a (t, w) -CFF(b, n) such that $X = \{x_1, \dots, x_b\}$ and $\mathcal{A} = \{A_1, \dots, A_n\}$. The incidence matrix of $(X, \mathcal{A})$ is defined to be a $b \times n$ $(0, 1)$ -matrix $M = (\beta_{i,j})$ such that $\beta_{i,j} = 1$ if $x_i \in A_j$, and $\beta_{i,j} = 0$, otherwise.

Now we can regard a cover-free family as a special case of TKDP, as seen in the following lemma.

Lemma 3.1. Let $L = 1$, i.e., $\mathcal{T}$ consists of one vertex. Then $(t, w, \mathcal{T})$ -TKDPs are equivalent to (t, w) -cover-free families.

Proof. Let $M_T = (\alpha_{i,j})$ be a $(t, w, \mathcal{T})$ -TKDP(b, n). Since M_T is a $(0, \infty)$ -matrix, we can define a $b \times n$ matrix $M_C = (\beta_{i,j})$ such that

$$\begin{cases} \beta_{i,j} = 1, & \text{if } \alpha_{i,j} = 0; \\ \beta_{i,j} = 0, & \text{if } \alpha_{i,j} = \infty. \end{cases}$$

Then M_C is the incidence matrix of a (t, w) -CFF(b, n). The converse is proved in a similar way.

There is a method of constructing TKDPs based on rooted trees of depth 1 from CFFs. (From now on, rooted trees of depth 1 are called *star-like trees*.) Let $\mathcal{T}_n$ be a star-like tree with root 0 and n leaves (labeled from 1 to n).

Theorem 3.1. If there exists a $(t-1, w)$ -CFF(b, n), then there exists a $(t, w, \mathcal{T}_n)$ -TKDP(b, n).

Proof. Let $M_C = (\beta_{i,j})$ be the incidence matrix of a $(t-1, w)$ -CFF(b, n). Then we can define a $b \times n$ matrix $M_T = (\alpha_{i,j})$ as follows;

$$\begin{cases} \alpha_{i,j} = 0, & \text{if } \beta_{i,j} = 1; \\ \alpha_{i,j} = j, & \text{if } \beta_{i,j} = 0. \end{cases}$$

We claim that M_T is a $(t, w, \mathcal{T}_n)$ -TKDP(b, n).

The columns of M_C and M_T are indexed from 1 to n ; the rows of M_C and M_T are indexed from 1 to b . Let $P = \{j_1, \dots, j_t\}$ and $F = \{j'_1, \dots, j'_w\}$ be disjoint subsets of columns. Since M_C is the incidence matrix of a $(t-1, w)$ -CFF(b, n), there exists a row i such that $\beta_{i,j_1} = \dots = \beta_{i,j_{t-1}} = 1$, and $\beta_{i,j'_1} = \dots = \beta_{i,j'_w} = 0$. Since $\alpha_{i,j_1} = \dots = \alpha_{i,j_{t-1}} = 0$, we have $\alpha_{i,j} \leq \alpha_{i,j_t}$ for every $j \in P$. On the other hand, we have $\alpha_{i,j'} \not\leq \alpha_{i,j_t}$ for every $j' \in F$ since the symbols $\alpha_{i,j'}$ represent distinct leaves of the tree. (In the context of KPS, the group key of P contains a hashed share $h(s_i \parallel j_t)$ as a summand, while the coalition of F cannot compute it.)

Example 3.1. Let $t = w = 2$ and $b = n = 7$. Then we can construct a $(2, 2, \mathcal{T}_7)$ -TKDP(7, 7), say M_T , from the incidence matrix M_C of a $(1, 2)$ -CFF(7, 7), where

$$M_C = \begin{pmatrix} 1 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 & 0 \\ 1 & 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 & 0 & 0 & 1 \end{pmatrix} \quad \text{and} \quad M_T = \begin{pmatrix} 0 & 0 & 3 & 0 & 5 & 6 & 7 \\ 1 & 0 & 0 & 4 & 0 & 6 & 7 \\ 0 & 2 & 0 & 4 & 5 & 0 & 7 \\ 1 & 2 & 3 & 0 & 0 & 0 & 7 \\ 0 & 2 & 3 & 4 & 0 & 6 & 0 \\ 1 & 0 & 3 & 4 & 5 & 0 & 0 \\ 1 & 2 & 0 & 0 & 5 & 6 & 0 \end{pmatrix}.$$

4 Probabilistic Method to Construct TKDPs

We will show the existence of a $(t, w, \mathcal{T})$ -TKDP(b, n), denoted $M = (\alpha_{i,j})$, by using the probabilistic method [1]. We would like for the matrix M to have as few rows as possible.

Now we fix a tree $\mathcal{T}$ on a set $V = \{0, \dots, L-1\}$. Let $S = \{0, \dots, L-1, \infty\}^n$ be the set of all possible rows of length n with symbols from $\{0, \dots, L-1, \infty\}$. We define a probability distribution $\mathbf{P}$ on the set S , and a random variable $X(P, F)$ on S for any disjoint pair of $P \in \mathcal{P}$ and $F \in \mathcal{F}$ as follows:

$$X(P, F) = \begin{cases} 0 & \text{if the row protects } P \text{ against } F \\ 1 & \text{otherwise.} \end{cases}$$

Let $\mathcal{M}_{b \times n}$ be the set of all $b \times n$ matrices with symbols from $\{0, \dots, L-1, \infty\}$. By the probability distribution $\mathbf{P}$, we pick uniformly at random b rows to construct a matrix in $\mathcal{M}_{b \times n}$. The distribution $\mathbf{P}$ and the random variable $X(P, F)$ naturally define the inherited probability distribution $\mathbf{P}_b$ and random variable

$$X_b(P, F) = \begin{cases} 0 & \text{if there exists a row that protects } P \text{ against } F \\ 1 & \text{otherwise.} \end{cases}$$

on $\mathcal{M}_{b \times n}$. Then we have

$$\begin{aligned} \mathbf{Exp}[X_b(P, F)] &= \mathbf{Pr}[X_b(P, F) = 0] \times 0 + \mathbf{Pr}[X_b(P, F) = 1] \times 1 \\ &= (\mathbf{Pr}[X(P, F) = 1])^b. \end{aligned}$$

Finally, we define a random variable

$$X = \sum_{P \in \mathcal{P}, F \in \mathcal{F}, P \cap F = \emptyset} X_b(P, F).$$

If $\mathbf{Exp}[X] < 1$, then there exists a $(t, w, \mathcal{T})$ -TKDP(b, n) in $\mathcal{M}_{b \times n}$. In order to evaluate $\mathbf{Exp}[X]$, we define

$$\bar{p}_{t,w} = \min_{P \in \mathcal{P}, F \in \mathcal{F}, P \cap F = \emptyset} \mathbf{Pr}[X(P, F) = 0].$$

Then it is easy to see that

$$\begin{aligned}
 \mathbf{Exp}[X] &= \sum_{P \in \mathcal{P}, F \in \mathcal{F}, P \cap F = \emptyset} \mathbf{Exp}[X_b(P, F)] \\
 &= \binom{n}{t} \binom{n-t}{w} \mathbf{Exp}[X(P, F)]^b \\
 &\leq n^{t+w} (1 - \bar{p}_{t,w})^b.
 \end{aligned}$$

A simple computation shows

$$b > \frac{(t+w) \ln n}{-\ln(1 - \bar{p}_{t,w})} \implies \mathbf{Exp}[X] < 1, \quad (1)$$

where

$$\frac{(t+w) \ln n}{-\ln(1 - \bar{p}_{t,w})} \approx \frac{(t+w) \ln n}{\bar{p}_{t,w}},$$

for sufficiently small $\bar{p}_{t,w}$. Note that $\bar{p}_{t,w}$ is determined by the probability distribution $\mathbf{P}$ on S and the tree $\mathcal{T}$.

Example 4.1. Let $\mathcal{T}$ be a chain on n vertices. Suppose we define

$$\begin{aligned}
 \mathbf{P} : \{0, \dots, n-1\}^n &\longrightarrow [0, 1] \\
 \alpha &\mapsto 1/n! \quad \text{if } \alpha \text{ is a permutation of the } n \text{ distinct symbols,} \\
 &0 \quad \text{otherwise.}
 \end{aligned}$$

Since

$$\bar{p}_{t,w} = \frac{t!w!}{(t+w)!},$$

we have a $(t, w, \mathcal{T})$ -TKDP(b, n) for

$$b \approx \frac{(t+w)(t+w)! \ln n}{t!w!}.$$

In the following sections, we consider two kinds of probability distributions to generate random rows.

4.1 Independent Random Selection of Symbols

In this section, we fix a tree $\mathcal{T}$ on $V = \{0, \dots, L-1\}$, and find the optimal probability distribution such that each symbol is chosen independently; each symbol $i \in V \cup \{\infty\}$ is selected with probability p_i , where $0 \leq p_i \leq 1$ for $i = 0, \dots, L-1, \infty$, and

$$p_0 + \dots + p_{L-1} + p_\infty = 1.$$

Then the probability distribution $\mathbf{P}$ on $S = \{0, \dots, L-1, \infty\}^n$ is derived as follows:

$$\begin{aligned}
 \mathbf{P} : S &\longrightarrow [0, 1] \\
 \alpha &\mapsto \prod_{i=0}^{L-1} p_i^{wt_i(\alpha)},
 \end{aligned}$$

where $wt_i(\alpha)$ is the number of i 's appearing in α . Now we have the following lemma.

Lemma 4.1. *Let $\mathbb{P}(v)$ denote the path from the root 0 to $v \in V$ in $\mathcal{T}$ and let $\text{prt}(v)$ denote the parent of v . Then we have*

$$\Pr[X(P, F) = 0] = \sum_{v \in V} \left(\left(\sum_{i \in \mathbb{P}(v)} p_i \right)^t - \left(\sum_{i \in \mathbb{P}(\text{prt}(v))} p_i \right)^t \right) \left(1 - \sum_{i \in \mathbb{P}(v)} p_i \right)^w,$$

for any disjoint $P \in \mathcal{P}$ and $F \in \mathcal{F}$.

Proof. Note that $\Pr[X(P, F) = 0]$ is the probability that a row(sequence) protects P against F when it is chosen by the probability distribution $\mathbf{P}$. Suppose that we pick up a sequence $\alpha = (\alpha_1, \dots, \alpha_n)$. If α protects P against F , then there exists $j^* \in P$ such that $\alpha_j \leq \alpha_{j^*}$, for all $j \in P$ and $\alpha_j \not\leq \alpha_{j^*}$, for all $j \in F$. Since the symbol α_{j^*} is unique, we can define $\max(\alpha, P) = \alpha_{j^*}$ for any sequence α that protects P against F .

If $\max(\alpha, P) = v$, then the positions of P should consist of symbols of the path from the root to the vertex v , and contain at least one symbol of v . On the other hand, the positions of F should contain no symbol from the path. Therefore we have

$$\Pr[\max(\alpha, P) = v] = \left(\left(\sum_{i \in \mathbb{P}(v)} p_i \right)^t - \left(\sum_{i \in \mathbb{P}(\text{prt}(v))} p_i \right)^t \right) \left(1 - \sum_{i \in \mathbb{P}(v)} p_i \right)^w,$$

for every $v \in V$. Now the lemma is true since

$$\Pr[X(P, F) = 0] = \sum_{v \in V} \Pr[\max(\alpha, P) = v].$$

By using the above lemma, we can find optimal probability distributions for some special cases.

$L = 2$. We have a unique tree $\mathcal{T}$ on 2 vertices. We would like to maximize the probability to construct a $(t, w, \mathcal{T})$ -TKDP(b, n), denoted $M = (\alpha_{i,j})$. In this model, symbols 0, 1 and ∞ are selected in an independent random way with probabilities p_0 , p_1 and p_∞ , respectively. By Lemma 4.1, we have

$$\bar{p}_{t,w} = p_0^t(1 - p_0)^w + ((p_0 + p_1)^t - p_0^t)(1 - p_0 - p_1)^w.$$

Removing p_∞ , we have to solve the following problem to find an optimal probability distribution:

$$\begin{aligned} &\text{Maximize} && \bar{p}_{t,w} = p_0^t(1 - p_0)^w + ((p_0 + p_1)^t - p_0^t)(1 - p_0 - p_1)^w \\ &\text{subject to} && p_0 + p_1 \leq 1, \quad p_0, p_1 \geq 0. \end{aligned}$$

Example 4.2. Let $t = 2$, $w = 1$ and $n = 1000$. Then the above problem is reduced to

$$\begin{aligned} &\text{Maximize} && \bar{p}_{2,1} = -p_0^3 - 2p_0^2p_1 + p_0^2 - 3p_0p_1^2 + 2p_0p_1 - p_1^3 + p_1^2 \\ &\text{subject to} && p_0 + p_1 \leq 1, \quad p_0, p_1 \geq 0. \end{aligned}$$

By elementary calculus, we can show that $\bar{p}_{2,1}$ attains its maximum $\bar{p}_{2,1}^* \approx 0.204$ when

$$p_0 = \frac{12}{23}, \quad p_1 = \frac{6}{23}, \quad \text{and} \quad p_\infty = \frac{5}{23}.$$

By the condition (1), there exists a $(2, 1, \mathcal{T})$ -TKDP(91, 1000). On the other hand, if we use a probabilistic method [7] for cover-free families, we can show the existence of a $(2, 1)$ -CFF(130, 1000) at best. Therefore we have about 40% improvement in the number of rows.

Star-Like Trees with Many Leaves. Suppose $\mathcal{T}$ is a star-like tree with a leaves (labeled from 1 to a). We choose each leaf with the same probability p_1 , and set $p_\infty = 0$ and $t = 2$, for simple analysis. Then we have to solve the following problem:

$$\begin{aligned} &\text{Maximize} && \bar{p}_{2,w} = p_0^2(1 - p_0)^w + a((p_0 + p_1)^2 - p_0^2)(1 - p_0 - p_1)^w \\ &\text{subject to} && p_0 + ap_1 = 1, \quad p_0, p_1 \geq 0. \end{aligned}$$

Put $x = p_0$ and $y = ap_1$. For a sufficiently large a , we have the approximation

$$\begin{aligned} \bar{p}_{2,w} &= (1 - y)^2 y^w + a \left(\left(x + \frac{y}{a} \right)^2 - x^2 \right) \left(y - \frac{y}{a} \right)^w \\ &= (1 - y)^2 y^w + \left(2xy + \frac{y^2}{a} \right) \left(y - \frac{y}{a} \right)^w \\ &\approx \left((1 - y)^2 + 2xy \right) y^w \\ &= (1 - y^2) y^w. \end{aligned}$$

Since

$$\begin{aligned} \frac{d}{dy} \left((1 - y^2) y^w \right) &= (-2y) y^w + w(1 - y^2) y^{w-1} \\ &= (w - (w + 2) y^2) y^{w-1}, \end{aligned}$$

we see that $\bar{p}_{2,w}$ attains its (approximate) maximum

$$\bar{p}_{2,w}^* \approx \frac{2}{w + 2} \left(\frac{w}{w + 2} \right)^{w/2},$$

when $y = \sqrt{\frac{w}{w+2}}$, or equivalently,

$$p_0 = 1 - \sqrt{\frac{w}{w+2}}, \quad p_1 = \frac{1}{a} \sqrt{\frac{w}{w+2}}, \quad (\text{and } p_\infty = 0).$$

By the condition (1), we know there exists a $(2, w, \mathcal{T})$ -TKDP(b, n) if

$$b > \frac{1}{2} (w+2)^2 \left(1 + \frac{2}{w}\right)^{w/2} \ln n,$$

for sufficiently large w . It is instructive to observe that a similar method shows the existence of $(2, w)$ -CFF(b, n) for

$$b > \frac{1}{4} (w+2)^3 \left(1 + \frac{2}{w}\right)^w \ln n.$$

4.2 Random Permutations

In this section, we fix a probabilistic distribution, and find the optimal tree ordering. We extend Example 4.1 to a (general) tree $\mathcal{T}$ on a set $V = \{0, \dots, L-1\}$ of vertices such that $L+1 \geq n$. We define a probability distribution $\mathbf{P}$ on $S = \{0, \dots, L-1, \infty\}^n$ as follows:

$$\begin{aligned} \mathbf{P} : S &\longrightarrow [0, 1] \\ \alpha &\mapsto \frac{(L+1-n)!}{(L+1)!} \text{ if } \alpha \text{ is a permutation of an } n\text{-subset of } V \cup \{\infty\}, \\ &0 \text{ otherwise.} \end{aligned}$$

Then we have the following lemma.

Lemma 4.2. *Let m_d be the number of vertices whose distance from the root is d . Then we have*

$$\Pr[X(P, F) = 0] = \frac{\sum_{d \geq t-1} m_d \binom{d+1}{t} \binom{L-d}{w} t! w!}{\binom{L+1}{t+w} (t+w)!},$$

for any disjoint $P \in \mathcal{P}$ and $F \in \mathcal{F}$.

Example 4.3. Let $t = 2$, $w = 1$, and let $\mathcal{T}$ be a binary tree on $L = 7$ vertices as seen in Fig. 1(b). Since $m_0 = 1$, $m_1 = 2$ and $m_2 = 4$, we have

$$\begin{aligned} \Pr[X(P, F) = 0] &= \frac{2 \cdot \binom{2}{2} \binom{6}{1} \cdot 2! + 4 \cdot \left(\binom{3}{2} - \binom{2}{2}\right) \binom{5}{1} \cdot 2!}{\binom{8}{3} \cdot 3!} \\ &= \frac{31}{84} \approx 0.369. \end{aligned}$$

by Lemma 4.2. Therefore, when we choose a sequence from $S = \{0, \dots, 6, \infty\}^n$ by the probability distribution $\mathbf{P}$ (for any $n \leq 8$), the sequence protects any two users against another single user with the probability ≈ 0.369 .

Proof. Note that $\Pr[X(P, F) = 0]$ is the probability that a row(sequence) protects P against F when it is chosen by the probability distribution $\mathbf{P}$. Suppose that we choose a sequence $\alpha = (\alpha_1, \dots, \alpha_n)$. If α protects P against F , then there exists $j^* \in P$ such that $\alpha_j \leq \alpha_{j^*}$, for all $j \in P$ and $\alpha_j \not\leq \alpha_{j^*}$, for all $j \in F$.

Since the symbol α_{j^*} is unique, we can define $\max(\alpha, P) = \alpha_{j^*}$ for any sequence α that protects P against F .

If $\max(\alpha, P) = v$, then the positions of P should consist of symbols of the path from the root to the vertex v , and contain exactly one symbol of v . On the other hand, the positions of F should contain no symbol from the path. Therefore we have

$$\Pr[\max(\alpha, P) = v] = \frac{\left(\binom{d+1}{t} - \binom{d}{t} \right) \binom{L-d}{w} t! w!}{\binom{L+1}{t+w} (t+w)!},$$

where $d \geq t-1$ is the distance from the root to the vertex v . If $d < t-1$, then $\Pr[\max(\alpha, P) = v] = 0$ since the symbols of P cannot be all distinct. Now the lemma follows since

$$\Pr[X(P, F) = 0] = \sum_{v \in V} \Pr[\max(\alpha, P) = v].$$

In order to find an optimal tree which maximizes $\bar{p}_{t,w}$, we have to solve the following optimization problem:

$$\text{Maximize} \quad \bar{p}_{t,w} = \frac{\sum_{d \geq t-1} m_d \left(\binom{d+1}{t} - \binom{d}{t} \right) \binom{L-d}{w} t! w!}{\binom{L+1}{t+w} (t+w)!} \quad (2)$$

$$\text{subject to} \quad \sum_{d=1}^D m_d = L-1, \quad (3)$$

$$D, m_1, \dots, m_D \in \mathbb{Z}^+. \quad (4)$$

Let

$$\begin{aligned} C_d &= \frac{\left(\binom{d+1}{t} - \binom{d}{t} \right) \binom{L-d}{w} t! w!}{\binom{L+1}{t+w} (t+w)!} \\ &= \left(\frac{(d+1)!}{(d+1-t)!} - \frac{d!}{(d-t)!} \right) \frac{(L-d)!}{(L-d-w)!} \cdot \frac{(L+1-t-w)!}{(L+1)!} \\ &= \frac{t \cdot d!}{(d+1-t)!} \cdot \frac{(L-d)!}{(L-d-w)!} \cdot \frac{(L+1-t-w)!}{(L+1)!}. \end{aligned}$$

C_d increases at d (as a function of d) if

$$C_{d-1} < C_d \Leftrightarrow d < \frac{(L+1)(t-1)}{t+w-1}.$$

Therefore C_d attains its maximum at

$$d^* = \left\lceil \frac{(L+1)(t-1)}{t+w-1} - 1 \right\rceil,$$

and we can solve the problem (2) with an optimal solution

$$(D^*, m_1^*, \dots, m_D^*) = (d^*, \overbrace{1, \dots, 1}^{d^*-1}, L - d^*),$$

and the optimal value

$$\begin{aligned} \bar{p}_{t,w,L}^* &= \frac{\sum_{d=t-1}^{d^*} m_d^* \left(\binom{d+1}{t} - \binom{d}{t} \right) \binom{L-d}{w} t! w!}{\binom{L+1}{t+w} (t+w)!} \\ &= \frac{\sum_{d=t-1}^{d^*} m_d^* ((d+1)! / (d+1-t)! - d! / (d-t)!)((L-d)! / (L-d-w)!)}{(L+1)! / (L+1-t-w)!} \\ &= \frac{(L+1-t-w)!}{(L+1)!} \sum_{d=t-1}^{d^*} \frac{m_d^* t \cdot d!}{(d+1-t)!} \cdot \frac{(L-d)!}{(L-d-w)!} \\ &\geq \frac{(L+1-t-w)!}{(L+1)!} \cdot \frac{(L-d^*)t \cdot d^*!}{(d^*+1-t)!} \cdot \frac{(L-d^*)!}{(L-d^*-w)!} \\ &= \frac{(L-d^*)t \cdot \overbrace{d^*(d^*-1) \dots (d^*-t+2)}^{t-1} \cdot \overbrace{(L-d^*)(L-d^*-1) \dots (L-d^*-w+1)}^w}{\underbrace{(L+1)L \dots (L-t-w+2)}_{t+w}} \\ &= w\Omega \left(\frac{t!w!}{(t+w)!} \right), \end{aligned}$$

since

$$\frac{d^*}{L} \approx \frac{t}{t+w}$$

for a sufficiently large L .

The optimal tree looks like a claw, as seen in Fig. 3. Now we conclude that

1. the probability $\bar{p}_{t,w}$ based on the optimal tree is greater than the one based on a chain (asymptotically) by a factor of w , and
2. the maximum hash depth applied to a secret seed is $d^* = \lceil (L+1)(t-1)/(t+w-1) - 1 \rceil$, which is relatively small for a large w .

Fig. 3. Optimal tree

Example 4.4. Let $n = L = 1000$, $t = 2$ and $w = 100$. Then we have $d^* = \lceil \frac{1001}{101} - 1 \rceil = 9$, and $\bar{p}_{t,w,L}^* \geq 0.00764$, while $\frac{t!w!}{(t+w)!} = 0.00019$. By the condition (1), there exists a $(2, 100, \mathcal{T}^*)$ -TKDP(91872, 1000), where $\mathcal{T}^*$ is a claw of depth $d^* = 9$.

5 Conclusion

We presented a new framework of tree-based key distribution pattern (TKDP). We constructed TKDPs from cover-free families. We note that other combinatorial structures such as orthogonal arrays, ordered designs and covering arrays also yield TKDPs. (This will be addressed in later work.) We showed the existence of TKDPs by the probabilistic method. Furthermore, we reduced the upper bounds on the minimum number of rows of $(t, w, \mathcal{T})$ -TKDPs, which are obtained from probabilistic methods, asymptotically by a factor of w as compared to Leighton and Micali's schemes by choosing optimal trees $\mathcal{T}$ instead of chains. The TKDP-based schemes are expected to have applications to sensor networks since they involve only hash computations in the establishment of pairwise keys, which are known to be more energy-efficient than RSA or elliptic curve operations.

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