An Operational Characterization of Strong Normalization*

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Abstract. This paper introduces the Φ -calculus, a new call-by-value version of the λ -calculus, following the spirit of Plotkin's $\lambda\beta_v$ -calculus. The Φ -calculus satisfies some interesting properties, in particular that its set of solvable terms coincides with the set of β -strongly normalizing terms in the classical λ -calculus.

1 Introduction

The standard λ -calculus equipped with the β -reduction is the paradigmatic language for the call-by-name functional computation. Its call-by-value version, historically called $\lambda\beta_v$ -calculus, has been introduced by Plotkin in 1975 [11]. The $\lambda\beta_v$ -calculus is based on a restriction of the β -rule, firing it only when the argument belongs to a particular subset of terms, called values. In [10, 13] two co-authors of this paper, in order to treat these two different calculi in a uniform way, introduced the $\lambda\Delta$ -calculus, parametric with respect to a subset Δ of terms, called *input values*, which generalizes the idea of Plotkin's values. Some conditions on Δ have been stated assuring some good properties for the calculus, in particular confluence and standardization. These conditions are very natural: the set of input values must contain the set of variables and it must be closed under substitution and Δ -reduction (a further condition is necessary for having standardization). Note that, in this setting, the standard λ -calculus now can be seen as a degenerated case of a $\lambda\Delta$ -calculus, where all terms are input values (that is, the set of input values is Λ). Plotkin's calculus coincides with the $\lambda\Gamma$ -calculus, where $\Gamma = \text{Var} \cup \{\lambda x.M \mid M \in \Lambda\}$.

The formalization of the $\lambda\Delta$ -calculus, where both call-by-name and call-by-value calculus can be uniformly representable, is an useful tool for studying the relationship between these two notions of computations. Some interesting properties relating $\lambda\Lambda$ and $\lambda\Gamma$ -calculus have been already proved. For example, it turns out that, in the $\lambda\Gamma$ -calculus, the notion of normal form is meaningless since there are different Γ -normal forms that can be consistently equated (see [13]). But the notion of β -normal form has an important meaning also in this calculus: in [7] it was proved that two different $\beta\eta$ -normal forms can be separated in the $\lambda\Gamma$ -calculus, and hence they cannot be consistently equated in any model.

In [8] we further explored the relationship between β -normal forms and Γ -evaluation. In that paper, it was proved that the set of strongly β -normalizing

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terms, with respect to a *lazy* reduction¹, coincides with the set of potentially Γ -valuable terms. A term is potentially Δ -valuable (where Δ is any set of input values) if and only if there is a substitution, replacing variables by closed input values, such that the substituted term reduces to an input value. Being input values the only terms that can be manipulated by the Δ -calculus (and that can be argument of a function) this class of terms is particularly interesting. For example, in Plotkin's operational semantics for the $\lambda\Gamma$ -calculus [11] based on the SECD machine [6], a potentially Γ -valuable term is always different from a non potentially Γ -valuable term. Moreover, all the non potentially Γ -valuable terms are equated [13].

A first natural question that arises is if this analogy between a call-by-name strong normalization and a call-by-value evaluation can be further developed. In particular, we are interested to know if *there is a set of input values Φ such that the set of potentially Φ -valuable terms coincides with the set of strongly β -normalizing terms.*

The set of lazy strong β -normalizing terms and the set Γ of input values have an interesting structural analogy: in order to find a lazy β -normal form of a λ -term it is not necessary to reduce under a λ -abstraction. The same happens when checking if a λ -term is a Γ -input value: it is not necessary to look under a λ -abstraction since all λ -abstractions are in Γ . We will call *weak* any set of input values containing all λ -abstractions. It turns out that “to be weak” has some consequences. Extending the notion of solvability to a generic Δ -calculus in the natural way, a term M is called Δ -solvable if and only if there is a sequence $P_1, \dots, P_n$ of Δ -input values such that $(\lambda\vec{x}.M)P_1\dots P_n =_{\Delta} \lambda x.x$ (where $\lambda\vec{x}.M$ is the term obtained from M by abstracting it with respect to all its free variables). In the $\lambda\Gamma$ -calculus, the set of solvable terms is a proper subset of the set of potentially valuable terms. This is a consequence of the fact that Γ is weak: if U is a closed Γ -unsolvable term then $\lambda x.U \in \Gamma$ is a potentially valuable, but unsolvable term.

This yields to a second question: *is there a $\lambda\Delta$ -calculus such that the set of Δ -potentially valuable terms coincides with the set of Δ -solvable terms?*

Certainly, such a Δ could not be weak. In fact, in order to check if a term is solvable, it is necessary to perform the evaluation *under* the λ -abstraction.

In this paper, we give an answer to the two questions posted above, presenting the $\lambda\Phi$ -calculus, where Φ is a set of input values which is the minimal solution of a recursive equation. Φ is not weak, since it is a proper subset of the set of Φ -normal forms. It turns out that the $\lambda\Phi$ -calculus enjoys, besides confluence, the standardization property. Moreover, Φ is a proper subset of the set of strongly β -normalizing terms, and we prove that Φ is minimal between all sets of terms answering the first question. Hence our result can be rephrased as: the whole set

¹ Following [13], lazy β -reduction is defined as the closure of the β -rule under *application*, but not under *abstraction*. This corresponds operationally to do not perform reduction under the λ -abstraction. In the field of real functional languages, “lazy” is used with a different meaning.

of strongly β -normalizing terms can be operationally described through a proper subset of it.

A further comment is in order. Plotkin's motivation on designing the $\lambda\Gamma$ -calculus was to propose a paradigmatic language for the call-by-value evaluation in real programming languages and, from this point of view, the choice of a weak set of input values is natural for modeling the notion of closure in the sense of Landin [6]. On the other hand, our motivation is purely theoretical, and the $\lambda\Phi$ -calculus presented here *is not* an alternative proposal for designing new call-by-value languages. In any case, implementing the Φ -calculus would be difficult, being the set Φ just semi-decidable. But we believe that this study is interesting by itself. In fact, Plotkin in [11], posed the question of an existence of call-by-value λ -language alternative to $\lambda\Gamma$. He said that the natural proposal was to choose the set of the β -normal forms as an alternative to Γ . Unfortunately, the set of β -normal forms induces a calculus lacking the confluence property, in fact β -normal forms are not input values in our sense. Hence the $\lambda\Phi$ -calculus, enjoying both confluence and standardization, gives an answer to this further question as well.

The rest of the paper is organized as follows: Section 2 contains basic notions of the parametric $\lambda\Delta$ -calculus; in Section 3 the $\lambda\Phi$ -calculus is introduced and finally, in Section 4 the main theorem is stated and proved.

2 The Parametric λ -Calculus

A calculus is a language equipped with some reduction rules. We will consider here calculi sharing the same language, the language of λ -calculus, while they differ from each other in their reduction rules. In order to treat them in a uniform way we will use the notion of parametric calculus, the $\lambda\Delta$ -calculus, that gives rise to different calculi by different instantiations of the parameter Δ . The $\lambda\Delta$ -calculus has been studied in [10, 13]. We use the terminology of [2, 13].

Definition 1 (The language Λ). *Let Var be a countable set of variables. The set Λ of λ -terms is defined by the following grammar:*

$$M ::= x \mid MM \mid \lambda x.M$$

λ -terms will be ranged over by Latin capital letters. Sets of λ -terms will be denoted by Greek capital letters. If Θ denotes a set of terms $(\Theta)^0$ is the set of closed terms belonging to Θ .

Sometimes, we will refer to λ -terms simply as terms. As usual, terms will be considered modulo α -conversion, i.e., modulo names of bound variables. The symbol $\equiv$ will denote syntactical identity of terms, up to α -equivalence.

We will use the following abbreviations, in order to avoid an excessive number of parentheses, thereby $\lambda x_1 \dots x_n.M$ will stand for $(\lambda x_1 (\dots (\lambda x_n.M) \dots))$ and $MN_1N_2 \dots N_n$ will stand for $(\dots ((MN_1)N_2) \dots N_n)$. Moreover $\vec{M}$ will denote a sequence of terms $M_1, \dots, M_n$, for some $n \geq 0$, and $\lambda \vec{x}.M$ and $\vec{M}\vec{N}$, will denote

respectively $\lambda x_1 \dots x_n.M$ and $M_1 \dots M_m N_1 \dots N_n$, for some $n, m \geq 0$. The length of the sequence $\vec{N}$ is denoted by $\|\vec{N}\|$.

The $\lambda\Delta$ -calculus consists of the language Λ equipped with a set $\Delta \subseteq \Lambda$ of input values, satisfying some closure conditions. Informally, input values represent already evaluated terms, that can be passed as arguments. The set Δ of input values and the reduction $\rightarrow_\Delta$, induced by it, are defined below.

Definition 2. Let $\Delta \subseteq \Lambda$.

- (i) The Δ -reduction ($\rightarrow_\Delta$) is the contextual closure of the following rule:

$$(\lambda x.M)N \rightarrow M[N/x] \text{ if and only if } N \in \Delta.$$

$(\lambda x.M)N$ is a Δ -redex (or simply redex).

- (ii) $\rightarrow_\Delta^+$, $\rightarrow_\Delta^*$ and $=_\Delta$ are respectively the transitive closure of $\rightarrow_\Delta$, the reflexive and transitive closure of $\rightarrow_\Delta$ and the symmetric, reflexive and transitive closure of $\rightarrow_\Delta$.
- (iii) A set $\Delta \subseteq \Lambda$ is a set of input values, when the following conditions are satisfied:
- $\text{Var} \subseteq \Delta$ (Var-closure);
 - $P, Q \in \Delta$ implies $P[Q/x] \in \Delta$, for each $x \in \text{Var}$ (substitution closure);
 - $M \in \Delta$ and $M \rightarrow_\Delta N$ imply $N \in \Delta$ (reduction closure).

The closure conditions on the set of input values assure us that the $\lambda\Delta$ -calculus enjoys the confluence property for every Δ , i.e., the following theorem holds.

Theorem 3 (Confluence). [10, 13] Let $M \rightarrow_\Delta^* N_1$ and $M \rightarrow_\Delta^* N_2$. There is Q such that both $N_1 \rightarrow_\Delta^* Q$ and $N_2 \rightarrow_\Delta^* Q$.

Two particular instantiations of Δ give rise to the call-by-name and the call-by-value λ -calculus. The call-by-name λ -calculus (i.e., the standard λ -calculus equipped with the β -reduction) coincides with the $\lambda\Delta$ -calculus. The call-by-value λ -calculus (defined by Plotkin in [11]) coincides with the $\lambda\Gamma$ -calculus, where $\Gamma = \text{Var} \cup \{\lambda x.M \mid M \in \Lambda\}$.

Let Δ be a set of input values. A term of the $\lambda\Delta$ -calculus is in Δ -normal form if and only if it does not contain occurrences of Δ -redexes. A term M is *strongly Δ -normalizing* if both M has a Δ -normal form and every reduction sequence starting from M eventually stops.

The set Δ -NF of Δ -normal forms can be defined in the following recursive way:

$$\begin{aligned} \Delta\text{-NF} = & \text{Var} \cup \{xM_1 \dots M_n \mid M_k \in \Delta\text{-NF} \ (1 \leq k \leq n)\} \\ & \cup \{\lambda \vec{x}.M \mid M \in \Delta\text{-NF}\} \\ & \cup \{(\lambda x.P)QM_1 \dots M_n \mid P, Q, M_k \in \Delta\text{-NF}, Q \notin \Delta \ (1 \leq k \leq n)\}. \end{aligned}$$

Note that for the $\lambda\Delta$ -calculus, being Λ its set of input values, the last case cannot happen, i.e., there are no normal forms of the shape $(\lambda x.P)QM_1 \dots M_n$, so $\Lambda\text{-NF} \subseteq \Delta\text{-NF}$, for all Δ .

In the $\lambda\Gamma$ -calculus, the notion of normal form is meaningless. In fact, there are different Γ -normal forms that can be consistently equated. The key notion, in a call by value setting, is the one of (potential) valuability, given in the next definition (see [9],[13]).

- Definition 4.** (i) *A term M is Δ -valuable if and only if there is $N \in \Delta$ such that $M \rightarrow_{\Delta}^* N$.*
 (ii) *A term M is potentially Δ -valuable if and only if there is a substitution $\mathbf{s}$, replacing variables by closed terms belonging to Δ , such that $\mathbf{s}(M)$ is Δ -valuable.*

It is immediate to verify that a closed term is potentially Δ -valuable if and only if it is Δ -valuable. Note that the notion of Δ -normal form and that one of potentially Δ -valuable are orthogonal. As an example, consider the $\lambda\Gamma$ -calculus, and the term $M \equiv (\lambda z.D)(yI)D$, where $I \equiv \lambda x.x$ and $D \equiv (\lambda z.zz)$. M is in Γ -normal form, but it is neither an input value nor potentially Γ -valuable. In fact, consider $M[Q/y]$, for some $Q \in (\Gamma)^0$. If QI reduces to an element in Γ then $M[Q/y] \equiv (\lambda z.D)(QI)D$ reduces to DD , which is not an input value. Otherwise $M[Q/y] \rightarrow_{\Gamma}^* (\lambda z.D)Q'D$, for every Q' such that $QI \rightarrow_{\Gamma}^* Q'$, which is not an input value. Thus $(\lambda z.D)(QI)D$ is not Γ -valuable. We call *Δ -liar-normal forms* terms which are in Δ -normal form but that are not potentially Δ -valuable.

In the $\lambda\Delta$ -calculus, the notion of solvability plays an important role, since in some sense the solvable terms represent the meaningful computations [2]. In [9], Γ -solvable and potentially Γ -valuable terms has been characterized. This notion has been extended to the parametric $\lambda\Delta$ -calculus in [13].

- Definition 5.** (i) *A context $C[.]$ is Δ -valuable if and only if $C[.] \equiv (\lambda \vec{x}.[.])\vec{P}$ where each $P \in \vec{P}$ is such that $P \in \Delta$.*
 (ii) *A term M is Δ -solvable if and only if there is a Δ -valuable context $C[.]$ such that:*

$$C[M] =_{\Delta} I.$$

- (iii) *A term is Δ -unsolvable if and only if it is not Δ -solvable.*

Note that $(\lambda \vec{x}.M)\vec{N} =_{\Delta} I$ means $(\lambda \vec{x}.M)\vec{N} \rightarrow_{\Delta}^* I$, since I is in Δ -NF, for every Δ .

3 The $\lambda\Phi$ -Calculus

As observed in the introduction, we are interested to know if there is a set of input values Φ such that the set of potentially Φ -valuable terms coincides with the set of strongly β -normalizing terms. Such a set cannot be weak i.e., it cannot contain all λ -abstractions. Since input values represent already evaluated terms, the natural choice would be to take Δ such that Δ coincides with its set of normal forms (i.e. $\Delta = \Delta$ -NF). From the recursive definition of Δ -normal form

restricted to the case where Δ is a set of input values, the following equations would be obtained:

$$(1.1) \quad \Delta = \text{Var} \cup \{xM_1 \dots M_n \mid M_k \in \Delta \ (1 \leq k \leq n)\} \cup \{\lambda \vec{x}.M \mid M \in \Delta\} \cup \\ \{(\lambda x.P)QM_1 \dots M_n \mid P, Q, M_k \in \Delta \ (1 \leq k \leq n), Q \notin \Delta\};$$

$$(1.2) \quad M, P \in \Delta \text{ implies } M[P/x] \in \Delta.$$

Note that the reduction closure for Δ is trivially satisfied, since we asked that terms in Δ are Δ -normal forms.

The only solution to the equation (1.1) is $\Delta = \Lambda\text{-NF}$. In fact, the set $\{(\lambda x.P)QM_1 \dots M_n \mid P, Q, M_k \in \Delta \ (1 \leq k \leq n), Q \notin \Delta\}$ is empty, due to the contradictory condition on Q . Unfortunately, $\Delta = \Lambda\text{-NF}$ does not satisfy the equation (1.2).

On the other hand, the simpler way of forcing Δ to satisfy equation (1.2) would be to restrict Δ so that it contains, besides variables, only closed terms. That is, to choose $\Delta^\dagger = (\Lambda\text{-NF})^0 \cup \text{Var}$. But $\Delta^\dagger$ does not solve our problem. In fact, the term $I(\lambda x.I(xx))$ is a strongly Λ -normalizing term, but it is not $\Delta^\dagger$ -solvable. Indeed $I(\lambda x.I(xx))$ is a $\Delta^\dagger$ -NF which cannot be reduced (in the $\lambda\Delta^\dagger$ -calculus), since $\lambda x.I(xx) \notin \Delta^\dagger$.

The discussion above suggests that we should look for a set Δ which is a proper subset of the Δ -normal forms, and a proper superset of $(\Lambda\text{-NF})^0 \cup \text{Var}$. Let us maintain the choice that Δ contains, besides variables, only closed terms. The previous pair of equations now become:

$$(2.1) \quad \Theta = \text{Var} \cup \{xM_1 \dots M_n \mid M_k \in \Theta \ (1 \leq k \leq n)\} \cup \{\lambda \vec{x}.M \mid M \in \Theta\} \cup \\ \{(\lambda x.P)QM_1 \dots M_n \mid P, Q, M_1 \dots M_n \in \Theta, Q \notin \Delta\};$$

$$(2.2) \quad \Delta = \text{Var} \cup (\Theta)^0.$$

Note that $\Theta = \Delta\text{-NF}$ and Δ is a set of input values.

But the last condition on equation (2.1) is too weak again, since now the set Θ may contain some Δ -liar-normal forms. As an example, $M \equiv (\lambda z.D)(yI)D$, where $D \equiv (\lambda z.zz)$, is a Δ -liar-normal form satisfying both the previous equations. Since Δ -liar-normal forms are Δ -unsolvable, such a set cannot supply an answer to our second question, i.e., cannot be such that the set of Δ -solvable terms coincide with the set of Δ -potentially valuable terms.

The following easy to prove property will help us on excluding such dangerous terms.

Property 6. If there is a substitution $\mathbf{s}$ such that $\mathbf{s}(P[Q/x]\vec{M}) \rightarrow_\Delta^* R \in \Delta$ and $\mathbf{s}(Q) \rightarrow_\Delta^* Q' \in \Delta$ then $\mathbf{s}((\lambda x.P)Q\vec{M}) \rightarrow_\Delta^* R$.

Taking into account the previous property, the pair of equations now becomes:

$$(3.1) \quad \Theta = \text{Var} \cup \{xM_1 \dots M_n \mid M_k \in \Theta \ (1 \leq k \leq n)\} \cup \{\lambda \vec{x}.M \mid M \in \Theta\} \cup \\ \{(\lambda x.P)QM_1 \dots M_n \mid Q, M_1, \dots, M_n \in \Theta, Q \notin \Delta, P[Q/x]M_1 \dots M_n \rightarrow_\Delta^* R \in \Theta\};$$

$$(3.2) \quad \Delta = \text{Var} \cup (\Theta)^0.$$

The minimal solution of this pair of recursive equations is defined next.

Definition 7. The sets of λ -terms $\mathcal{Y}_i, \Phi_i$ ($i \in \mathbb{N}$) are defined by mutual induction, as follows

$$\begin{aligned} \mathcal{Y}_0 &= \text{Var} \\ \Phi_i &= \text{Var} \cup (\mathcal{Y}_i)^0 \\ \mathcal{Y}_{i+1} &= \text{Var} \cup \{xM_1 \dots M_n \mid M_k \in \mathcal{Y}_i (1 \leq k \leq n)\} \cup \{\lambda \vec{x}.M \mid M \in \mathcal{Y}_i\} \\ &\quad \cup \left\{ (\lambda x.P)Q M_1 \dots M_n \mid \begin{array}{l} Q \in \mathcal{Y}_i - (\Lambda^0 \cup \text{Var}), \quad M_1, \dots, M_n \in \mathcal{Y}_i, \\ P[Q/x]M_1 \dots M_n \rightarrow_{\Phi_i}^* R \in \mathcal{Y}_i \end{array} \right\} \end{aligned}$$

Moreover, we define $\mathcal{Y} = \cup_i \mathcal{Y}_i$ and $\Phi = \text{Var} \cup (\mathcal{Y})^0$.

For example, $\Phi_0 = \text{Var}$, $\mathcal{Y}_1 = \text{Var} \cup \{xy_1 \dots y_n \mid y_i \in \text{Var}\} \cup \{\lambda \vec{x}.y \mid y \in \text{Var}\}$ and $\Phi_1 = \text{Var} \cup \{\lambda x_1 \dots x_m.x_j \mid 1 \leq j \leq m\}$.

The following result holds trivially:

- Lemma 8.** (i) For all $i \in \mathbb{N}$, Φ_i is a set of input value.
(ii) $\Phi = \cup_i \Phi_i$.
(iii) Φ is a set of input values.
(iv) For all $i \in \mathbb{N}$, $\mathcal{Y}$ and $\mathcal{Y}_i$ are not sets of input values.

Also, it is easy to check that the following properties hold.

- Property 9.* (i) $\mathcal{Y}_i \subset \mathcal{Y}_{i+1}$, $\Phi_i \subset \Phi_{i+1}$ and $\rightarrow_{\Phi_i} \subseteq \rightarrow_{\Phi_{i+1}}$, for all $i \in \mathbb{N}$;
(ii) $\mathcal{Y}_i \subset \mathcal{Y}$, $\Phi_i \subset \Phi$ and $\rightarrow_{\Phi_i} \subseteq \rightarrow_{\Phi}$, for all $i \in \mathbb{N}$;
(iii) $\mathcal{Y}^0 = \Phi^0$;
(iv) $M \in \Phi^0$ implies $M \equiv \lambda z.P$, for some $z \in \text{Var}$ and $P \in \mathcal{Y}$ (note that $\Phi \subseteq \mathcal{Y}$);
(v) $\Phi \subseteq \mathcal{Y}$ and $\mathcal{Y} \subseteq \Phi\text{-NF}$;
(vi) $\Phi\text{-NF} \not\subseteq \mathcal{Y}$ and $\Phi\text{-NF} \not\subseteq \Phi$.

Proof. (vi) Let $M \equiv \lambda z.(\lambda x.D)(zI)D$. $M \in \Phi\text{-NF}$ since $zI \notin \Phi$. But $M \notin \mathcal{Y}$ and $M \notin \Phi$. $\square$

Lemma 8.(iii) implies that the $\lambda\Phi$ -calculus enjoys the confluence property. Moreover it is possible to check that it also satisfies the additional necessary condition for standardization, stated in [10, 13].

4 The Main Result

The $\lambda\Phi$ -calculus, besides confluence and standardization, has some further interesting properties. The most important one is that the set of potentially Φ -valuable terms coincides with the set of strongly Λ -normalizing terms. Other properties characterize completely the operational behavior of the $\lambda\Phi$ -calculus. In particular, a term is Φ -solvable if and only if it is potentially Φ -valuable if and only if it Φ -reduces to a term in $\mathcal{Y}$.

Theorem 10 (Main Theorem). The following statements are equivalent:

- (i) M is strongly Λ -normalizing;
- (ii) M is Φ -solvable;
- (iii) M is potentially Φ -valuable;
- (iv) $M \rightarrow_{\Phi}^* R \in \mathcal{Y}$;

Proof. (i) implies (ii) by Theorem 26.

(ii) implies (iii) by Theorem 16.(ii).

(iii) implies (iv) by Theorem 16.(i).

(iv) implies (i) by Theorem 20.

where theorems 16, 20 and 26 will be proved later in this Section. $\square$

Notice that Φ is a *proper* subset of the set of strongly Λ -normalizing terms. In fact, a strongly Λ -normalizing term of the shape $M_1 \dots M_n$ where M_i is closed ($1 \leq i \leq n$) does not belong to Φ . Moreover Φ is minimal between all sets of input values which answer our first question.

Property 11. Let Δ^* be a set of input values.

If the set of potentially Δ^* -valuable terms coincides with the set of the strongly Λ -normalizing terms and $\Delta^* \subset \Phi$ then $\Delta^* = \Phi$.

Proof. Clearly $\Delta^* = \Phi$ if and only if $(\Delta^*)^0 = \Phi^0$, since $\Delta^* \subset \Phi$ and $\Phi = \text{Var} \cup \Phi^0$. Thus suppose $M \in \Phi^0$. Note that M is Φ -valuable, potentially Φ -valuable and also in Φ -normal form. Moreover M is a closed strongly Λ -normalizing term, by the Main Theorem. Thus, M is potentially Δ^* -valuable by hypothesis and $M \in \Lambda^0$ implies that M is Δ^* -valuable. But $\Delta^* \subset \Phi$ implies $\Phi\text{-NF} \subset \Delta^*\text{-NF}$, hence $M \in \Delta^*\text{-NF}$. Then $M \in \Delta^*$ and the proof is done. $\square$

Finally, it is worthy to say that, although Φ is minimal, it is not *the* minimum set supplying an answer to questions stated in the introduction. In fact, the minimum solution to the following equations:

$$\begin{aligned} \Theta &= \{ \lambda x_0 \dots x_n. y \mid y \neq x_i \ (0 \leq i \leq n) \} \cup \\ &\quad \{ x M_1 \dots M_n \mid M_k \in \Theta \ (1 \leq k \leq n) \} \cup \{ \lambda \vec{x}. M \mid M \in \Theta \} \cup \\ &\quad \{ (\lambda x. P) Q M_1 \dots M_n \mid Q, M_1, \dots, M_n \in \Theta, Q \notin \Delta, P[Q/x] M_1 \dots M_n \rightarrow^*_{\Delta} R \in \Theta \} \\ \Delta &= \{ \lambda x_0 \dots x_n. y \mid y \neq x_i \ (0 \leq i \leq n) \} \cup (\Theta)^0 \end{aligned}$$

also answers our questions and it is not comparable with Φ .

The rest of the paper is devoted to the proof of the Main Theorem.

4.1 Potential Φ -Valuability and Φ -Solvability

First of all, we will introduce the *weight* of a terms, as measure for carrying out some inductive proofs. The weight of a term M is an upper bound to the number of symbols of M , to the length of its leftmost Λ -reduction sequence and to the length of its Φ -reduction sequence according to the standard strategy [10].

Definition 12. The weight $\langle _ \rangle : \Lambda \longrightarrow \mathbb{N}$ is the partial function defined as follows:

- $\langle \lambda x. M' \rangle = 1 + \langle M' \rangle$.
- $\langle x M_1 \dots M_m \rangle = 1 + \langle M_1 \rangle + \dots + \langle M_m \rangle$.
- $\langle (\lambda x. M_0) M_1 \dots M_m \rangle = 1 + \langle M_1 \rangle + \langle M_0[M_1/x] M_2 \dots M_m \rangle$.

As examples $\langle x \rangle = 1$, $\langle xx \rangle = 2$, $\langle \lambda x. xx \rangle = 3$. It is easy to check that $M \rightarrow_{\Phi} N$ implies $M[P/z] \rightarrow_{\Phi} N[P/z]$, if $P \in \Phi$.

Lemma 13. (i) If $M \in \mathcal{Y}$ then $\langle M \rangle \in \mathbb{N}$.

(ii) If $M \rightarrow_{\Phi}^+ N$ and $\langle N \rangle \in \mathbb{N}$ then, both $\langle M \rangle \in \mathbb{N}$ and $\langle N \rangle < \langle M \rangle$.

(iii) If $M \rightarrow_{\Phi}^+ N$ and $\langle M \rangle \in \mathbb{N}$ then, both $\langle N \rangle \in \mathbb{N}$ and $\langle N \rangle < \langle M \rangle$.

Proof. (i) The proof can be done by induction on the $\mathcal{Y}$ stratification.

(ii) The proof is given by induction on $\langle N \rangle$.

– If $M \equiv \lambda x.P \rightarrow_{\Phi}^+ \lambda x.P' \equiv N$ then $\langle P' \rangle \in \mathbb{N}$ implies $\langle P' \rangle < \langle P \rangle \in \mathbb{N}$ by induction, hence $\langle N \rangle < 1 + \langle P \rangle = \langle M \rangle$.

– Let $M \equiv xM_1 \dots M_m \rightarrow_{\Phi}^+ xN_1 \dots N_m \equiv N$ ($m \geq 1$) where either $M_k \rightarrow_{\Phi}^+ N_k$ or $M_k \equiv N_k$ ($1 \leq k \leq m$). Note that there is at least one $h \in \mathbb{N}$ such that $M_h \rightarrow_{\Phi}^+ N_h$ and $\langle N_h \rangle < \langle M_h \rangle$ ($1 \leq h, k \leq m$). Thus the proof follows easily by induction.

– Let $M \equiv (\lambda z.M_0)M_1 \dots M_m \rightarrow_{\Phi}^+ N$ for some $m \geq 1$.

Either $M \rightarrow_{\Phi}^* (\lambda x.N_0)N_1 \dots N_m \rightarrow_{\Phi} N_0[N_1/x]N_1 \dots N_m \rightarrow_{\Phi}^* N$ or $M \rightarrow_{\Phi}^+ (\lambda x.N_0)N_1 \dots N_m \equiv N$, where $M_k \rightarrow_{\Phi}^+ N_k$ or $M_k \equiv N_k$ ($1 \leq k \leq m$). In all cases, the proof follows by induction.

(iii) The proof is given by induction on $\langle M \rangle$.

– If $M \equiv \lambda x.M_0 \rightarrow_{\Phi}^+ \lambda x.N_0 \equiv N$ then $\langle M_0 \rangle \in \mathbb{N}$. Hence $\langle N_0 \rangle < \langle M_0 \rangle$ by induction, thus $\langle N \rangle = 1 + \langle N_0 \rangle < 1 + \langle M_0 \rangle = \langle M \rangle$.

– Let $M \equiv xM_1 \dots M_m \rightarrow_{\Phi}^+ xN_1 \dots N_m \equiv N$ ($m \geq 1$), where either $M_k \rightarrow_{\Phi}^+ N_k$ or $M_k \equiv N_k$ ($1 \leq k \leq m$). Note that there is $h \in \mathbb{N}$ such that $M_h \rightarrow_{\Phi}^+ N_h$ and $\langle N_h \rangle < \langle M_h \rangle$. Thus the proof follows easily by induction.

– Let $M \equiv (\lambda z.M_0)M_1 \dots M_m \rightarrow_{\Phi}^+ N$ ($m \geq 1$).

Either $M \rightarrow_{\Phi}^* (\lambda x.N_0)N_1 \dots N_m \rightarrow_{\Phi} N_0[N_1/x]N_1 \dots N_m \rightarrow_{\Phi}^* N$ or $M \rightarrow_{\Phi}^+ (\lambda x.N_0)N_1 \dots N_m \equiv N$, where $M_k \rightarrow_{\Phi}^+ N_k$ or $M_k \equiv N_k$ ($1 \leq k \leq m$).

If $M_1 \rightarrow_{\Phi}^+ N_1$ then $\langle N_1 \rangle < \langle M_1 \rangle$ and the proof follows by induction.

Otherwise $M_1 \equiv N_1$ and $M_0[M_1/x]M_2 \dots M_m \rightarrow_{\Phi}^+ N_0[M_1/x]N_2 \dots N_m$, hence $\langle M_0[M_1/x]M_2 \dots M_m \rangle < \langle N_0[M_1/x]N_2 \dots N_m \rangle$ and the proof follows immediately, in all cases. $\square$

It is possible to characterize the terms for which the weight is defined.

Corollary 14. $\langle M \rangle$ is defined if and only if $M \rightarrow_{\Phi}^* R$, for some $R \in \mathcal{Y}$.

Proof. $\Leftarrow$ The proof follows by Lemma 13.(i) and Lemma 13.(ii).

$\Rightarrow$ The proof is given by induction on $\langle M \rangle$.

If $M \equiv \lambda x.M_0$ or $M \equiv xM_1 \dots M_m$ ($m \in \mathbb{N}$) then the proof follows by induction. Let $M \equiv (\lambda z.M_0)M_1 \dots M_m$ ($m \geq 1$), so $\langle M_1 \rangle < \langle M \rangle$ and by induction $M_1 \rightarrow_{\Phi}^* S \in \mathcal{Y}$.

– If $M \in \Phi\text{-NF}$ then $M_1 \in \mathcal{Y}$, but $M_1 \notin \Phi = \mathcal{Y}^0 \cup \text{Var}$. Furthermore $\langle M_0[M_1/x]M_2 \dots M_m \rangle < 1 + \langle M_1 \rangle + \langle M_0[M_1/x]M_2 \dots M_m \rangle = \langle M \rangle$ implies that $M_0[M_1/x]M_2 \dots M_m \rightarrow_{\Phi}^* R'$, for some $R' \in \mathcal{Y}$. The proof follows by definition of $\mathcal{Y}$.

– Otherwise $M \rightarrow_{\Phi}^+ N$, so the proof follows by Lemma 13.(iii) and by induction. $\square$

If the weight of a term is defined, then the weight of all its subterms is also defined. The next lemma proves this statement in some particular cases.

Lemma 15. (i) If $M[N/z] \rightarrow_{\Phi}^* R \in \mathcal{Y}$ then $M \rightarrow_{\Phi}^* S \in \mathcal{Y}$.
(ii) If $MN \rightarrow_{\Phi}^* R \in \mathcal{Y}$ then $M \rightarrow_{\Phi}^* S \in \mathcal{Y}$.

Proof. (i) We will prove that $\langle M[N/z] \rangle \in \mathbb{N}$ implies $\langle M \rangle \in \mathbb{N}$ by induction on $h = \langle M[N/z] \rangle$, thus the proof follows by Corollary 14.

Let $M \equiv \lambda x.M_0$. $\langle M_0[N/z] \rangle \in \mathbb{N}$ implies $\langle M_0 \rangle < h$ by induction and the proof follows. If $M \equiv xM_1 \dots M_m$ ($m \geq 1$) then, in all cases, $\langle M_i[N/z] \rangle < h$ ($1 \leq i \leq m$) and the proof follows by induction. If $M \equiv (\lambda x.M_0)M_1 \dots M_m$ ($m \geq 1$) then $\langle M[N/z] \rangle = 1 + \langle M_1[N/z] \rangle + \langle (M_0[M_1/x]M_2 \dots M_m)[N/z] \rangle$. Thus $\langle M_1 \rangle$ and $\langle M_0[M_1/x]M_2 \dots M_m \rangle$ are defined by induction and the proof follows.

(ii) We will prove that $\langle MN \rangle \in \mathbb{N}$ implies $\langle M \rangle \in \mathbb{N}$ by induction on $\langle MN \rangle$, thus the proof follows by Corollary 14.

If $M \equiv \lambda x.M_0$ then $\langle MN \rangle = 1 + \langle N \rangle + \langle M_0[N/z] \rangle$, so $\langle M_0[N/z] \rangle \in \mathbb{N}$ and the proof follows by the previous point of this Theorem and the definition of weight. The other cases are easier. $\square$

Theorem 16. (i) M is potentially Φ -valuable implies that $M \rightarrow_{\Phi}^* R \in \mathcal{Y}$, for some $R \in \mathcal{Y}$.

(ii) M is Φ -solvable implies that M is potentially Φ -valuable.

Proof. (i) M is potentially Φ -valuable means that there is a substitution $\mathbf{s}$, replacing variables by closed terms belonging to Φ , such that $\mathbf{s}(M) \rightarrow_{\Phi}^* N \in \Phi$. Since $\Phi \subseteq \mathcal{Y}$, the proof follows by Lemma 15(i).

(ii) M is Φ -solvable means that there is a Φ -valuable context $C[\cdot] \equiv (\lambda \vec{x}[\cdot])\vec{N}$ such that $C[M] \rightarrow_{\Phi}^* I$. Since $C[M] \rightarrow_{\Phi}^* I$ implies $C[M]I \dots I \rightarrow_{\Phi}^* I$, we can assume $\|\vec{x}\| \leq \|\vec{N}\|$ without loss of generality. Moreover $C[M] \rightarrow_{\Phi}^* I$ implies $C[M][N/z] \rightarrow_{\Phi}^* I \equiv I[N/z]$ for all $N \in \Phi^0$, so we can also assume that $C[M] \in \Lambda^0$.

If $\vec{N} \equiv \vec{N}_0 \vec{N}_1$ and $\|\vec{x}\| = \|\vec{N}_0\|$ then $C[M] \rightarrow_{\Phi}^* M[\vec{N}_0/\vec{x}]\vec{N}_1 \rightarrow_{\Phi}^* I \in \Phi$, thus $M[\vec{N}_0/\vec{x}] \rightarrow_{\Phi}^* S' \in \mathcal{Y}^0$, by Lemma 15.(ii). The proof is done, since $\mathcal{Y}^0 = \Phi^0$. $\square$

4.2 Strong Λ -Normalization and Φ -Reduction

In order to prove both that the terms strongly Λ -normalizing are also Φ -solvable and that terms which Φ -reduce to an element of $\mathcal{Y}$ are strongly Λ -normalizing, we will use an intersection type assignment system [1, 4] that types all and only the strongly Λ -normalizing terms [5, 12].

Definition 17. (i) Let $\mathcal{C}$ be a countable set of type-constants (ranging over $\alpha, \beta, \dots$). The set $T(\mathcal{C})$ of types, ranging over by $\sigma, \tau, \pi, \rho, \dots$ is inductively defined as follows:

$$\begin{aligned} \sigma \in \mathcal{C} &\Rightarrow \sigma \in T(\mathcal{C}) \\ \sigma, \tau \in T(\mathcal{C}) &\Rightarrow (\sigma \rightarrow \tau) \in T(\mathcal{C}) \\ \sigma, \tau \in T(\mathcal{C}) &\Rightarrow (\sigma \wedge \tau) \in T(\mathcal{C}). \end{aligned}$$

We use the convention that the constructor $\wedge$ take precedence over $\rightarrow$.

- (ii) A basis is a partial function from Var to $T(\mathbb{C})$ having a finite domain of definition. If B is a basis then $B[\sigma/x]$ denotes the basis such that

$$B[\sigma/x](y) = \begin{cases} \sigma & \text{if } y \equiv x, \\ B(y) & \text{otherwise.} \end{cases}$$

Furthermore, the basis B such that $\text{dom}(B) = \{x_1, \dots, x_n\}$ and $B(x_i) = \sigma_i$, for $1 \leq i \leq n$ will be often denoted by $[\sigma_1/x_1, \dots, \sigma_n/x_n]$.

- (iii) The type assignment system $\vdash$ is a formal system proving typing judgments of the shape:

$$B \vdash M : \sigma$$

where M is a term, $\sigma \in T(\mathbb{C})$ and B is a basis.

The type assignment system $\vdash$ consists of the following rules:

$$\begin{array}{c} \frac{}{B[\sigma/x] \vdash x : \sigma} \text{ (var)} \\[10pt] \frac{B[\sigma/x] \vdash M : \tau}{B \vdash \lambda x.M : \sigma \rightarrow \tau} (\rightarrow I) \quad \frac{B \vdash M : \sigma \rightarrow \tau \quad B \vdash N : \sigma}{B \vdash MN : \tau} (\rightarrow E) \\[10pt] \frac{B \vdash M : \sigma \quad B \vdash M : \tau}{B \vdash M : \sigma \wedge \tau} (\wedge I) \\[10pt] \frac{B \vdash M : \sigma \wedge \tau}{B \vdash M : \sigma} (\wedge E_l) \quad \frac{B \vdash M : \sigma \wedge \tau}{B \vdash M : \tau} (\wedge E_r) \end{array}$$

If B, B' are bases then $B \cap B'$ is the basis defined as follows:

$$(B \cap B')(y) = \begin{cases} B(y) \wedge B'(y) & \text{if both } B(y) \text{ and } B'(y) \text{ are defined,} \\ B(y) & \text{if } B(y) \text{ is defined and } B'(y) \text{ is undefined,} \\ B'(y) & \text{if } B'(y) \text{ is defined and } B(y) \text{ is undefined,} \\ \text{undefined} & \text{otherwise.} \end{cases}$$

The type systems $\vdash$ enjoys the subject-reduction property and a restricted form of subject-expansion.

Property 18 (Subject-Reduction and Typed Subject-Expansion).

- (i) If $B \vdash M : \sigma$ and $M \rightarrow_A N$ then $B \vdash N : \sigma$.
 (ii) Let $C[\cdot]$ be a context. Then $B \vdash C[P[Q/x]] : \sigma$ and $B' \vdash Q : \tau$ imply $B \cap B' \vdash C[(\lambda x.P)Q] : \sigma$.

Proof. See [4]. □

Lemma 19. *If $M \in \mathcal{T}$ then $B \vdash M : \sigma$, for some basis B and $\sigma \in T(\mathcal{C})$.*

Proof. The proof is given by induction on the stratification of $\mathcal{T}$. The case $M \in \mathcal{T}_0$ is trivial. If $M \in \mathcal{T}_{i+1}$, the cases $M \equiv xM_1 \dots M_m$ and $M \equiv \lambda x.P$ follow easily from the inductive hypothesis, using the rules of the system. If $M \equiv (\lambda x.P)QM_0 \dots M_m$ then both $Q \in \mathcal{T}_i - (\Lambda^0 \cup \text{Var})$ and $P[Q/x]M_1 \dots M_m \rightarrow_{\Phi_i}^* R \in \mathcal{T}_i$. Therefore $B \vdash R : \sigma$, by induction. Since the $\rightarrow_{\Phi_i}^*$ reduction reduces only in case the argument belongs to Φ_i this can be typed by induction, and hence $B \vdash P[Q/x]M_1 \dots M_m : \sigma$ by Property 18.(ii). Since $B' \vdash Q : \tau$ by induction, the proof follows by Property 18.(ii). $\square$

Theorem 20. *$M \rightarrow_{\Phi}^* R \in \mathcal{T}$ implies M is Λ -strongly normalizing.*

Proof. By Lemma 19, $B \vdash R : \sigma$ for some basis B and $\sigma \in T(\mathcal{C})$. Thus $B \vdash M : \sigma$ by Property 18.(ii). Then the proof follows from the fact that the system characterizes the strongly Λ -normalizing terms [5, 12]. $\square$

4.3 Strong Λ -Normalization and Φ -Solvability

In order to prove that Λ -strong normalization implies Φ -solvability, we will use a computability argument, which is an adaptation to intersection types of the reducibility candidates method [3].

Let M be a term, let $\text{FV}(M) \subseteq \{x_1, \dots, x_m\}$ for some $m \geq 0$ and let O^i be $\lambda x_0 \dots x_i. x_i$ for all $i \in \mathbb{N}$.

The meaning of $\mathcal{P}(M)$ will be: “there is $r \in \mathbb{N}$ such that, for all $h, k \geq r$,

$$M[O^h/x_1, \dots, O^h/x_m] \underbrace{O^h \dots O^h}_k \rightarrow_{\Phi}^* O^s \text{ for some } s \in \mathbb{N}.”$$

Property 21. (i) $\mathcal{P}(M)$ implies M is Φ -solvable.

(ii) $\mathcal{P}(x\vec{M})$ and $\mathcal{P}(N)$ imply $\mathcal{P}(x\vec{M}N)$.

Proof. (i) $M[O^h/x_1, \dots, O^h/x_m] \underbrace{O^h \dots O^h}_k \rightarrow_{\Phi}^* O^s$ implies that

$$M[O^h/x_1, \dots, O^h/x_m] \underbrace{O^h \dots O^h}_k \underbrace{I \dots I}_s \rightarrow_{\Phi}^* I.$$

(ii) Let $\vec{M} \equiv M_1 \dots M_n$ ($n \in \mathbb{N}$) and let $\text{FV}(x\vec{M}N) \subseteq \{x_1, \dots, x_m\}$ ($m \in \mathbb{N}$).

$$\exists r_0 \in \mathbb{N}, \forall h_0, k_0 \geq r_0, \exists s_0 \in \mathbb{N}, x\vec{M}[O^{h_0}/x_1, \dots, O^{h_0}/x_m] \underbrace{O^{h_0} \dots O^{h_0}}_{k_0} \rightarrow_{\Phi}^* O^{s_0},$$

$$\exists r_1 \in \mathbb{N}, \forall h_1, k_1 \geq r_1, \exists s_1 \in \mathbb{N}, N[O^{h_1}/x_1, \dots, O^{h_1}/x_m] \underbrace{O^{h_1} \dots O^{h_1}}_{k_1} \rightarrow_{\Phi}^* O^{s_1},$$

by hypothesis. So the proof follows by putting $r = \max\{r_0, r_1, n + 1\}$. $\square$

The predicate $\mathcal{P}$ is used to define the computability predicate.

Definition 22. The predicate *Comp* is defined by induction on types as follows:

- *Comp*(B, α, M) if and only if $\mathcal{P}(M)$, $\alpha \in \mathbb{C}$ and B is a basis;
- *Comp*($B, \sigma \rightarrow \tau, M$) if and only if, for all $N \in \Lambda$, *Comp*(B', σ, N) implies *Comp*($B \cap B', \tau, MN$);
- *Comp*($B, \sigma \wedge \tau, M$) if and only if *Comp*(B, σ, M) and *Comp*(B, τ, M).

We prove that $B \vdash M : \sigma$ implies *Comp*(B, σ, M), which in its turn implies $\mathcal{P}(M)$. It is easy to check that *Comp*(B, σ, M) does not imply $B \vdash M : \sigma$.

Lemma 23. (i) $\mathcal{P}(x\vec{M})$ implies *Comp*($B, \sigma, x\vec{M}$), for all B and $\sigma \in T(\mathbb{C})$.
 (ii) *Comp*(B, σ, M) implies $\mathcal{P}(M)$, for all B and $\sigma \in T(\mathbb{C})$.

Proof. The proof is by mutual induction on σ . The only case which is not obvious is when $\sigma = \tau \rightarrow \rho$.

- (i) We will prove that *Comp*(B', τ, N) implies *Comp*($B \cap B', \rho, x\vec{M}N$), thus *Comp*($B, \tau \rightarrow \rho, x\vec{M}$) follows by definition. *Comp*(B', τ, N) implies $\mathcal{P}(N)$, by induction. But $\mathcal{P}(x\vec{M})$ by hypothesis, thus $\mathcal{P}(x\vec{M}N)$ by Property 21.(ii).
- (ii) $\mathcal{P}(z)$ holds so, in particular, *Comp*(B', τ, z) holds by induction on (i). Thus *Comp*($B \cap B', \rho, Mz$) by definition of *Comp* and this implies $\mathcal{P}(Mz)$ by induction. That is, there is $r \in \mathbb{N}$ such that, for all $h, k \geq r$,

$$Mz[O^h/z, O^h/x_1, \dots, O^h/x_m] \underbrace{O^h \dots O^h}_k \rightarrow_{\Phi}^* O^s$$

where $s \in \mathbb{N}$, $O^h \equiv \lambda x_0 \dots x_h. x_h$ and $\text{FV}(Mz) \subseteq \{z, x_1, \dots, x_m\}$ ($m \geq 0$). So, if $r' = r + 1$, for all $h, k \geq r'$, $M[O^h/x_1, \dots, O^h/x_m] \underbrace{O^h \dots O^h}_k \rightarrow_{\Phi}^* O^s$, for

some s . Hence $\mathcal{P}(M)$ holds. $\square$

Property 24. Let Q be such that $\mathcal{P}(Q)$.

If *Comp*($B, \sigma, P[Q/x]\vec{Q}$) then *Comp*($B, \sigma, (\lambda x.P)Q\vec{Q}$).

Proof. The proof is by induction on the structure of types.

Assume $\sigma \in \mathbb{C}$. Note that $\mathcal{P}(Q)$ and Lemma 15.(i) imply

$$\exists r_0 \in \mathbb{N}, \forall h_0, k_0 \geq r_0, Q[O^{h_0}/x_1, \dots, O^{h_0}/x_m] \rightarrow_{\Phi}^* M \in \Upsilon^0 = \Phi^0.$$

Moreover, *Comp*($B, \sigma, P[Q/x]\vec{Q}$) implies $\mathcal{P}(P[Q/x]\vec{Q})$ by definition, thus

$$\exists r_1, \forall h_1, k_1 \geq r_1, \exists s_1, (P[Q/x]\vec{Q})[O^{h_1}/x_1, \dots, O^{h_1}/x_m] \underbrace{O^{h_1} \dots O^{h_1}}_{k_1} \rightarrow_{\Phi}^* O^{s_1}.$$

Let $r = \max\{r_0, r_1\}$. Then, $\forall h, k \geq r$, $((\lambda x.P)Q\vec{Q})[O^h/x_1, \dots, O^h/x_m] \underbrace{O^h \dots O^h}_k$

$$=_{\Phi} (P[Q/x]\vec{Q})[O^h/x_1, \dots, O^h/x_m] \underbrace{O^h \dots O^h}_k \rightarrow_{\Phi}^* O^s, \text{ for some } s \in \mathbb{N}.$$

Hence $\mathcal{P}(Q)$ and $\sigma \in \mathbb{C}$ imply *Comp*($B, \sigma, (\lambda x.P)Q\vec{Q}$).

Let $\sigma = \tau \rightarrow \rho$. Thus $Comp(B, \tau \rightarrow \rho, P[Q/x]\vec{Q})$ implies that $\forall N$ such that $Comp(B', \tau, N)$, $Comp(B \cap B', \rho, P[Q/x]\vec{Q}N)$ holds. Therefore by induction, $Comp(B \cap B', \rho, (\lambda x.P)Q\vec{Q}N)$ and hence $Comp(B \cap B', \tau \rightarrow \rho, (\lambda x.P)Q\vec{Q})$ by definition of $Comp$. The case $\sigma = \tau \wedge \rho$ is trivial. $\square$

Lemma 25. *Let $FV(M) \subseteq \{x_1, \dots, x_n\}$ and $B(x_i) = \sigma_i$ ($1 \leq i \leq n$). If $Comp(B_i, \sigma_i, N_i)$ ($1 \leq i \leq n$) and $B \vdash M : \tau$, then*

$$Comp(B_1 \cap \dots \cap B_n, \tau, M[N_1/x_1, \dots, N_n/x_n]).$$

Proof. By induction on the derivation d of $B \vdash M : \tau$. The most interesting case is when the last rule applied on d is $(\rightarrow I)$. Let $M \equiv \lambda x.M'$, $\tau = \mu \rightarrow \rho$ and

$$\frac{B[\mu/x] \vdash M' : \rho}{B \vdash \lambda x.M' : \mu \rightarrow \rho} (\rightarrow I)$$

Suppose that $Comp(B', \mu, N)$ holds. Then $\mathcal{P}(N)$ holds by Lemma 23.(ii). Thus, by induction

$$Comp(B' \cap B_1 \cap \dots \cap B_n, \rho, M'[N_1/x_1, \dots, N_n/x_n, N/x])$$

and $Comp(B' \cap B_1 \cap \dots \cap B_n, \rho, (\lambda x.M'[N_1/x_1, \dots, N_n/x_n])N)$ by Property 24. Hence, $Comp(B_1 \cap \dots \cap B_n, \mu \rightarrow \rho, M[N_1/x_1, \dots, N_n/x_n])$ by definition of $Comp$. All other cases follow directly from the inductive hypothesis. $\square$

Theorem 26. *M is Λ -strongly normalizing implies that M is Φ -solvable.*

Proof. In [12, 5] it is proved that the system $\vdash$ characterizes the strongly Λ -normalizing terms. So, let $B \vdash M : \sigma$, $FV(M) \subseteq \{x_1, \dots, x_n\}$ and $B(x_i) = \sigma_i$. Since $Comp(B, \sigma_i, x_i)$ ($1 \leq i \leq n$) by Lemma 23.(i), then $Comp(B, \sigma, M)$ by Lemma 25. Thus $\mathcal{P}(M)$ by Lemma 23.(ii). The proof follows by Property 21.(i). $\square$

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