Rough Contraction Through Partial Meets

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Abstract. The paper addresses the problem of constructing rough belief change functions. Contraction and revision postulates using the notion of *rough consequence* and *rough consistency* have been proposed in [2]. The base logic is a reasoning framework $\mathcal{L}_{\mathcal{R}}$, that has a semantics of rough truth. We demonstrate here that functions satisfying the contraction postulates can be constructed through the method of *partial meets* [1]. As a result, a construction of rough revision functions is also obtained.

1 Introduction

Belief revision models the changes in beliefs of a person when some new information is provided or some earlier information has to be retracted. This kind of study helps in building intelligent agents, who have to manage information about the real world in order to achieve their goals. Beliefs may not represent complete or consistent information in all contexts, so one should be able to revise them when new correct or complete information is acquired. Thus a fundamental issue in belief revision is to decide upon the beliefs which should to be retracted in order to avoid contradiction.

The problem of belief change is well studied in the classical background, where any statement is either true or false – there are well-defined functions which model belief change. But clearly, there is need for appropriate methodology to deal with belief change in situations where information cannot be directly classified into a ‘reject’ or ‘accept’ category. Our idea is to be able to express through the base logic, vague/rough statements, and incorporate their properties in belief change. For instance, it is observed that a rough belief and its negation may not lead to a contradiction – motivating a relaxation in the definition of ‘contradiction’. In this article, we further investigate an instance of *paraconsistent* belief change that is based on the logic $\mathcal{L}_{\mathcal{R}}$ of rough truth [2]. $\mathcal{L}_{\mathcal{R}}$ is, in fact, an alternative formulation of the well-known paraconsistent logic J due to Jaśkowski. It should be mentioned that one of the first studies in paraconsistent belief change is presented in [7], where a 4-valued base logic is used. Other work in the direction can be found in [6,9]. A comprehensive study on belief change in 3-valued and rough logics has been recently done in [8].

Postulates that rough belief change functions should satisfy, have been proposed in [2]. In this work, we show that rough contraction functions can be constructed through the method of *partial meets* [1]. Functions are defined using

maximal subsets of belief sets that *do not* imply a belief A , and it is proved that they satisfy the rough contraction postulates with respect to A . The next section gives the requisite preliminaries. Section 3 presents the results on partial meet contraction in the rough context. Section 4 concludes the article.

2 Preliminaries

2.1 Partial Meet Contraction in Classical Belief Change

The base consequence relation $\vdash$ or operator Cn is assumed to be supraclassical (including classical consequence), and to satisfy cut, deduction theorem, monotonicity and compactness [4]. A set K of well-formed formulae (wffs) is a *belief set*, if $K = Cn(K)$. The three functions modelling belief change are *expansion*, *revision* and *contraction*. Incoming information may produce new beliefs (which may also lead to inconsistencies). Given a belief set K and a belief A , the expansion $K + A$ of K with respect to A is $Cn(K \cup \{A\})$. Sets of postulates [1,4] regulate revision and contraction functions to give belief sets $K * A$, $K - A$, revised and contracted with respect to a belief A respectively. The contraction and revision functions can be interrelated through the *Levi* and *Harper* identities.

One of the methods of constructing a contraction function is through *partial meets*. The central idea is that a reasonable $K - A$ must not contain A . One considers *maximal* subsets m of K not implying A . Formally, m is such that

- (i) $m \subseteq K$,
- (ii) $A \notin Cn(m)$, and
- (iii) for any m' such that $m \subset m' \subseteq K$, $A \in Cn(m')$.

Let $K \perp A$ denote the collection of such sets m , and $M(K) = \bigcup_{A \in K} K \perp A$. Note that if $A \notin K$ then we take $K \perp A = \{K\}$, and if A is a theorem then $K \perp A = \emptyset$. A contraction $K - A$ is then the intersection of some ‘preferred’ elements of $K \perp A$. Let S denote the selection function for choosing these elements, i.e. $S(K \perp A) \subseteq K \perp A$. A contraction function can then be defined as:

$$K - A = \begin{cases} \bigcap S(K \perp A) & \text{if } \not\vdash A \\ K & \text{otherwise.} \end{cases} \quad (1)$$

It turns out that the most appropriate S is the one picking the top elements in $K \perp A$, based on a reflexive and transitive preference relation $\geq$ over the maximal subsets in $K \perp A$, i.e.

$$S(K \perp A) = \{m \in K \perp A : m \geq m', \text{ for all } m' \in K \perp A\}. \quad (2)$$

2.2 $\mathcal{L}_{\mathcal{R}}$

The notion of rough truth was introduced in [5] to reflect ‘inductive’ truth, that, with gain of knowledge, leads to total, deductive truth.

The language of the system $\mathcal{L}_{\mathcal{R}}$ is that of a normal modal propositional logic. $\mathcal{L}_{\mathcal{R}}$ is, in fact, based on the modal system S_5 . Let us notice that an S_5 (Kripke)

model $\mathcal{M} \equiv (X, R, \pi)$ is essentially an *approximation space* (X, R) , where $X \neq \emptyset$, with the function π interpreting every wff of S_5 as a rough set in (X, R) . If L, M denote the necessity and possibility connectives respectively, a modal wff $L\alpha(M\alpha)$, representing ‘definitely’ (‘possibly’) α , is interpreted by π as the lower (upper) approximation $\pi(\alpha) \equiv \{x \in X : x' \in X, \text{ for all } x' \text{ such that } xRx'\}$ ($\overline{\pi(\alpha)} \equiv (\pi(\alpha)^c)^c$) of the set $\pi(\alpha)$.

A wff α is termed *roughly true* in $\mathcal{M}$, if $\overline{\pi(\alpha)} = X$. The notion of rough truth can be extended to *rough validity*. In the following, Γ is any set of wffs.

Definition 1. An S_5 -model $\mathcal{M} \equiv (X, R, \pi)$ is a rough model of Γ , if and only if every member γ of Γ is roughly true in $\mathcal{M}$, i.e. $\overline{\pi(\gamma)} = X$.

Definition 2. α is a rough semantic consequence of Γ (denoted $\Gamma \models \alpha$) if and only if every rough model of Γ is a rough model of α . If Γ is empty, α is said to be roughly valid, written $\models \alpha$.

$\mathcal{L}_{\mathcal{R}}$ is defined through the *rough consequence* relation.

Definition 3. α is a rough consequence of Γ (denoted $\Gamma \vdash \alpha$) if and only if there is a sequence $\alpha_1, \dots, \alpha_n (\equiv \alpha)$ such that each $\alpha_i (i = 1, \dots, n)$ is either (i) a theorem of S_5 , or (ii) a member of Γ , or (iii) derived from some of $\alpha_1, \dots, \alpha_{i-1}$ by R_1 or R_2 given below. If Γ is empty, α is said to be a rough theorem, written $\vdash \alpha$.

$$\begin{array}{ll}
 R_1. & \frac{\alpha}{\beta} \\
 & \text{where } \vdash_{S_5} M\alpha \rightarrow M\beta \\
 R_2. & \frac{M\alpha}{M\beta} \\
 & \frac{M\alpha \wedge M\beta}{M\alpha \wedge M\beta} \\
 DR1. & \frac{\alpha}{\vdash_{S_5} \alpha \rightarrow \beta} \\
 & \frac{\vdash_{S_5} \alpha \rightarrow \beta}{\beta} \\
 DR2. & \frac{(M\alpha)L\alpha}{(M\alpha)L\alpha \rightarrow \beta} \\
 & \frac{(M\alpha)L\alpha \rightarrow \beta}{\beta} \\
 DR3. & \frac{M\alpha}{\alpha} \\
 DR4. & \frac{\alpha}{M\alpha}
 \end{array}$$

DR1 – 4 are some derived rules of the system. The usual deduction theorem holds in $\mathcal{L}_{\mathcal{R}}$, though not its converse. $\mathcal{L}_{\mathcal{R}}$ is paraconsistent, i.e. there are α, β such that $\{\alpha, \neg\alpha\} \not\vdash \beta$. A set Γ is *roughly consistent*, if $M\Gamma = \{M\alpha : \alpha \in \Gamma\}$ is S_5 -consistent: there is no $\alpha \in \mathcal{L}_{\mathcal{R}}$ such that $M\Gamma \vdash_{S_5} \alpha$ and $M\Gamma \vdash_{S_5} \neg\alpha$.

2.3 The Rough Belief Change Postulates

The changed belief set is denoted as $K *_r A$ ($K -_r A$) representing *rough revision* (*contraction*) of K with respect to A . The *expansion* $K +_r A$ of K by the wff A is the set $C_r(K \cup \{A\})$, where $C_r(K) = \{\alpha : K \vdash \alpha\}$. The *rough equality* connective $\approx$ [3] is used: $A \approx B \equiv (LA \leftrightarrow LB) \wedge (MA \leftrightarrow MB)$.

Postulates for Rough Revision

- K^*1 For any wff A and belief set K , $K *_r A$ is a belief set.
- K^*2 $A \in K *_r A$.
- K^*3 $K *_r A \subseteq K +_r A$.
- K^*4 If $\neg A \notin K$, then $K +_r A \subseteq K *_r A$.
- K^*5 If $K *_r A$ is not roughly consistent, then $\vdash \neg A$.
- K^*6 If $\vdash A \approx B$, then $K *_r A = K *_r B$.
- K^*7 $K *_r (MA \wedge MB) \subseteq K *_r A +_r B$.
- K^*8 If $\neg B \notin K *_r A$ then $K *_r A +_r B \subseteq K *_r (MA \wedge MB)$.

Postulates for Rough Contraction

- K^-1 For any wff A and belief set K , $K -_r A$ is a belief set.
- K^-2 $K -_r A \subseteq K$.
- K^-3 If $A \notin K$, then $K -_r A = K$.
- K^-4 If $A \in K -_r A$, then $\vdash A$.
- K^-5 $K \subseteq (K -_r A) +_r A$, if A is of the form LB or MB for some wff B .
- K^-6 If $\vdash A \approx B$, then $K -_r A = K -_r B$.
- K^-7 $K -_r A \cap K -_r B \subseteq K -_r (MA \wedge MB)$.
- K^-8 If $A \notin K -_r (MA \wedge MB)$ then $K -_r (MA \wedge MB) \subseteq K -_r A$.

Among other results, we can prove the following (partly done in [2]).

Theorem 1. *Let the Levi identity give $*_r$, i.e. $K *_r A \equiv (K -_r \neg A) +_r A$, where the contraction function $-_r$ satisfies $K^-1 - 8$. Then $*_r$ satisfies $K^*1 - 8$.*

Observation 1. *Rough belief revision and contraction coincide with the corresponding classical notions if $\vdash_{S_5} A \leftrightarrow LA$ for every wff A , i.e. all beliefs are definable/describable (S_5 collapses into classical propositional logic).*

3 Partial Meet Contraction in Rough Belief Change

Similar to the classical case, we can construct partial meet contractions in the rough background. We refer to definitions (1), (2) of Section 2.1, modified to use C_r or $\vdash$. The derived rules $DR1 - 4$ (cf. Section 2.2) are used in the proofs.

Lemma 1. *Let m be in $K \perp A$ for some A in K . Then (i) m is closed, and (ii) $C_r(m \cup \{\alpha\}) = K$ where $\alpha \in K \setminus m$.*

Proof. (i) Let $m \vdash \alpha$ and assume $\alpha \notin m$. Now $m \vdash \alpha$ implies $m \vdash M\alpha$. So $M\alpha \in K$, as $m \subset K$ and K is closed. Hence $m \subset m \cup \{M\alpha\} \subseteq K$, which implies $m \cup \{M\alpha\} \vdash A$ or $m \vdash M\alpha \rightarrow A$. But then $m \vdash A$, as $m \vdash M\alpha$ and $M\alpha$ is modal – a contradiction. Hence $\alpha \in m$.

(ii) We already have $C_r(m \cup \{\alpha\}) \subseteq K$. $K \subseteq C_r(m \cup \{\alpha\})$: let $\beta \in K \setminus m$, and we show that $MA \rightarrow M\beta \in m$. Then any belief set containing MA and m (in particular $C_r(m \cup \{\alpha\})$) would have $M\beta$ and hence β . Suppose $MA \rightarrow M\beta \notin m$. Then $m \subset m \cup \{MA \rightarrow M\beta\} \subseteq K$ as $\vdash_{S_5} M\beta \rightarrow (MA \rightarrow M\beta)$ and $\beta(M\beta) \in K$. Thus $(MA \rightarrow M\beta) \rightarrow MA \in m$. But $\vdash_{S_5} ((MA \rightarrow M\beta) \rightarrow MA) \leftrightarrow MA$. So MA or $A \in m$, a contradiction. $\square$

Importantly, Lemma 1(ii) says that all the beliefs in K outside m , behave identically with respect to m . Further, we have

Lemma 2. *Let $A \in K$ and $m \in K \perp A$. Then for all B in $K \setminus m$, $m \in K \perp B$.*

Using this result, we compare $K \perp (MA \wedge MB)$, $K \perp A$ and $K \perp B$. Note that $\{MA \wedge MB\} \sim A, B$, and $\{A, B\} \sim MA \wedge MB$. One can easily see that if $m \in K \perp (MA \wedge MB)$ then it is in $K \perp A$ or $K \perp B$ because $MA \wedge MB \notin m$ implies $A \notin m$ or $B \notin m$. What if m is in $K \perp A$? Conditions (i) and (ii) for membership in $K \perp MA \wedge MB$ are satisfied, and Lemma 2 implies (iii). Suppose $m \subset m' \subseteq K$. Then in addition to $A \in C_r(m')$, $C_r(m') = K$ also, implying that $B \in C_r(m')$ (as $B \in K$). Hence $MA \wedge MB$ is in $C_r(m')$.

Lemma 3. *Let A and B be in K . Then $K \perp (MA \wedge MB) = K \perp A \cup K \perp B$.*

As S is to select the ‘best’ elements of $K \perp A$, let us assume that it does so with the help of a reflexive and transitive *preference* relation $\geq$ over the maximal subsets in $K \perp A$. So S is defined through (2). The following then proves that such a selection works – definition (1) gives a rough contraction function.

Theorem 2. *Let $\geq$ be a reflexive and transitive relation on $M(K)$. Then the function defined through (1) with S as in (2) is a rough contraction function, i.e. it satisfies $K^{-1} - 8$.*

Proof. Let $A \sim B$ abbreviate $\{A\} \sim B$, $\{B\} \sim A$.

K^{-1} : Each maximal subset m in $S(K \perp A)$ is closed and intersection of closed sets is closed.

K^{-2} : Follows from the definition.

K^{-3} : If $A \notin K$ then $K \perp A = \{K\}$, and hence $K -_r A = K$.

K^{-4} : If $\not\sim A$ then $K \perp A$ is non-empty. Hence $A \notin K -_r A$.

K^{-5} : In fact, we obtain the classical recovery axiom (viz. $K \subseteq (K -_r A) +_r A$ for any A), and so, in particular, K^{-5} . Let $\alpha \in K$. Assume $\not\sim A$. We show that $(MA \rightarrow \alpha) \in K -_r A$. Using Lemma 1(ii), we get for all $m \in K \perp A$, $C_r(m \cup \{A\}) = K$, or $C_r(m \cup \{MA\}) = K$. Thus $MA \rightarrow \alpha \in m$, for all $\alpha \in K$, implying $MA \rightarrow \alpha \in K -_r A = \bigcap S(K \perp A)$. Hence $\alpha \in (K -_r A) + A$.

K^{-6} : Use, $\sim A \approx B$ implies $K \perp A = K \perp B$. In fact, $A \sim B$ gives the result.

K^{-7} : If $\sim A$ or $\sim B$ then obviously this holds: if $\sim A$ then $K -_r A = K$ and $K -_r B = K -_r (MA \wedge MB)$ (as $MA \wedge MB \sim B$). So let us assume $\not\sim A$ and $\not\sim B$. We have to show $(\bigcap S(K \perp A)) \cap (\bigcap S(K \perp B)) \subseteq \bigcap S(K \perp (MA \wedge MB))$. But $S(K \perp A \cup K \perp B) = S(K \perp A) \cap S(K \perp B) \subseteq S(K \perp A) \cup S(K \perp B)$. Hence $\bigcap (S(K \perp A \cup K \perp B)) \supseteq \bigcap (S(K \perp A) \cup S(K \perp B))$. Now $\bigcap (S(K \perp A) \cup S(K \perp B)) = (\bigcap S(K \perp A)) \cap (\bigcap S(K \perp B))$. By Lemma 3 we get $(\bigcap S(K \perp A)) \cap (\bigcap S(K \perp B)) = \bigcap (S(K \perp A) \cup S(K \perp B)) \subseteq \bigcap (S(K \perp A \cup K \perp B)) \cap S(K \perp MA \wedge MB)$.

K^{-8} : Let $A \notin K -_r (MA \wedge MB)$. Thus there exists an $m' \in K \perp (MA \wedge MB)$ such that $m' \geq m$, for all $m \in K \perp (MA \wedge MB)$ and $A \notin m'$. Note that $m' \in K \perp A$ as $A \notin m'$. Now we show $S(K \perp A) \subseteq S(K \perp (MA \wedge MB))$, whence $\bigcap S(K \perp (MA \wedge MB)) \subseteq \bigcap S(K \perp A)$. If $m'' \in S(K \perp A)$ then $m'' \geq m' \geq m$, for all $m \in K \perp MA \wedge MB$. By transitivity of $\geq$, $m'' \geq m$, for all $m \in K \perp (MA \wedge MB)$. Hence $m'' \in S(K \perp (MA \wedge MB))$, giving $S(K \perp A) \subseteq S(K \perp (MA \wedge MB))$. $\square$

A rough revision function satisfying all the postulates $K^*1 - 8$ can thus be obtained through this method as well (using Theorems 2 and 1).

4 Conclusions

In rough belief change, the main consideration is to perform the change operations preserving rough consistency and rough truth. It is not surprising that the classical ‘exact’ notions are achieved as a special case, when a belief A can be identified with LA or MA . In this article, we show that the method of partial meets can be used to construct rough contraction and revision functions satisfying all the proposed rough postulates.

The next goal would be to study, in the rough context, the methods of Grove’s spheres – focussing on *maximally consistent* sets of beliefs, and *epistemic entrenchment* – using an ordering on the beliefs. The latter, in particular, is expected to offer a more computationally tractable solution to the problem of constructing rough belief change functions.

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