Mammogram Analysis Using Two-Dimensional Autoregressive Models: Sufficient or Not?

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Abstract. Two-dimensional ($2 - D$) autoregressive (AR) models have been used as one of the methods to characterise the textures of tumours in mammograms. Previously, the $2 - D$ AR model coefficients were estimated for the block containing the tumour and the blocks in its 3×3 neighbourhood. In this paper, the possibility of having the estimated set of AR model coefficients of the block containing the tumour as a unique set of AR model coefficients for the entire mammogram is looked into. Based on the information given from the MiniMammography database, the possible number of blocks of the same size of the block containing the tumour is obtained from the entire mammogram and for each block a set of AR model coefficients is estimated using a method that combines both the Yule-Walker system of equations and the Yule-Walker system of equations in the third-order statistical domain. These sets of AR model coefficients are then compared. The simulation results show that 98.6% of the time we can not find another set of AR model coefficients representing the blocks of pixels in the possible neighbourhood of the entire mammogram for the data (95 mammograms with 5 of them having two tumours) available in the MiniMammography database.

1 Introduction

The two-dimensional ($2 - D$) autoregressive modelling technique has been used as one of the statistical methods to characterise and analyse the textures in images [4][2]. The images is divide into blocks of pixels and for each block a set of AR model coefficients is estimated. Alternatively, for the area of interest a set of AR model coefficients is estimated.

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A number of methods are available in the literature for estimating the AR model coefficients, including the Yule-Walker system of equations (YW) [1], the Yule-Walker system of equations in the third-order statistical domain (YWT) [8], the combined method using the two aforementioned methods [4], the constrained optimisation formulation with equality constraints (ConEq) [4] and the constrained optimisation formulation with inequality constraints (ConIneq) [4]. From [4], the comparison among these methods can be found for synthetic images which are built using a known stable $2 - D$ AR model and using a non-Gaussian driving input. The images are also contaminated with an external Gaussian noise. The YW method is capable of estimating the AR model coefficients when the external noise is small. The YWT method can give good estimations when the external noise is large, however, the variances calculated using the estimations from a number of realisations are higher than the YW method. The rest of aforementioned methods were proposed to improve these problems.

In [2] and [4], the $2 - D$ AR modelling technique is applied to mammogram analysis. The block containing the tumour was analysed and the coefficients representing the block were claimed to be symmetric. However, by taking a block of the same size of pixels in any of them in its possible neighbourhood, can we find another set of AR model coefficients which have values close to the one representing the tumour block? This is the question we try to answer in this paper.

All mammograms containing a tumour or tumours are taken from the database [5]. There are 95 of them to analyse, including 5 of mammograms containing two tumours. The combined method that employs both the Yule-Walker system of equations and the Yule-Walker system of equations in the third-order statistical domain [4] is used to estimate the AR model coefficients. The reason for choosing this method is that in [4] it was proven that this method is capable of estimating AR model coefficients in both low and high SNR environments.

The paper is organised as follows. In Section 2, the $2 - D$ AR model is revisited. In Sections 3, 4 and 5, the conventional Yule-Walker system of equations, the Yule-Walker system of equations in the third-order statistical domain and the combined method can be found respectively. How the possible neighbourhood around the tumour is defined is explained in Section 6. In Section 7, simulation results can be found. The conclusion and summary are given in Section 8.

2 Two-Dimensional Autoregressive Model

Let us consider a digitised image x of size $M \times N$. Each pixel of x is characterised by its location $[m, n]$ and can be represented as $x[m, n]$, where $1 \leq m \leq M$, $1 \leq n \leq N$ and $x[m, n]$ is a positive intensity (gray level) associated with it. A two-dimensional ($2 - D$) autoregressive (AR) model is defined as [1]

$$x[m, n] = - \sum_{i=0}^{p_1} \sum_{j=0}^{p_2} a[i, j] x[m - i, n - j] + w[m, n], \quad (1)$$

where $[i, j] \neq [0, 0]$, $a[i, j]$ is the AR model coefficient, $w[m, n]$ is the input driving noise, and $p_1 \times p_2$ is the order of the model.

The driving noise, $w[m, n]$, is assumed to be zero-mean, i.e., $E\{w[m, n]\} = 0$ and non-Gaussian. The AR model coefficient $a[0, 0]$ is assumed to be 1 for scaling purposes, therefore we have $[(p_1 + 1)(p_2 + 1) - 1]$ unknown coefficients to solve.

An external zero-mean Gaussian noise, $v[m, n]$, is added onto the system, so that the method will be able to deal with real images which are contaminated with such noise. Mathematically the new system can be written as

$$y[m, n] = x[m, n] + v[m, n]. \quad (2)$$

The signal-to-noise ratio (SNR) of the system is calculated by

$$SNR = 10 \log_{10} \frac{\sigma_x^2}{\sigma_v^2} \quad dB \quad (3)$$

where σ_x^2 is the variance of the signal and σ_v^2 is the variance of the noise.

3 Yule-Walker System of Equations

The conventional Yule-Walker equations are given by [1][6]

$$\sum_{i=0}^{p_1} \sum_{j=0}^{p_2} a[i, j] r_{yy}[i - k, j - l] = -r_{yy}[-k, -l] \quad (4)$$

for $k = 0, \dots, p_1$ and $l = 0, \dots, p_2$, where $[k, l] \neq [0, 0]$, $[i, j] \neq [0, 0]$, $a[i, j]$ is the AR model coefficient, $1 \leq m \leq M, 1 \leq n \leq N$, $M \times N$ is the size of the given image and $r_{yy}[i, j] = E\{y[m, n]y[m + i, n + j]\}$.

Equation (4) can be written in vector-matrix form as

$$\underline{\mathbf{R}}\underline{\mathbf{a}} = -\underline{\mathbf{r}}, \quad (5)$$

where $\underline{\mathbf{R}}$ is a $(p_1 p_2 + p_1 + p_2) \times (p_1 p_2 + p_1 + p_2)$ matrix and $\underline{\mathbf{a}}$ and $\underline{\mathbf{r}}$ are both $(p_1 p_2 + p_1 + p_2) \times 1$ vectors.

These equations give good AR model coefficient estimations when the SNR is high. However, the error increases with σ_v^2 .

4 Yule-Walker System of Equations in the Third-Order Statistical Domain

The equations that relate the AR model parameters to the third-order moment samples are [6][8]:

$$\sum_{i=0}^{p_1} \sum_{j=0}^{p_2} a[i, j] C_{3y}([i - k, j - l], [i - k, j - l]) = -C_{3y}([-k, -l], [-k, -l]) \quad (6)$$

for $k = 0, \dots, p_1, l = 0, \dots, p_2$ and $[k, l] \neq [0, 0]$, where $C_{3y}([i_1, j_1], [i_2, j_2]) = E\{y[m, n]y[m + i_1, n + j_1]y[m + i_2, n + j_2]\}$ for zero-mean process $y[m, n]$.

These equations are insensitive to external Gaussian noise. The equations can be written in matrix form as

$$\underline{\mathbf{C}}\underline{\mathbf{a}} = -\underline{\mathbf{c}}, \quad (7)$$

where $\underline{\mathbf{C}}$ is a $(p_1p_2 + p_1 + p_2) \times (p_1p_2 + p_1 + p_2)$ matrix and $\underline{\mathbf{a}}$ and $\underline{\mathbf{c}}$ are both $(p_1p_2 + p_1 + p_2) \times 1$ vectors.

5 The Method Combining the YW and the YWT

The method using both the YW and the YWT techniques was defined as [3][4]

$$\underline{\mathbf{D}} \begin{pmatrix} \underline{\mathbf{R}} \\ \underline{\mathbf{C}} \end{pmatrix} \underline{\mathbf{a}} = -\underline{\mathbf{D}} \begin{pmatrix} \underline{\mathbf{r}} \\ \underline{\mathbf{c}} \end{pmatrix} \quad (8)$$

where the matrix $\underline{\mathbf{R}}$ and vector $\underline{\mathbf{r}}$ are defined in (5),

the matrix $\underline{\mathbf{C}}$ and vector $\underline{\mathbf{c}}$ are defined in (7),

$\underline{\mathbf{D}}$ is a diagonal weighting matrix, and

$\underline{\mathbf{a}}$ is the vector of the unknown AR model coefficients,

$[a[0, 1], \dots, a[0, p_2], \dots, a[p_1, p_2]]^T$.

Let us consider the formulation where the system is contaminated with external Gaussian noise.

$$\underline{\mathbf{R}}_{yy}\underline{\mathbf{a}} + \underline{\mathbf{r}}_{yy} = \sigma_v^2 \underline{\mathbf{I}}\underline{\mathbf{a}} \quad (9)$$

where the AR model coefficients estimation $\underline{\mathbf{a}}$ is obtained from (7) using

$$\underline{\mathbf{a}} = -\underline{\mathbf{C}}^{-1}\underline{\mathbf{c}}. \quad (10)$$

Let $\underline{\mathbf{r}}_1 = \underline{\mathbf{R}}_{yy}\underline{\mathbf{a}} + \underline{\mathbf{r}}_{yy}$, where $\underline{\mathbf{a}}$ is obtained from (10). The variance of the noise $v[m, n]$ can be calculate using

$$\sigma_v^2 = (\underline{\mathbf{a}}^T \underline{\mathbf{a}})^{-1} \underline{\mathbf{a}}^T \underline{\mathbf{r}}_1 \quad (11)$$

The weighting diagonal matrix, $\underline{\mathbf{D}}$, is determined as (12).

$$D[i, i] = \begin{cases} 1 & \text{for } 1 \leq i \leq (p_1 + 1)(p_2 + 1) - 1 \\ \lceil 50\sigma_v^2 \rceil & \text{for } (p_1 + 1)(p_2 + 1) \leq i \leq 2(p_1 + 1)(p_2 + 1) - 2 \end{cases} \quad (12)$$

where $\lceil x \rceil$ denotes rounding toward infinity.

6 Possible Neighbourhood Around Tumour

The main purpose of this paper is to find out whether the set of AR model coefficients that represents the texture of the block containing an tumour can be

Fig. 1. The possible neighbourhood of the block containing a tumour in a mammogram

found anywhere else in the mammogram. For all the mammograms used in the simulations the information on the location and the size of the tumour is given. From this piece of information, we establish the possible neighbourhood by calculating the number of the blocks with the same size of the block containing the tumour we can have to four corners of the mammogram. In general, for the small tumours, we have a large number of blocks in the possible neighbourhood, whereas for the big tumours, we have small number of blocks in the possible neighbourhood. However, the total number of blocks in the possible neighbourhood also depends on the location of the tumour. All the blocks in the possible neighbourhood are restricted to have equal size, so some pixels near the borders might be neglected. An illustration can be found in Fig. 1. The smallest block in these simulations is 13×13 and the largest block is 395×395 .

7 Simulation Results

From the database [5], the mammograms with tumours are extracted and their identity number remains as shown under (1) in Table 1. For each mammogram, the square block size that contains the tumour can be found under (2) in Table 1. The possible neighbourhood is defined and the number of blocks in the neighbourhood can be found under (3). A set of AR model coefficients is estimated using the method in Section 5. The order of the model is assumed to be 1×1 . Each set of AR model coefficient is compared with the set representing the block with the tumour. The number of sets of AR model coefficients that exceed $\pm 3\%$ of the coefficients representing the tumour is counted for each mammogram and the results can be found under (4) in Table 1. The percentage of these sets in all blocks in the possible neighbourhood can be obtained under (5) in the same table.

For 100 tumours in 95 mammograms from the database, 38 of them contain no repeating AR model coefficients in their possible neighbourhood. For those

Table 1. The results arisen from the comparison of two-dimensional autoregressive model coefficients of the tumour block and its possible neighbourhood

(1)	(2)	(3)	(4)	(5)	(1)	(2)	(3)	(4)	(5)
mdb001	395 × 395	1	0	0	mdb144b	55 × 55	323	4	1.24
mdb002	139 × 139	41	2	4.88	mdb145	99 × 99	99	0	0
mdb005a	61 × 61	255	1	0.39	mdb148	349 × 349	4	0	0
mdb005b	53 × 53	341	2	0.59	mdb150	125 × 125	48	2	4.17
mdb010	67 × 67	209	1	0.48	mdb152	99 × 99	99	2	2.02
mdb012	81 × 81	144	0	0	mdb155	191 × 191	19	0	0
mdb013	63 × 63	255	1	0.39	mdb158	177 × 177	24	0	0
mdb015	137 × 137	35	0	0	mdb160	123 × 123	48	1	2.08
mdb017	97 × 97	99	1	1.01	mdb163	101 × 101	89	4	4.49
mdb019	99 × 99	99	2	2.02	mdb165	85 × 85	120	2	1.67
mdb021	99 × 99	80	0	0	mdb167	71 × 71	168	0	0
mdb023	59 × 59	271	6	2.21	mdb170	165 × 165	24	0	0
mdb025	159 × 159	29	0	0	mdb171	125 × 125	55	5	9.09
mdb028	113 × 113	63	1	1.59	mdb175	67 × 67	209	4	1.91
mdb030	87 × 87	120	5	4.17	mdb178	141 × 141	41	0	0
mdb032	133 × 133	48	0	0	mdb179	135 × 135	41	4	2.88
mdb058	55 × 55	323	3	0.93	mdb181	109 × 109	71	0	0
mdb063	67 × 67	209	9	4.31	mdb184	229 × 229	16	0	0
mdb069	89 × 89	109	0	0	mdb186	99 × 99	100	0	0
mdb072	57 × 57	305	11	3.61	mdb188	123 × 123	48	1	2.08
mdb075	47 × 47	461	5	1.08	mdb190	63 × 63	239	0	0
mdb080	41 × 41	624	3	0.48	mdb191	83 × 83	120	0	0
mdb081	263 × 263	8	0	0	mdb193	265 × 265	8	0	0
mdb083	77 × 77	155	1	0.65	mdb195	53 × 53	341	5	1.47
mdb090	99 × 99	89	2	2.25	mdb198	187 × 187	15	1	6.67
mdb091	41 × 41	599	0	0	mdb199	63 × 63	239	2	0.84
mdb092	87 × 87	120	2	1.67	mdb202	75 × 75	155	3	1.94
mdb095	59 × 59	288	10	3.47	mdb204	43 × 43	528	5	0.95
mdb097	69 × 69	195	2	1.03	mdb206	35 × 35	840	6	0.71
mdb099	47 × 47	461	6	1.3	mdb207	39 × 39	675	4	0.59
mdb102	77 × 77	143	7	4.9	mdb209	175 × 175	24	0	0
mdb104	101 × 101	99	1	1.01	mdb211	27 × 27	1481	28	1.89
mdb105	197 × 197	19	0	0	mdb213	91 × 91	109	0	0
mdb107	223 × 223	15	1	6.67	mdb214	23 × 23	2069	1	0.0483
mdb110	103 × 103	80	3	3.75	mdb218	17 × 17	3968	2	0.0504
mdb115	235 × 235	9	0	0	mdb219	59 × 59	271	2	0.74
mdb117	169 × 169	24	0	0	mdb222	35 × 35	869	0	0
mdb120	159 × 159	35	0	0	mdb223a	59 × 59	271	0	0
mdb121	165 × 165	19	0	0	mdb223b	13 × 13	7055	18	0.2551
mdb124	67 × 67	209	4	1.91	mdb227	19 × 19	3135	63	2.01
mdb125	121 × 121	63	0	0	mdb231	89 × 89	120	3	2.5
mdb126	47 × 47	440	8	1.82	mdb236	29 × 29	1259	3	0.24
mdb127	97 × 97	89	1	1.12	mdb238	35 × 35	840	4	0.4762
mdb130	57 × 57	288	8	2.78	mdb239a	81 × 81	131	0	0
mdb132a	105 × 105	72	0	0	mdb239b	51 × 51	360	23	6.39
mdb132b	37 × 37	728	6	0.82	mdb240	47 × 47	440	45	10.23
mdb134	99 × 99	89	0	0	mdb241	77 × 77	168	2	1.19
mdb141	59 × 59	288	6	2.08	mdb244	105 × 105	63	2	3.17
mdb142	53 × 53	341	1	0.29	mdb248	21 × 21	2499	6	0.24
mdb144a	59 × 59	288	0	0	mdb249	97 × 97	89	0	0

(1) Mammogram identity number (as in the Database [7])

(2) The size of the block that contains a tumour

(3) The number of blocks available in the entire mammogram.

(4) The number of blocks that can be represented in 2-D AR model coefficients similar to the set representing the tumour block.

(5) The percentage of blocks that contain similar 2-D AR model coefficients compared to the set representing the tumour block.

who do have a set or sets of AR model coefficients within the range of $\pm 3\%$ of the AR model coefficients representing the tumour block, mammogram mdb240 contains 10.23% of the AR model coefficients in all 440 sets estimated which are within this range, followed by mammogram mdb171, which has 9.9% of its 55 blocks in the neighbourhood within this range of the AR model coefficients representing the tumour. For all the tumours in mammograms we took into account, 98.6% of the time, the AR model coefficients (within $\pm 3\%$) representing the tumour block does not appear again in any other blocks in the possible neighbourhood.

8 Conclusion and Summary

In this work, we looked into the possibility of representing the tumour area of a mammogram using two-dimensional ($2 - D$) autoregressive (AR) models. The possible neighbourhood with blocks of the same size as the one containing a tumour is defined and a set of AR model coefficients is estimated for each block. For all the sets of coefficients obtained, we compared them with the set representing the tumour. It was found from the simulations results that 98.6% of the time, the AR model coefficients representing the tumour can not be found anywhere else in the same mammogram using the same block size. This lead to the conclusion that $2 - D$ AR model coefficients can provide sufficient data for texture analysis of mammograms.

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