Approximate Steerability of Gabor Filters for Feature Detection

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Abstract. We discuss the connection between Gabor filters and steerable filters in pattern recognition. We derive optimal steering coefficients for Gabor filters and evaluate the accuracy of the approximative orientation steering numerically. Gabor filters can be well steerable, but the error of the approximation depends heavily on the parameters. We show how a rotation invariant feature similarity measure can be obtained using steerability.

1 Introduction

Many different types of oriented filters have been proposed for low-level vision tasks. In this paper we consider only the suitability of filters to the task of generic visual feature detection. A central task in feature detection is to internally represent the local gray level structure of the image at specific fiducial locations in a robust and generic way. Of specific interest are systems such as the bunch graph matching model [12] which consists of local feature descriptors in a global graph structure, representing the current state of the art in face alignment and recognition.

Gabor filters are oriented filters which achieve the lower limit of joint uncertainty in spatial and frequency domains, described by the Heisenberg-Weyl (Gabor) uncertainty principle. In this sense they appear to be optimal for pattern recognition [2]. There exists also strong psychophysical evidence that mechanisms employing oriented linear filters are involved in mammalian vision, and they are well approximated with Gabor filters [1]. Despite being somewhat difficult to use in applications requiring reconstructions from the filter responses, Gabor filters can form a relatively good approximation of a tight wavelet frame and reconstructions using iterative methods are possible [7].

Steerable filters are a flexible and computationally efficient method to compute the response of an oriented filter in an arbitrary orientation [3]. They have found applications in many different low level vision tasks. The steerability property of the feature detectors facilitates strong rotational invariance, as orientation estimates can be computed from the filter responses. However, exactly steerable filters of the form $P(x)G(\sqrt{x^2 + y^2})$, where $P(x)$ is a polynomial, such as the examples considered in [3] are less flexible, because orientation and radial frequency

selectivities of such filters cannot be easily chosen independently. Polar-separable oriented steerable filters were already proposed in [6] as generic image processing operators. Polar-separable oriented filters are typically constructed in frequency space, and have no closed-form expression in the spatial domain.

Using the theory of Lie groups, a steerable basis can be found for arbitrary parameter groups by representing or approximating the filters in an equivariant function space [8], [5]. For single-parameter 2D rotation expressed in polar coordinates (r, θ) , this function space is $\{f(r)\exp(ik\theta)\}$, $k \in \mathbb{Z}$, i.e. complex harmonics together with an arbitrary (real-valued) radial component $f(r)$ [10]. A viable approach for filter design is to start with an ideal filter prototype (for example, a Gabor filter) and approximate it in the appropriate equivariant function space, which is guaranteed to be closed under the same transformational group [11]. Before performing the approximation it is preferable to know the steering capability of the original Gabor filter.

2 Gabor Filters

The generic form of the Gabor filter with a non-spherical Gaussian envelope function is described by

$$f(\xi; \mu, S, R_\theta) = \frac{|\mu|^2}{\sqrt{\det(S)}} \exp\left(-\frac{|\mu|^2}{2}(R_\theta \xi)^T S^{-1} R_\theta \xi\right) \exp(i\mu^T R_\theta \xi) \quad (1)$$

where $\xi = [x \ y]^T$ are the spatial coordinates, the wave vector $\mu = [f_c \ 0]^T$ determines the center frequency f_c of the filter and also acts as a scaling factor in this parameterization, θ determines the orientation of the filter via the rotation matrix

$$R_\theta = \begin{bmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{bmatrix} \quad (2)$$

and the covariance matrix

$$S = \begin{bmatrix} \sigma_x^2 & 0 \\ 0 & \sigma_y^2 \end{bmatrix} \quad (3)$$

determines the frequency bandwidths of the filter along the axes in Cartesian coordinates. Leaving the normalization term out, we can write the complex Gabor filter as a product of a Gaussian distribution modulated by a complex exponential,

$$f(\xi; \mu, S, R_\theta) \propto N(0, |\mu|^2 R_\theta^T S R_\theta) \cdot \exp(i\mu^T R_\theta \xi). \quad (4)$$

An important property of this parameterization is that the filters retain their shape regardless of rotation angle θ and center frequency μ .

The Fourier transform of a Gaussian function is also a Gaussian function (although no longer normalized), and modulation by complex plane wave corresponds to a shift from the origin in the Fourier plane by the amount described by $R_\theta^T \mu$. The rotation property of 2D Fourier transform states that rotations in the spatial plane correspond directly to rotations in the Fourier plane. As a

result, the Fourier transform of the complex Gabor filter is a single Gaussian function

$$\mathcal{F}\{f\} \propto N(R_\theta^T \mu, R_\theta^T S^{-1} R_\theta). \quad (5)$$

We denote the real-valued even and odd Gabor filters with $g = \text{Re}\{f\}$ and $h = \text{Im}\{f\}$, respectively, so that $f = g + ih$. Fourier transforms of real and imaginary parts of the complex filter are sums of two Gaussian functions,

$$\mathcal{F}\{g\} \propto N(R_\theta^T \mu, R_\theta^T S^{-1} R_\theta) + N(-R_\theta^T \mu, R_\theta^T S^{-1} R_\theta) \quad (6)$$

and

$$\mathcal{F}\{h\} \propto N(R_\theta^T \mu, R_\theta^T S^{-1} R_\theta) - N(-R_\theta^T \mu, R_\theta^T S^{-1} R_\theta). \quad (7)$$

The real and imaginary parts of the complex Gabor filter are approximately in quadrature, allowing independent analysis of signal amplitude and phase. Low values of the filter shape parameter σ_x cause a prominent DC component to the even filter, which makes the quadrature approximation inexact. In practice this means that the filter phase responses are directly dependent on the signal amplitude, which is usually undesirable. One way to remove DC component is to subtract it from the modulating complex exponential, giving the equation

$$N(0, |\mu|^2 R_\theta^T S R_\theta) \cdot \left(\exp(i\mu^T R_\theta \xi) - \exp\left(-\frac{\sigma_x^2}{2}\right) \right), \quad (8)$$

which is no longer exactly a Gabor filter. Because of brevity reasons we will not consider this filter type in this paper further, but only note that its behavior is highly similar to the results presented here for Gabor filters.

3 Steerability

Freeman originally derived the conditions for exact steerability by considering the Fourier series of the angular component of the filter in polar coordinates [3]. An alternative approach is to use linear algebra to find the optimal linear steering functions for an arbitrary set of filters [4], which we review here briefly.

Orientation steerability of a (real-valued) linear filter g means that arbitrary filter orientations can be computed (or at least approximated) by computing the sum of a set of basis filters $\mathbf{g} = \{g_{\theta_1}, g_{\theta_2}, \dots, g_{\theta_N}\}$ weighted with steering coefficients $\mathbf{k} = \{k_1(\theta), k_2(\theta), \dots, k_N(\theta)\}$,

$$g(\theta) \approx \sum_{j=1}^N k_j(\theta) g_{\theta_j} = \mathbf{k}^T \mathbf{g}. \quad (9)$$

In the case of complex-valued filters in quadrature, separate real-valued steering coefficients $k_j(\theta)$ are needed for the real and imaginary parts of the filter. We assume that the basis filters g_{θ_j} share the same shape parameters S and frequency μ .

Let us define the inner product between real-valued functions u and v as $\langle u, v \rangle = \int_{\omega \in R^2} u(\omega)v(\omega) d\omega$. Let the functions u and v be normalized without

loss of generality so that $\langle u, u \rangle = \langle v, v \rangle = 1$. The optimal steering coefficients k can be solved analytically by minimizing the L2 norm of the error $e = g(\theta) - \mathbf{k}^T \mathbf{g}$,

$$\begin{aligned} \arg \min_{\mathbf{k}} \|e\|^2 &= \arg \min_{\mathbf{k}} \langle g(\theta) - \mathbf{k}^T \mathbf{g}, g(\theta) - \mathbf{k}^T \mathbf{g} \rangle \\ &= \arg \min_{\mathbf{k}} \langle g(\theta), g(\theta) \rangle - 2 \langle g(\theta), \mathbf{k}^T \mathbf{g} \rangle + \langle \mathbf{k}^T \mathbf{g}, \mathbf{k}^T \mathbf{g} \rangle. \end{aligned} \quad (10)$$

The minimum of this expression is obtained by differentiating Eq. 10 and setting the result to zero, leading to the matrix equation

$$\mathbf{G} \mathbf{k} = \gamma \quad (11)$$

where the matrix $\mathbf{G}$ and vector γ have the elements $\mathbf{G}_{i,j} = \langle g_{\theta_i}, g_{\theta_j} \rangle$ and $\gamma_i = \langle g(\theta), g_{\theta_i} \rangle$, respectively. The vector of optimal steering coefficients $\mathbf{k}$ can be solved using straightforward matrix inversion of $\mathbf{G}$, which is constant relative to the steering angle θ . Another possibility is to compute the Singular Value Decomposition of $\mathbf{G}$ and use it to calculate the SVD inverse. Unlike previous approaches using the SVD approach ([4], [9]), we compute the inner products $u(\theta) = u(\alpha - \beta) = \langle g_\alpha, g_\beta \rangle$ analytically.

4 Inner Product of Rotated Gabor Filters

The inner product integral $u(\theta)$ of two even Gabor filters in the frequency space is

$$\begin{aligned} \langle g, g_\theta \rangle &= \langle \mathcal{F}\{g\}, \mathcal{F}\{g_\theta\} \rangle = \int (N(\mu, S^{-1}) + N(-\mu, S^{-1})) \\ &\quad \cdot (N(R_\theta^T \mu, R_\theta^T S^{-1} R_\theta) + N(-R_\theta^T \mu, R_\theta^T S^{-1} R_\theta)) d\omega \end{aligned} \quad (12)$$

which, using a symmetry argument, becomes

$$= 2 \int N(\mu, S^{-1}) N(R_\theta^T \mu, R_\theta^T S^{-1} R_\theta) + N(\mu, S^{-1}) N(-R_\theta^T \mu, R_\theta^T S^{-1} R_\theta) d\omega. \quad (13)$$

The inner product of two Gaussian functions (the normalization constant of a product of two Gaussian functions) is also Gaussian with respect to the parameters of the functions,

$$\langle N(a, A) \cdot N(b, B) \rangle \propto \sqrt{\frac{|C|}{|A||B|}} \exp \left(-\frac{(a^T A^{-1} a + b^T B^{-1} b - c^T C^{-1} c)}{2} \right), \quad (14)$$

with $C = (A^{-1} + B^{-1})^{-1}$ and $c = CA^{-1}a + CB^{-1}b$. Applying this result gives after some manipulation

$$\langle g, g_\theta \rangle = \frac{\sqrt{|U|}}{Z_g} \exp \left(\frac{\nu^T (U + R_\theta^T U R_\theta) \nu}{2} \right) \cdot \cosh \left(-\frac{\nu^T (U R_\theta^T + R_\theta U) \nu}{2} \right) \quad (15)$$

where $U = (S + R_\theta S R_\theta^T)^{-1}$, $\nu = [\sigma_x^2 \ 0]^T$ and Z_g is a normalization factor. It is most conveniently computed by requiring that the inner product equals to one at $\theta = 0$, yielding the result $Z_g = \frac{1}{2} \sigma_x^{-1} \sigma_y^{-1} \exp \left(\frac{1}{2} \sigma_x^2 \right) \cosh \left(-\frac{1}{2} \sigma_x^2 \right)$.

Inner product function of two odd Gabor filters h and h_θ is obtained similarly,

$$\langle h, h_\theta \rangle = \frac{\sqrt{|U|}}{Z_h} \exp\left(\frac{\nu^T(U + R_\theta^T U R_\theta)\nu}{2}\right) \cdot \sinh\left(-\frac{\nu^T(UR_\theta^T + R_\theta U)\nu}{2}\right), \quad (16)$$

with the normalization factor $Z_h = \frac{1}{2}\sigma_x^{-1}\sigma_y^{-1} \exp\left(\frac{1}{2}\sigma_x^2\right) \sinh\left(-\frac{1}{2}\sigma_x^2\right)$.

The inner product values define the elements of $\mathbf{G}$ and γ , and optimal steering coefficients $\mathbf{k}$ for an arbitrary steering angle θ can be computed via the matrix multiplication $\mathbf{k}(\theta) = \mathbf{G}^{-1}\gamma(\theta)$.

5 Steering Error

The steering property of Gabor-type filters is not exact, but only approximate. The goodness of the steering approximation depends heavily on the number of basis filters and shape parameters S . Let us define the measure for steering error by

$$E_s = \max_{\theta} \sqrt{\frac{\langle g(\theta) - \mathbf{k}(\theta)^T \mathbf{g}, g(\theta) - \mathbf{k}(\theta)^T \mathbf{g} \rangle}{\langle g(\theta), g(\theta) \rangle}}, \quad (17)$$

that is, the 2-norm distance of the maximum relative impulse response error. The same error measure was used in [4]. In an evenly spaced filter bank the maximum error occurs always exactly between known filter orientations, that is, if filters are in orientations $\theta_i = \pi \frac{i}{N}, i \in \{0, 1, \dots, N-1\}$, maximum error is reached at $\theta = \frac{\pi}{2N}$. It is straightforward then to evaluate numerically the maximum steering error with different filter shape parameters S . Since we have separate steering functions $\mathbf{k}(\theta)$ for even and odd filters of the quadrature pair, we define the total steering error as the average of the even and odd filter errors,

$$E_s^{avg} = \frac{E_s^{even} + E_s^{odd}}{2}. \quad (18)$$

The error behavior of Gabor filters in banks of three to eight filters is shown in Fig. 1. The overall effects of the filter shape parameters S are similar with any number N of basis filters. Because the angular bandwidth of Gabor filters depends nonlinearly on both σ_x and σ_y , there exists no single optimal value for the shape parameters, but rather only tradeoffs between approximate steerability, exactness of the quadrature pair and frequency resolution in the angular and radial domains. Steering error becomes progressively lower as more basis filters are present, and while the spatial domain sidelobes are not prominent. Most importantly, spherical Gabor filters are in general not optimal in terms of steering, and slightly flattened filters with $\sigma_y/\sigma_x < 1$ have considerably better error performance. In other words, steerability is improved if the filters are less specific in the angular dimension than in the frequency dimension. This behavior is compatible with the properties of derivative of Gaussian filters, which are similarly flattened in the frequency space although their envelope function is a spherically symmetric Gaussian.

Fig. 1. Base-10 logarithm of steering error in impulse responses of Gabor filters. In typical applications 1% error (the -2 contour) may be considered acceptable

6 Rotation Invariant Feature Similarity

The Gabor transform G_I of the image I is defined by convolving the image with the filter bank $\mathbf{f}$,

$$G_I = \mathbf{f} * I = \{f_{\theta_1} * I, f_{\theta_2} * I, \dots, f_{\theta_N} * I\}. \quad (19)$$

The *Gabor jet* $J_{x,y} = G_I(x,y) = \{j_1, j_2, \dots, j_N\}$ of a feature is the Gabor transform vector at the feature location (x,y) . In feature detection, a *similarity function* which assigns a numerical value for the similarity between two perceived

patterns, is needed. For example, the normalized inner product between the feature vectors consisting of filter responses can be used [12], defined as

$$S_1 = \frac{\langle J^{(1)}, J^{(2)} \rangle}{\|J^{(1)}\| \|J^{(2)}\|}. \quad (20)$$

A straightforward way to extend this similarity measure to be rotation invariant is to compute the inner product in all $2N$ relative discrete rotation angles of the filter jets and choose the largest,

$$S_2 = \frac{1}{\|J^{(1)}\| \|J^{(2)}\|} \left(\max_{i=-N, \dots, N-1} \sum_{k=0}^{N-1} j_{k-i}^{(1)} j_k^{(2)} \right). \quad (21)$$

Here a negative index filter $-i$ is interpreted as complex conjugate filter of the positive index $N - i$ (this is because the response of the Gabor filter has equal amplitude and opposite phase when a 180 degree rotation is performed) and indices $N + i, i \geq 0$ wrap around to $-N + i$.

This discrete scheme can be easily extended to continuous space by using steerability. Let us expand the discrete filter response $J^{(1)}$ into a continuous one with rotation angle ϕ using the steering coefficients $\hat{k}_i(\phi)$, so that the jet $J^{(1)}$ consists of filter responses $j_k^{(1)}(\phi) = \sum_i \hat{k}_i(\phi) j_{i+k}^{(1)}$. Now we can compute the similarity between $J^{(1)}$ and $J^{(2)}$ in any *continuous* relative orientation angle and choose the largest,

$$S_3 = \max_{\phi} \frac{1}{\|J^{(1)}\| \|J^{(2)}\|} \sum_k \left(\left(\sum_i \hat{k}_i(\phi) j_{i+k}^{(1)} \right) j_k^{(2)} \right). \quad (22)$$

The norm of the steered jet $J^{(1)}$ is preserved exactly only if the steering is exact. Approximate steering causes slight variation in the norm, but normalization may

Fig. 2. Behavior of three different similarity functions with a synthetic test image a), with the test feature marked with a red cross. b) Normalized inner product similarity S_1 . c) Discrete angle rotation invariant similarity S_2 . d) Continuous angle rotation invariant similarity S_3

still be performed only after maximization to lessen the computational cost, if the maximum steering error is small.

We illustrate the behavior of the different similarity functions with two example images. Fig. (2) shows the similarity fields of a corner feature (marked with a white cross) in a synthetic test image, computed using Eqs. (20), (21) and (22). A filter bank with four orientations is used, with shape parameters $\sigma = [2.5 \ 1.75]$.

The similarity function S_1 provides best localization of the correct feature, but withstands only small rotations. The similarity function S_2 has generally an unequal response with respect to rotation, and only orientations which are present in the filter bank provide the correct response. The similarity function S_3 has equal response in all orientation angles. However, as a consequence of rotation invariance, feature specificity is lower than with the two other measures, and the localization capability is worse.

The normalization factor $\frac{1}{\|J^{(1)}\| \|J^{(2)}\|}$, which is present in all three similarity measures, provides contrast invariance and causes medium similarity values to be found also among background noise.

Similarity functions are not highly sensitive to the filter shape parameter values, and while the filters used in this example do not retain the shape of their impulse responses very well under steering (error is approximately 6%), there

Fig. 3. Similarity of a reference mouth corner marked with a white cross, with five independent rotated test images. *b*) Normalized inner product similarity S_1 . *c*) Discrete angle rotation invariant similarity S_2 . *d*) Continuous angle rotation invariant similarity S_3 . Even a small rotation in the image necessitates rotation invariance

is hardly any noticeable variation due to rotation in the similarity field of the function S_3 .

Fig. (3) shows the similarity fields of S_1 , S_2 and S_3 of real-world face images. Here, eight filters with shape parameters $\sigma = [5 \ 2.8]$ were used. The reference Gabor jet feature was obtained from the mouth corner of the leftmost face, marked again with a white cross. The five test images have been rotated 13 degrees, and the manually annotated feature locations are marked with crosses in their similarity fields. Eight basis filters are useful in making the features specific so that the mouth corners are recognized well, but without rotation invariance even small rotations cause the similarity to drop drastically, and detection is not possible. In contrast, similarity functions S_2 and S_3 provide very good feature localization, with a clear maximum near the annotated feature location in all test images. The differences between the performance of S_2 and S_3 are masked by the large variation occurring in natural images. The similarity measures alone do not suffice in solving the face alignment problem, and the disambiguities caused by feature variability have to be resolved by including the information in the relative locations of the detected features.

7 Conclusions

The main contribution of this work is to make the methods of steerable filters available when using Gabor filters. The advantages of Gabor filters are mainly that they have a simple analytical form, making the filter bank design problem easier, and that the filters can be tuned for different applications by adjusting the shape parameters.

We have shown that the methods for steerable filters are available also for some Gabor filters in applications which do not require exact steering. The steering error depends dramatically on the shape parameters, and non-spherical Gabor filters (with $\sigma_x \neq \sigma_y$) have significantly better error performance than the more commonly used spherical Gabor filters.

Steerability allows a straightforward implementation of rotation invariance in the image plane. Without rotation invariance, features detected in high angular resolution become also very sensitive to their particular orientation. Rotation invariance removes this fundamental tradeoff, but makes the features also less specific using the same angular resolution. Demanding feature detection applications may require filter banks with high angular resolution. Our second example with natural face images implies that some form of rotation invariance is a necessity with such filters. The natural variation in the features of the test images is large enough to mask the errors of the discrete rotation invariance, and little additional gain is obtained with more accurate continuous rotation invariance.

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