Optimal Encoding of Vector Data with Polygonal Approximation and Vertex Quantization

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Abstract. Problem of lossy compression of vector data is considered. We attack the problem by jointly considering data reduction by polygonal approximation and quantization of the prediction errors for approximation nodes. Optimal algorithms proposed for vector data encoding with minimal distortion for given target bit-rate, and with minimal bit-rate for given maximum deviation.

1 Introduction

We consider the problem of vector data compression, which is important for vector map storage and transmission to end-user devices, such as desktop, notebook or PDA computers. The problem has been addressed in two complementary ways: vector data reduction by polygonal approximation [2], [7], [8], [9], [11], [13], [16], [17], [18] [20], and quantization of vector data [3], [4], [12], [14], [19]. We propose a rate-distortion optimal algorithm for vector data encoding, which combines polygonal approximation of vector data and quantization of approximation nodes.

Originally the problem of optimal polygonal approximation was formulated in two basic forms [6]:

- a) *Min- ϵ problem:* Given an open N -vertex polygonal curve P , approximate it by another polygonal curve S with a given number of segments M so that the *approximation error* $E(P)$ is *minimized*.
- b) *Min-# problem:* Given an open N -vertex polygonal curve P , approximate it by another polygonal curve S with the *minimum number* of linear *segments* M that the approximation error $E(P)$ does not exceed a given maximum tolerance ϵ .

Many heuristic algorithms have been designed for these problems but they lack optimality. Optimal solution is based on dynamic programming algorithm for shortest path in graph for *min-# problem* [1], [5], [6], and *k-link* in weighted graph for *min- ϵ problem* [15], [10].

In [2], [7], [8], [9], [11], [13], [16], [17], [18], [20] polygonal approximation was applied to lossy compression of *digitized contours*. The algorithms were designed for the encoding of object contours for MPEG-4 standard where the vertices of object boundaries are defined on *uniform square grid* and can be represented with integer

undaries are defined on *uniform square grid* and can be represented with integer coordinates or by chain code. Relative coordinates of approximation nodes are encoded with variable length codes.

In the general case, vertex coordinates are given as *real numbers*. Lossless compression of such data type is not very efficient: some vertices of input polygonal curve are discarded, and the coordinates of the approximating nodes are stored with unnecessarily high accuracy. With vector quantization of the approximation nodes we achieve better compression in comparison to *lossless* encoding of the approximating data.

Straightforward quantization of the approximating nodes can destroy optimality of the polygonal approximation solution. For instance, in the case of *min-#* approximation the maximum deviation in a segment after quantization of approximating vertices may exceed the given threshold. On the other hand, applying only quantization of the vertices without reduction of the number of vertices we cannot achieve good rate-distortion performance, especially at a low bit-rate.

To get an optimal solution of the problem in question, we offer to take into joint consideration the effects caused by *polygonal approximation* and *quantization of the approximating nodes*. Given a bit budget, we have to decide whether we should invest more bits for having more approximating nodes, or for better spatial resolution of the nodes.

2 Lossy Compression of Digital Curves

An *open N-vertex polygonal curve* P in 2-dimensional space is the ordered set of vertices $P = \{p_1, p_2, \dots, p_N\}$, and $p \in \mathbb{R}^2$. The polygonal curve P is approximated by another polygonal curve $S = \{s_1, \dots, s_{M+1}\}$, where $s \in \mathbb{R}^2$, and $M < N$. End points of S are end points of P : $s_1 = p_1$, $s_{M+1} = p_N$; other nodes of S are defined from vertices of P by polygonal approximation and quantization procedures.

2.1 Vertex Prediction

To use spatial correlation of successive approximating nodes for the data compression, at first a predictor $p^*(j)$ for vertex $p(j)$ is calculated, then corresponding *prediction error* (or *residual*) is encoded. Some of preceding vertices can be used as predictor for vertex $p(j)$: $p^*(j) = p(i)$, $i < j$. Prediction error $\Delta p(j)$ is defined as difference between the current vertex $p(j)$ and its predictor $p^*(j)$:

$$\Delta p(j) = p(j) - p^*(j). \quad (1)$$

In case of vertices given on uniform square grid, the prediction errors are integer numbers that can be encoded with lossless encoder. In our case, coordinates of vector map vertices are real numbers, $\Delta p(j) \in \mathbb{R}^2$, and quantization of the prediction error induces quantization error. According to the scheme, prediction error $\Delta p(j)$ for vertex

$p(j)$ is defined as difference between the node $p(j)$ and *decoded* value $p_q(i)$ of the node $p(i)$:

$$\Delta p(j) = p(j) - p_q(i). \quad (2)$$

In its turn, the decoded value $p_q(j)$ of the point $p(j)$ is defined by quantized value $Z[\Delta p(j)]$ of the prediction error $\Delta p(j)$ and by the decoded value $p_q(i)$ of the vertex $p_q(i)$ as follows:

$$p_q(j) = p_q(i) + Z[p(j) - p_q(i)] \quad (3)$$

With the scheme we avoid accumulation of quantization errors, because every time we quantize difference between target vertex $p(j)$ and *decoded* point $p_q(i)$.

2.2 Polygonal Approximation Error

Quantization of prediction errors induces displacement of decoded approximating nodes $s(m) \equiv p_q(i_m)$ relatively its original location at point $p(i_m)$. This distortion affects on error of polygonal approximation with the nodes and should be taken into account. Approximation error for a curve segment $(p(i), p(i+1), \dots, p(j))$ with measure L_2 is defined as sum of squared distances from vertices $p(k)$ of the curve segment to the approximating line defined by end points $p_q(i)$ and $p_q(j)$ of the segment (see Fig. 1):

$$e_2(p_q(i), p_q(j)) = \sum_{k=i}^j d^2(p(k); (p_q(i), p_q(j))). \quad (4)$$

Fig. 1. Scheme of polygonal approximation of curve segment $(p(i), \dots, p(j))$ (circles) by line segment (solid line) with end points $p_q(i)$ and $p_q(j)$ (black dots).

The distance from point $p(k) = (x_k, y_k)$ to the line $(p_q(i), p_q(j))$ is given with following expression:

$$d(p(k); (p_q(i), p_q(j))) = \frac{|ax_k + by_k + c|}{a^2 + b^2} \quad (5)$$

where the coefficients a , b and c of the line can be computed from the values $p_q(i)$ and $p_q(j)$ as end points of the line segment. The approximation error $e_2(p_q(i), p_q(j))$ can be calculated in $O(1)$ time using pre-computed cumulants for coordinates x^2 , x ,

xy , y , and x^2 [15]. The total distortion (approximation error) $E_2(P)$ for the input curve P is defined as the sum of approximation errors for all approximating segments:

$$E_2(P) = \sum_{m=1}^M e_2(p_q(i_m), p_q(i_{m+1})) \quad (6)$$

The total approximation error with additive error measure L_2 for K curves $\{P_1, P_2, \dots, P_K\}$ is defined as sum of the distortions for the curves.

Approximation error for a curve segment $(p(i), p(i+1), \dots, p(j))$ with measure L_∞ is defined as maximum distance from vertices $p(k)$ of the curve segment to the approximating line with end points $p_q(i)$ and $p_q(j)$:

$$e_\infty(p_q(i), p_q(j)) = \max_{i \leq k \leq j} \{d(p(k); (p_q(i), p_q(j)))\} \quad (7)$$

Complexity of algorithm for computation of the maximal deviation for one segment is $O(N)$. The total distortion $E_\infty(P)$ for the input curve P is defined as maximum of the deviations for all segments:

$$E_\infty(P) = \max_{\{m\}} \{e_\infty(p_q(i_m), p_q(i_{m+1}))\} \quad (8)$$

Distortion with error measure L_∞ for K curves $\{P_1, P_2, \dots, P_K\}$ is defined as maximal distortions for the curves.

2.3 Encoding of Prediction Errors

To compress the data, prediction errors have to be quantized with given codebook, and indices of the codecells are to be encoded with entropy encoder; the coordinates of the start vertex $p(1)$ can be stored without compression. Vector quantizer is defined by set of codecells $\{C_1, \dots, C_L\}$ and corresponding codevectors $\{c_1, \dots, c_L\}$ [3], [4]. If prediction error $\Delta p(j)$ falls into a codecell C_l , decoded value $Z(\Delta p(j))$ of the prediction error $\Delta p(j)$ is defined by codevector of the codecell:

$$\Delta p(j) \in C_l \Rightarrow Z(\Delta p(j)) = c_l. \quad (9)$$

The integer index l of the codecell can be compressed with variable length codes. To encode a prediction error $\Delta p(j)$ with given vector quantizer, we have to test all code vectors from codebook to find the nearest one.

Uniform product quantizer is defined by quantization step q . For the quantizer the pair of indices (l_x, l_y) and quantized values $Z(\Delta x), Z(\Delta y)$, where Δx and Δy are Cartesian coordinates of $\Delta p(j)$, are defined as follows:

$$\{l_u = \lceil \Delta u / q \rceil, \quad Z(\Delta u) = q \cdot l_u, \quad (10)$$

Bit-rate for the encoding the current approximating node $p_q(i_{m+1})$ is defined by number bits $r(p_q(i_m), p_q(i_{m+1}))$ to encode the quantized prediction error $Z(\Delta p(i_{m+1}))$. The total bit-rate $R(P)$ for the approximating polygon P is sum of bit-rates for all the approximating nodes of S :

$$R(P) = \sum_{m=1}^M r(p_q(i_m), p_q(i_{m+1})), \quad (11)$$

here bit-rate for encoding the start vertex is omitted for sake of simplicity, because it does not affect on optimization procedure.

For uniform product quantizer with step q the bit-rate $r(p_q(i_m), p_q(i_{m+1}))$ can be estimated by entropy H of the prediction error:

$$H = \lceil \log_2(2|l_x|) \rceil + \lceil \log_2(2|l_y|) \rceil, \quad (12)$$

where Δx and Δy are Cartesian coordinates of prediction error $\Delta p(i_{m+1})$.

2.4 Optimal Encoding of Approximating Nodes with L_2 Error Measure

The problem of optimal polygonal approximation and quantization of prediction errors for approximating nodes can be stated as follows: approximate the polygonal curve P by the polygonal curve S so that total approximation error $E_2(P)$ is minimized, and bit-rate $R(P)$ does not exceed the given threshold $R_{\max}$:

$$\begin{aligned} E_2(P) &= \min_{\{i_m\}} \left\{ \sum_{m=1}^M e_2(p_q(i_m), p_q(i_{m+1})) \right\} \\ \text{subject to: } &\sum_{m=1}^M r(p_q(i_m), p_q(i_{m+1})) < R_{\max}. \end{aligned} \quad (13)$$

For sake of simplicity we omit term connected with start point encoding. The found set of vertices $\{p_q(i_m)\}$ give us vertices of approximating curve S : $s(m)=p_q(i_m)$.

The *constrained optimization problem* can be converted into *unconstrained problem* with Lagrange multiplier λ introducing modified cost function D :

$$D = E_2 + \lambda \cdot R = \min_{\{i_m\}} \left\{ \sum_{m=1}^M (e_2(p_q(i_m), p_q(i_{m+1})) + \lambda \cdot r(p_q(i_m), p_q(i_{m+1}))) \right\}. \quad (14)$$

To find solution of the *unconstrained* problem, a directed acyclic graph G is constructed. Nodes of the graph correspond to vertices $(p(1), p(2), \dots, p(N))$ of P . Edge $v(i, j)$ of G corresponds to approximation of curve segment $(p(i), p(i+1), \dots, p(j))$ by line segment $(p_q(i), p_q(j))$. Weight $W(i, j)$ of the edge $v(i, j)$ is defined as value of modified cost function $D(i, j)=e(i, j)+\lambda r(i, j)$. Solution of the problem under is given by shortest path on the weighted graph G with given cost function. To find solution which satisfies the constraint on bit-rate $R < R_{\max}$ we have to repeat construction of the shortest path in G for different values of λ . The optimal value of λ can to be found by bisection, for example.

To calculate the shortest path in directed acyclic graph G we need two 1-dimensional arrays: one array $D[j]$ for cost function D , another one is array $A[j]$ of pointers to parent nodes which provides optimal solution for the current node. To find the best parent node $n(i_{\text{opt}})$ for a node $n(j)$ we have to calculate cost functions for all

parent nodes $n(i)$, where $i < j$, and select that one which provides minimum of the cost function D for the node $n(j)$. In the case under consideration, weight $W(i, j)$ of edge $v(i, j)$ is defined by *decoded* values of vertex $p_q(j)$ and parent vertex $p_q(i)$. The decoded value $p_q(j)$ is defined by $p_q(i)$, see (3), and this value is different for different parent vertices considered as candidates to approximating node. To reduce processing time, for every node $n(j)$ we keep decoded value $p_q(i)$ of approximating node along with pointer to the best parent node $A[j]=n(i_{\text{opt}})$.

Weights of edges cannot be calculated in advance for the whole graph as in the case without quantization, but the weights can be calculated sequentially during the vertices processing. current state. However, weight of graph is completely defined by (decoded) previous approximation vertex, so principle of optimality is not violated, and dynamic programming algorithm can be applied for the shortest path construction in the case under consideration.

Complexity of the algorithm is defined by complexity of algorithm for the shortest path in graph, which is $O(N^2)$ and the number of iterations to calculate λ . The time complexity of approximation error calculation is $O(1)$; complexity of vector quantization is $O(L)$, where L is size of codebook (the number of codecells).

The algorithm for compression of one curve can be extended on the case of K curves. All the curves are processed sequentially, the total distortion is defined as sum of distortions for all the objects, and the total bit-rate is sum of bit-rates for the objects, too. Modified cost function $D=E_2+R$ for K objects is sum of cost functions for all the objects, and Lagrange multiplier λ is one for all shapes.

2.5 Optimal Encoding of Approximating Nodes with L_∞ Error Measure

Now let us extend the approach to the algorithm for vector data compression with L_∞ error measure. In such a case, the problem of optimal compression with polygonal approximation and prediction errors encoding can be stated as follows: approximate the polygonal curve P by the polygonal curve S so that total bit-rate $R(P)$ is minimized and approximation error $E_\infty(P)$ does not exceed the given threshold $D_{\max}$:

$$R(P) = \min_{\{i_m\}} \left\{ \sum_{m=1}^M r(p_q(i_m), p_q(i_{m+1})) \right\} \quad \text{subject to: } E_\infty(P) < D_{\max}. \quad (15)$$

Solution of the optimization problem can be found as shortest path in weighted acyclic graph G taking into consideration quantization of prediction errors for approximating nodes (vertices of S). Nodes of the *feasibility* graph G correspond to vertices of P . Edge $v(i, j)$ connecting two nodes $n(i)$ and $n(j)$ corresponds to approximation of segment $(p(i), p(i+1), \dots, p(j))$ by line segment $(p_q(i), p_q(j))$, where $p_q(i)$ and $p_q(j)$ are *decoded* values of end points $p(i)$ and $p(j)$. Weight $W(i, j)$ of the edge $v(i, j)$ is defined as follows:

$$W(i, j) = \begin{cases} r(p_q(i_m), p_q(i_{m+1})), & \text{if } \max_{i \leq k \leq j} \{d(p(k); (p_q(i_m), p_q(i_{m+1})))\} < D_{\max}, \\ \infty, & \text{otherwise.} \end{cases} \quad (16)$$

Again, in addition to two arrays to keep the current cost function (the bit-rate) values and parent nodes, we need one array to store *decoded* values of approximation nodes we need to calculate *real* distortions and bit-rates.

Complexity of the algorithm is defined by complexity of the algorithm for the shortest path construction, which is $O(N^2)$, by complexity of algorithm for maximum deviation calculation, $O(N)$, and by complexity of vector quantization. That gives us the overall time complexity $O(LN^3)$; the spatial complexity is $O(N)$. The algorithm for compression of one curve can be extended on the case of K curves in natural way: all the curves are processed individually, the total bit-rate is defined as sum of bit-rates obtained for the curves.

2.6 Construction of Vector Quantizer

Generally speaking, the distortion $E_2(P)$ or bit rate R can be minimized also by proper choice of vector quantizer. To achieve the minimum of the cost function for the given bit-rate, the optimal quantizer can be constructed basing on statistics of prediction error [3], [4], [12]. On the other hand, in the case under consideration, the statistics (probability distribution) of prediction error is defined by 1) polygonal approximation and 2) quantization of the residuals. To construct quantizer, we need statistics, but the statistics depends on the quantizer.

The problem can be solved with iterative algorithm. We can start with some vector quantizer (for instance, with uniform product scalar quantizer) to find optimal polygonal approximation for the quantizer and collect statistics of residuals. Then we design rate-distortion optimal vector quantizer using the collected statistics, and use the codebook to calculate the rate-distortion optimal polygonal approximation. The process is iterated until no improvement or some fixed number of runs.

In the case of uniform scalar product quantizer, the quantization steps q_x and q_y has to be found to minimize the cost function.

3 Results and Discussion

To illustrate proposed algorithm for *min-ε problem* and estimate performance of the algorithm we use 2903-vertex shape "Australia" (see Fig. 2). Prediction errors were quantized with uniform scalar product quantizer. Bit-rate was estimated as entropy of quantized prediction error for one point. For uniform product quantizer for relative coordinates, approximating nodes are located on regular square lattice with step q .

Result of lossy compression of the test shape is represented on Figs. 3. Original data are stored with 8 bytes per point. For target bit-rate $R=2$ bit/point, the optimal quantization step $q_{\text{opt}}=1/2^{4/5}$, the number of segments $M=1079$, and distortion $E=0.281$. Reduction of the vertex number provides 2.7:1 compression ratio, the further improvement of the ratio is result of quantization of relative coordinates.

Fig. 2. Test shape "Australia", $N=2903$. The labeled by arrow fragment depicted on Fig. 3.

Fig. 3. Result of decoding for the test shape (fragment): bit-rate $R=2$ bit/point. Vertices of original curve P are labeled with dots, approximation nodes of S are labeled with circles. The fragment is labeled by arrow on Fig. 2.

The rate-distortion functions were calculated for quantization steps $q=1/2^k$, where $k=3, 3\frac{1}{2}, 4, \dots, 11$ with step $\frac{1}{2}$ (see Fig. 4). As it follows from the Fig. 4, for the given bit-rate, bigger quantization step provides smaller distortion, because with bigger quantization step q we can afford the bigger number of approximating nodes; the bigger number of approximating nodes means smaller polygonal approximation error. On the other hand, the bigger quantization step means bigger quantization error, with big quantization error we cannot reduce the total distortion below some limit, even we still have got some reserve of bits.

With smaller quantization step we can further reduce quantization error by cost of bigger polygonal approximation component of the distortion, caused by smaller number of approximating nodes. With bigger target bit-rate we can afford smaller quantization step to invest more bits into spatial resolution of approximating nodes. So, for

given target bit-rate we have to find the optimal quantization step which provides minimum of the total distortion.

The optimal solutions of *min- ϵ problem* for different bit-rates are given by points on *lower envelope* of the curves. In the case under consideration, the dependence of logarithm of distortion $\log(D)$ on bit-rate R is almost linear in the interval 1-14 bit/point.

Fig. 4. Rate-distortion functions for quantization steps $q_k=1/2^k$, where $k=3, 3\frac{1}{2}, \dots, 11$. The corresponding rate-distortion curves are followed from left to right.

The entropy of relative coordinates that has been used for bit-rate estimation gives the theoretical low limit for the bit-rate. For real encoder the distortion should be bigger to satisfy the given constraint on bit-rate. On the other hand, with better vector quantizer we can reduce distortion and improve rate-distortion performance in comparison with the uniform product scalar quantizer in use.

4 Conclusions

The problem of optimal of vector data (vector maps) is considered. We formulate the problem by taking into joint consideration data reduction by polygonal approximation, and quantization of the prediction errors for approximation nodes. Optimal algorithms for vector data compression with minimal distortion for given target bit-rate, and with minimal bit-rate for given maximum deviation are suggested. The proposed approach can be generalized on the case of vector data approximation with non-linear functions (polynomials, splines, and wavelets).

References

1. Dunham, J.: Optimum uniform piecewise linear approximation of planar curves. *IEEE Trans. Pattern Analysis and Machine Intelligence*, 8 (1986) 67-75
2. Chung, J.-W., Lee, J.-H., Moon, J.-H., Kim, J.-K.: A new vertex-based binary shape coder for high coding efficiency. *Signal Processing: Image Communication*, 15 (2000) 665-684
3. Gersho, A., Gray, M.: *Vector Quantization and Signal Compression*. Kluwer Int. Series in Engineering and Computer Science, Vol. 152 (1992)
4. Gray, R.M., Neuhoff, D. L.: Quantization, *IEEE Trans. Information Theory*, 44 (1998) 2325-2383
5. Imai, H., Iri, M.: Computational-geometric methods for polygonal approximations of a curve. *Computer Vision, Graphics and Image Process.* 36 (1986) 31-41
6. Imai, H., Iri, M.: Polygonal approximations of a curve (formulations and algorithms). In: G.T.Toussaint, (Ed), *Computational Morphology*, North-Holland, Amsterdam, (1988) 71-86
7. Hu, M., Worrall, S., Sadka, A., Kondoz, A.M.: A scalable vertex-based shape intra-coding scheme for video objects. In: *Proc. Int. Conf. Acoustics, Speech, and Signal Process.-ICASSP04*. Vol. 3 (2004) 273-276
8. Katsaggelos, A.K., Kondi, L.P., Meier, F.W., Ostermann, J., Schuster, G.M.: MPEG-4 and rate-distortion-based shape-coding techniques. *Proc. IEEE*, 86 (1998) 1126-1154
9. Kim, J.I., Bovik, A.C., Evans, B.L.: Generalized predictive binary shape coding using polygon approximations. *Signal Processing: Image Communication*, 15 (2000) 643-663
10. Kolesnikov, A., Fränti, P.: Reduced-search dynamic programming for approximation of polygonal curves. *Pattern Recognition Letters*, 24 (2003) 2243-2254
11. Kolesnikov, A., Fränti, P.: Data reduction of large vector graphics. *Pattern Recognition*, vol. 38 (2005) 381-394
12. Akimov, A., Kolesnikov, A., Fränti, P.: Coordinate quantization in vector map compression. In: *Proc. IASTED Int. Conf. Visualization, Imaging and Image Process.-VIIP'04*, (2004) 748-753
13. Le Buhan, C., Ebrahimi, T.: Progressive polygon. encoding of shape contours. In: *Proc. Int. Conf. Image Processing and its Applications*, (1997) 1721
14. Li, Z., Openshaw, S.: Algorithms for objective generalization of line features based on the natural principle. *Int. J.Geographical Information Systems*, 6 (1992) 373-389
15. Perez, J.C., Vidal, E.: Optimum polygonal approximation of digitized curves. *Pattern Recognition Letters*, 15 (1994) 743-750
16. Servais, M., Vlachos, T.: Progressive polygon encoding of segmentation maps. In: *Proc. Int. Conf. Image Process.-ICIP'04*, Singapore (2004) 1121-1124
17. Schuster, G.M., Melnikov, G., Katsaggelos, A.K.: Operationally optimal vertex-based shape coding. *IEEE Signal Processing Magazine*, 15 (1998) 91-108
18. Schuster, G.M., Katsaggelos, A.K.: An optimal polygonal boundary encoding scheme in the rate-distortion sense. *IEEE Trans. Image Proc.* 7 (1998) 13-26
19. Shekhar, S., Huang, Y., Djugash, J., Zhou, C.: Vector map compression: a clustering approach. In: *Proc. 10th ACM Int. Symp. Advances in Geographic Inform. Syst.-GIS'02* (2002) 74-80
20. Zaletelj, J., Tasic, J.: Optimization and tracking of polygon vertices for shape coding. In: *Lecture Notes in Computer Science*, Vol. 2756. Springer-Verlag, Berlin Heidelberg New York (2003) 418-425