A LiE Subroutine for Computing Prehomogeneous Spaces Associated with Complex Nilpotent Orbits

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Abstract. We develop a LiE subroutine to compute the irreducible components of certain prehomogeneous spaces that are associated with complex nilpotent orbits. An understanding of these spaces is necessary for solving more general problems in the theory of nilpotent orbits and the representation theory of Lie groups. The output is a set of L^AT_EX statements that can be compiled in a L^AT_EX environment in order to produce tables. Although the algorithm is used to solve the problem in the case of exceptional complex reductive Lie groups [2], it does describe these prehomogeneous spaces for the classical cases also. Complete tables for the exceptional groups can be found at <http://www.math.umb.edu/~anoel/publications/tables/>.

1 Introduction

It has been long established that the representation theory of Lie groups is of fundamental importance to physics, chemistry and mathematics. However it is less known to scientists that this theory is currently being applied in many engineering applications, in computer science and in the development of financial mathematics. The theory is still being developed by mathematicians who are more and more participating in interdisciplinary projects. Consequently, progress in the representation theory of groups should be of interest to scientists and engineers working at the cutting edges of their fields.

The representation theory of semisimple Lie groups (and general reductive Lie groups) is especially important and challenging. Mathematicians have developed a large number of techniques to create different representations of a given Lie group. An understanding of the nilpotent orbits of such groups would provide important information about their possible representations. In the study and classification of such orbits the theory of prehomogeneous spaces plays an important role.

Definition 1 (M. Sato [5]). *Let G be a connected semisimple complex Lie group, V a finite dimensional vector space over $\mathbb{C}$ and ρ a rational representation of G in V . Then the triple (G, ρ, V) is called a prehomogeneous vector space if*

G admits a Zariski open dense orbit Ω in V . If ρ is irreducible then (G, ρ, V) is said to be irreducible.

We often write (G, V) instead of (G, ρ, V) and gx instead of $(\rho(g))(x)$.

Let $\mathfrak{g}$ be a semisimple complex Lie algebra and G its adjoint group. It is a fact that the number of nilpotent orbits of G in $\mathfrak{g}$ is finite. Let X be a representative of a nilpotent orbit $\mathcal{O}$ of G in $\mathfrak{g}$. Then from the Jacobson-Morozov theorem X can be embedded into a triple (H, X, Y) with H semisimple and Y nilpotent such that

$$[H, X] = 2X, \quad [H, Y] = -2Y, \quad [X, Y] = H.$$

Such an (H, X, Y) is called an $\mathfrak{sl}_2$ -triple. Given such a triple, the action of ad_H determines a grading $\mathfrak{g} = \bigoplus_{i \in \mathbb{Z}} \mathfrak{g}_i$, where $\mathfrak{g}_i = \{Z \in \mathfrak{g} : [H, Z] = iZ\}$.

It is a fact that $\mathfrak{g}_0$ is a reductive Lie subalgebra of $\mathfrak{g}$. Let G_0 be the connected subgroup of G such that $\text{Lie}(G_0) = \mathfrak{g}_0$. In 1959, Kostant [4] showed that $(G_0, \mathfrak{g}_2)$ is a prehomogeneous vector space. Later, Vinberg [7] generalized Kostant's result by showing that all $(G_0, \mathfrak{g}_i)$ pairs are prehomogeneous vector spaces.

Our goal is to describe the irreducible components of the G_0 -modules $\mathfrak{g}_i$ for all nilpotent orbits of the Lie group G in $\mathfrak{g}$.

It is enough to consider the simple complex Lie algebras. We have developed a general theory for classical complex simple Lie groups by exploiting the fact that their nilpotent orbits are parametrized by certain partitions of an integer related to their rank. Such parametrization is not available for exceptional groups. As a result our subroutine is needed in order to solve the problem in the cases where G is of exceptional type. As a byproduct we obtain another method of computing these prehomogeneous spaces for any fixed simple complex classical Lie group. All these results are found in [2].

In order to give the reader the flavor of the general result we provide a complete description of the result in the case of the general linear group.

2 Type GL_n

Let GL_n be the set of $n \times n$ complex invertible matrices. Then $\mathfrak{g} = \mathfrak{gl}_n$, the set of all $n \times n$ complex matrices. It is known that the nilpotent orbits of the group $G = GL_n$ on $\mathfrak{g}$ are parametrized by partitions of n . Therefore, if X is a nilpotent element of $\mathfrak{g}$ then up to conjugation we can take X as sum of Jordan blocks of size d_i where the d_i 's are parts of the partition $\mathbf{d} = [d_1, \dots, d_k]$ representing the class of X . For example if $n = 8$ and $\mathbf{d} = [3, 3, 2]$ then

$$X = \left(\begin{array}{ccc|ccc|cc} 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array} \right).$$

The Jacobson-Morozov theorem tells us that we can embed X in a triple (H, X, Y) . In this case we can choose H to be the following diagonal matrix of trace zero:

$$H = \text{diag}(d_1 - 1, d_1 - 3, \dots, -d_1 + 3, -d_1 + 1, \dots, d_k - 1, \dots, -d_k + 1)$$

Let $V = \mathbb{C}^n$ be the standard representation of $\mathfrak{g}$. Since H is semisimple we must have $V = \bigoplus_{i \in \mathbb{Z}} V_i$, where V_i is the H -eigenspace of eigenvalue i .

Theorem 1. *Maintaining the above notation, $G_0 \simeq \prod GL(V_i)$ and each G_0 -module $\mathfrak{g}_l \simeq \bigoplus_{i-j=l} V_i \otimes V_j^*$.*

Proof. Let G^H be the centralizer of H in G . Then G^H preserves each eigenspace V_i of H . Hence $G^H \simeq \prod GL(V_i)$. But under the adjoint action $G_0 = G^H$. Let $\mathfrak{g}_0 \simeq \bigoplus \mathfrak{gl}(V_i)$ be the Lie algebra of G_0 ; then as a $\mathfrak{g}_0$ -module we can identify $\mathfrak{g}$ with $V \otimes V^*$, which is equivalent to $\bigoplus_{i,j} V_i \otimes V_j^*$. Moreover, as $\mathfrak{g}_0$ -module, $\mathfrak{g} \simeq \bigoplus \mathfrak{g}_l$. It follows that each $\mathfrak{g}_0$ -module $\mathfrak{g}_l$ is identified exactly with $\bigoplus_{i-j=l} V_i \otimes V_j^*$ since ad_H is a derivation. $\square$

Here is an example. Let X be the representative of $\mathbf{d} = [3, 3, 2]$ given above. Then

$$H = \text{diag}(2, 0, -2, 2, 0, -2, 1, -1).$$

Since $\dim V_2 = \dim V_0 = \dim V_{-2} = 2$ and $\dim V_1 = \dim V_{-1} = 1$ we have

$$G_0 = GL_2 \times GL_2 \times GL_2 \times GL_1 \times GL_1$$

while

$$\begin{aligned} \mathfrak{g}_1 &\simeq (V_2 \otimes V_1^*) \oplus (V_1 \otimes V_0^*) \oplus (V_0 \otimes V_{-1}^*) \oplus (V_{-1} \otimes V_{-2}^*), \\ \mathfrak{g}_2 &\simeq (V_2 \otimes V_0^*) \oplus (V_1 \otimes V_{-1}^*) \oplus (V_0 \otimes V_{-2}^*), \\ \mathfrak{g}_3 &\simeq (V_2 \otimes V_1^*) \oplus (V_1 \otimes V_{-2}^*) \\ \mathfrak{g}_4 &\simeq (V_2 \otimes V_{-2}^*). \end{aligned}$$

3 Description of the Algorithm

Maintaining the above notations, let $\Delta = \{\alpha_1, \dots, \alpha_l\}$ be the Bourbaki set of simple roots of $\mathfrak{g}$ and R^+ the positive root system generated by Δ . Then there is a one-to-one correspondence between nilpotent orbits of G in $\mathfrak{g}$ and a subset of Dynkin diagrams whose nodes are labeled with elements of the set $\{0, 1, 2\}$. This is the *Dynkin-Kostant classification* [4]. Algorithms for labeling the Dynkin diagrams are found in [1]. For example, if $\mathfrak{g} = \mathfrak{sl}_8$ (the set of 8×8 matrices of trace 0) and $\mathcal{O}$ is the nilpotent orbit corresponding to the partition $\mathbf{d} = [3, 3, 2]$ then $\mathcal{O}$ is associated with the following diagram:

The labels represent the evaluations of the simple roots α_i on the diagonal matrix:

$$\hat{H} = \text{diag}(2, 2, 1, 0, 0, -1, -2, -2).$$

with $\alpha_i(\hat{H}) = \hat{h}_{i,i} - \hat{h}_{i+1,i+1}$. Moreover H is conjugate to $\hat{H}$ under the Weyl group S_8 (the symmetric group on eight letters). See [1] for more details. It follows that $\mathfrak{g}_i \simeq \bigoplus_{\alpha(\hat{H})=i} \mathfrak{g}^\alpha$ where $\mathfrak{g}^\alpha$ is the root space associated with the root α .

It is useful to observe that the type of the semisimple part of the reductive Lie group G_0 can be read from the diagram. It corresponds to the group generated by the roots labeled with zero. In the above example the semisimple part of G_0 is of type $A_1 \oplus A_1 \oplus A_1$.

Definition 2. Let v be a non-zero element of the root space $\mathfrak{g}^\lambda \subseteq \mathfrak{g}_i$ and X_α a non-zero vector in any $\mathfrak{g}^\alpha \subseteq [\mathfrak{g}_0, \mathfrak{g}_0]$ with $\alpha > 0$. If $[X_\alpha, v] = 0$ for all such X_α 's then we say that v is a highest weight vector of G_0 on $\mathfrak{g}_i$ and we call λ a highest weight of $\mathfrak{g}_i$.

We shall describe an algorithm which computes the highest weights of $\mathfrak{g}_i$ given a labeling of the Dynkin diagram corresponding to the nilpotent orbit under consideration. The subroutine is written in the language LiE. Readers who are not familiar with LiE may consult [6]. The highest weights will be expressed in terms of the fundamental weights. For the reader's convenience all LiE generic functions will be written in boldface characters. The subroutine name is `irrdmodule()`. Its input consist of a labeling string (usually the Bala-Carter label attached to the nilpotent orbit), the Dynkin-Kostant labels and the type of the simple group. The subroutine returns a set of L^AT_EX statements that generate a table containing all the highest weights of all non-zero $\mathfrak{g}_i$ for $i > 0$.

3.1 Algorithm

```
irrdmodule(tex bala;vec label; grp g ) =
{
# Formatting statements to be used for exceptional groups
# bala: Bala-Carter notation of the nilpotent orbit
# label: Dynkin-Kostant label of the nilpotent orbit
# g: complex simple Lie algebra
grptype = “ ”; if g == G2 then grptype = “\gtwo”;fi; if g == F4 then
grptype = “\ffour”;fi; if g == E6 then grptype = “\esix”;fi; if g == E7
then grptype = “\eseven”;fi; if g == E8 then grptype = “\eeight”;fi;
printflag = 0;
setdefault (g); n=n_pos_roots; l = Lie_rank; alpha = pos_roots;
if size(label) == 1 then
lookfordegree = 0; modcounter = n;
while modcounter > 0 do
degree = null(n); spaceofinterest = null(n,l);
```

```

# Compute degree positive roots
counter = 0; for i = 1 to n do for j = 1 to l do
degree[i] = degree[i] + label[j] * alpha[i][j];od;

# Build the  $g_i$  vector space specified by lookfordegree
if degree[i] == lookfordegree then counter = counter+1;
spaceofinterest[counter] = alpha[i]; fi;od;
space = null( counter, l);
for i = 1 to counter do space[i] = spaceofinterest[i]; od;
dimspace = counter;
modcounter = modcounter - dimspace;
if modcounter < 0 then print(“ Negative modcounter”); break;fi;
if dimspace > 0 && lookfordegree > 0 then

# Set up the output LATEX statement
latexline = “ ”;
if printflag == 1 then latexline = “ \cline3-5 &”; fi;
if printflag == 0 then printflag = 1;
latexline = “\hline ” + bala + “ & ” + grptype;
for i = 1 to l do
latexline = latexline+“{”+label[i] + “}” ; od; fi;
latexline = latexline + “ & ” + lookfordegree + “ & ” + dimspace +
“ & \ba”;

# Find all the highest weights of  $g_i$ 
hw = all_one ( dimspace );
for i = 1 to l do
if label[i] == 0 then
for j = 1 to dimspace do candidate = alpha[i] + space[j];

# Check if candidate is in space. If yes then it is not a highest weight
for k = 1 to dimspace do
if candidate == space [k] then hw[j] = 0; break;fi; od; fi; od; fi; od;

# Compute highest weights
numhw = 0;
for i = 1 to dimspace do numhw = numhw + hw[i]; od;
rhighestweights = null(numhw,l);
counter = 0;
for i = 1 to dimspace do
if hw[i] == 1 then
counter = counter+1; rhighestweights[counter] = space[i]; fi; od;

# Convert to fundamental weights
if numhw > 0 then
highestweights = rhighestweights * Cartan;
for i = 1 to numhw do latexline = latexline + “(” ;
for j = 1 to l do

```

```

latexline = latexline + highestweights[i][j];
if  $j < l$  then latexline = latexline + “,” fi; od;
latexline = latexline + “) \\ ”; od;
latexline = latexline + “\ea \\”; fi;
print (latexline); fi;

# Process the next  $\mathfrak{g}_i$  vector space
lookfordegree = lookfordegree+1;
od;
else print (“label is incompatible with group type”);
fi;
}

```

Example 1. We compute the highest weights of each of the four $\mathfrak{g}_i$ modules associated with the nilpotent orbit $\mathbf{d} = [3, 3, 2]$ when $\mathfrak{g} = \mathfrak{sl}_8$. Since $\mathfrak{g}_0$ is reductive we only need to consider the action of its semisimple part $[\mathfrak{g}_0, \mathfrak{g}_0]$. The subroutine is placed in a file called *phmod* which has to be loaded in the LiE environment. Here is the LiE session that produces Table 1:

LiE version 2.2 created on Nov 22 1997 at 16:50:29 Authors: Arjeh M. Cohen, Marc van Leeuwen, Bert Lisser. Mac port by S. Grimm Public distribution version

type ‘?help’ for help information
type ‘?’ for a list of help entries.

```

> read phmod
> irrdmodule(“2,2,1,0,0,-1,-2,-2”, [0,1,1,0,1,1,0], A7)

```

Observe that since the number of positive roots of $\mathfrak{sl}_8$ is 28 and the semisimple part of $\mathfrak{g}_0$ is of type $A_1 \oplus A_1 \oplus A_1$ the sum of the dimensions of the $\mathfrak{g}_i$ ’s is correct. By the “Theorem of the Highest Weight” [3–p. 279], $\mathfrak{g}_i$ is completely determined as $\mathfrak{g}_0$ -module by its highest weights. Moreover, when interpreting the results given in the table, one should be aware that the action of the semisimple part of $\mathfrak{g}_0$ on $\mathfrak{g}_i$ is completely determined by those coefficients associated with the nodes of Dynkin-Kostant label 0; the other coefficients affect only the action of the center of $\mathfrak{g}_0$, which in any case must act by scalars on each irreducible component of $\mathfrak{g}_i$. Let us denote the seven fundamental weights of $\mathfrak{sl}_8$ by ω_i for $1 \leq i \leq 7$. Then (disregarding the action of the center) we see that

$$\begin{aligned}
 \mathfrak{g}_1 &\simeq V^{\omega_1} \oplus V^{\omega_4} \oplus V^{\omega_4} \oplus V^{\omega_7} \\
 \mathfrak{g}_2 &\simeq V^0 \oplus V^{\omega_1+\omega_4} \oplus V^{\omega_4+\omega_7} \\
 \mathfrak{g}_3 &\simeq V^{\omega_1} \oplus V^{\omega_7} \\
 \mathfrak{g}_4 &\simeq V^{\omega_1+\omega_7}.
 \end{aligned}$$

Table 1.

Example in <code>sls</code> revisited				
Elementary divisors	Label	i	$\dim \mathfrak{g}_i$	Highest weights of $\mathfrak{g}_i$
2, 2, 1, 0, 0, -1, -2, -2	0110110	1	8	(1, 1, -1, 0, 0, 0, 0) (0, -1, 1, 1, -1, 0, 0) (0, 0, -1, 1, 1, -1, 0) (0, 0, 0, 0, -1, 1, 1)
		2	9	(0, -1, 1, 0, 1, -1, 0) (1, 0, 0, 1, -1, 0, 0) (0, 0, -1, 1, 0, 0, 1)
		3	4	(1, 0, 0, 0, 1, -1, 0) (0, -1, 1, 0, 0, 0, 1)
		4	4	(1, 0, 0, 0, 0, 0, 1)

where V^λ denotes the irreducible $[\mathfrak{g}_0, \mathfrak{g}_0]$ -module of highest weight λ . We note that since $\mathfrak{g}_{-i}$ is dual to $\mathfrak{g}_i$, the module structure of $\mathfrak{g}_{-i}$ is easily determined from that of $\mathfrak{g}_i$.

The correctness of the algorithm should be evident to anyone familiar with the representation theory of complex reductive Lie groups. The computation of highest weights is accomplished by finding the set of vectors in each $\mathfrak{g}_i$ which are annihilated by all positive simple root spaces in $[\mathfrak{g}_0, \mathfrak{g}_0]$. Such root spaces correspond to the simple roots of Dynkin-Kostant label 0. Since $\mathfrak{g}_0$ acts by the adjoint action, the subroutine returns only the roots β such that $\mathfrak{g}^\beta$ lies in $\mathfrak{g}_i$ and $[X_\alpha, X_\beta] = 0$ for all positive α with $\mathfrak{g}^\alpha \subset [\mathfrak{g}_0, \mathfrak{g}_0]$. This is exactly the set of highest weights of $\mathfrak{g}_i$, which is recorded in the array *rhighestweights*. Finally, we express the highest weights in terms of the fundamental weights using the Cartan matrix of $\mathfrak{g}$.

A worst case analysis of the code reveals the algorithm is $\mathcal{O}(\text{rank}(\mathfrak{g}) \times (n_{\text{pos.roots}})^3)$. This is due to the fact that the control variable of the outer **while** loop, *modcounter*, is bounded by the number of positive roots and that the more intensive internal **for** loop is at worst executed $(\text{rank}(\mathfrak{g}) \times (n_{\text{pos.roots}})^2)$ times. Of course we are assuming that the LiE internal functions are very fast. From our experience we believe that they are optimal. On average the subroutine performs very well. We use it to compute the irreducible modules of the prehomogeneous spaces associated to all the nilpotent orbits of the exceptional complex simple groups. Complete tables for the exceptional groups can be found at <http://www.math.umb.edu/~anoel/publications/tables/>. For more information see [2]. The computations were carried on an iMac G4 with speed 1GHz and 1Gb SDRAM of memory.

4 Conclusion and Future Work

We presented a LiE implementation of a simple algorithm for computing the module structures of a large class of prehomogeneous spaces, namely those asso-

ciated with nilpotent orbits in the adjoint representations of complex reductive Lie groups. In [2] we have developed general methods for the classical types, taking advantage of the parametrization of the nilpotent orbits by partitions. We developed the algorithm presented here to solve the problem for groups of exceptional type. The algorithm as implemented in LiE is stable and of a low complexity. Our implementation is clear, which makes analysis very easy. The correctness of the algorithm is a consequence of the well-known “Theorem of the Highest Weight.” Several other authors have treated special cases of this problem; our results generalize theirs to a complete solution for complex reductive groups.

We are currently working on extending these results to non-compact real reductive Lie groups. In light of the Kostant-Sekiguchi correspondence, it suffices to consider the action of the subgroup $K_{\mathbb{C}} \subset G_{\mathbb{C}}$ whose real form is maximal compact in G on the complex symmetric space $G_{\mathbb{C}}/K_{\mathbb{C}}$. This work has already started and we have obtained preliminary results.

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