Welfare for Economy Under Awareness

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Abstract. We present the extended notion of pure exchange economy under uncertainty, called an *economy with awareness structure*, where each trader having a strictly monotone preference makes decision under his/her awareness and belief, and we introduce a generalized notion of equilibrium for the economy, called an *expectations equilibrium in awareness*. We show the existence theorem of the equilibrium and the fundamental welfare theorem for the economy, i.e., an allocation in the economy is ex-ante Pareto optimal if and only if it is an expectations equilibrium allocation in awareness.

1 Introduction

This article relates economies and distributed belief. We shall present a generalized notion of economy under uncertainty, called an *economy with awareness structure*, where each trader makes decision in his/her awareness and belief under incomplete information. The purposes are: First, to introduce an extended notion of expectations equilibrium for the economy, called an *expectations equilibrium in awareness*. Secondly, to show the fundamental welfare theorem for the extended economy under expectations equilibrium in awareness.

Main Theorem. *In a pure exchange economy under uncertainty, the traders are assumed to have an awareness structure and they are risk averse. Then an allocation in the economy is ex-ante Pareto optimal if and only if it is an expectations equilibrium allocation in awareness for some initial endowment with respect to some price system.*

In Economic theory and its related fields, many authors have investigated several notions of equilibrium in an economy under asymmetric information.¹ They have studied the relationships between these equilibrium concepts (e.g.:

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¹ See the literatures cited in F. Forges, E. Minelli, and R. Vohla, *Incentive and the core of exchange economy - Survey*, Journal of Mathematical Economics 38 (2002), 1–41.

The existence theorem of equilibrium, the core equivalence theorem and the no trade theorem etc.) One of the serious limitations of their analysis is to assume ‘partition’ structure as information the traders receive. From the epistemic point of view, the partition structure represents the traders’ knowledge satisfying the postulates: ‘Truth’ **T** (what is known is true), the ‘positive introspection’ **4** (that we know what we do) and the ‘negative introspection’ **5** (that we know what we do not know). The postulate **5** is indeed so strong that describes the hyper-rationality of traders, and thus it is particularly objectionable.

This raises the question to what extent results on the information partition structure (or the equivalent postulates of knowledge). The answer is to strengthen the results: We shall weaken the conditions in the partition. This relaxation can potentially yield important results in a world with imperfectly Bayesian agents.

The idea has been performed in different settings. Geanakoplos [5] showed the no speculation theorem in the extended rational expectations equilibrium under the assumption that the information structure is reflexive, transitive and *nested*. The condition ‘nestedness’ is interpreted as a requisite on the ‘memory’ of the trader. Einy et al [4] extended the core equivalence theorem of Aumann [1] to the equivalence theorem between the ex-post core and the rational expectations equilibria for an economy under asymmetric information. Recently, Matsuhisa [6] gives an extension of the theorem into an economy under awareness structure. In his line we establish the fundamental theorem for welfare into the generalized economy.

This article is organized as follows: In Section 2 we propose the model: An economy with awareness structure and an expectations equilibrium in awareness. In Section 3 we state explicitly the fundamental theorem for welfare economics and sketch the proof. Finally we conclude by remarks.

2 The Model

Let Ω be a non-empty *finite* set called a *state space* and 2^Ω the field 2^Ω consisting of all subsets of Ω . Each member of 2^Ω is called an *event* and each element of Ω called a *state*. We denote by T the set of the *traders*. We shall present a model of awareness according to E. Dekel et al [3].²

2.1 Awareness, Belief and Information

A *belief structure* is a tuple $\langle \Omega, (B_t)_{t \in T} \rangle$ in which $B_t : 2^\Omega \rightarrow 2^\Omega$ is trader t ’s *belief operator*. The interpretation of the event $B_t E$ is that ‘ t believes E .’ An *awareness structure* is a tuple $\langle \Omega, (A_t)_{t \in T}, (B_t)_{t \in T} \rangle$ in which $\langle \Omega, (B_t)_{t \in T} \rangle$ is a belief structure and A_t is t ’s *awareness operator* on 2^Ω defined by

$$\text{PL} \quad A_t E = B_t E \cup B_t(\Omega \setminus B_t E) \quad \text{for every } E \text{ in } 2^\Omega.$$

² A different approach of awareness models is discussed in R. Fagin, J.Y. Halpern, Y. Moses and M.Y. Vardi, *Reasoning about Knowledge*. The MIT Press, Cambridge, Massachusetts, London, England, 1995.

The interpretation of $A_t E$ is that ‘ t is aware of E .’ The property **PL** says that t is aware of E if he believes it or if he believes that he does not believe it.

We shall give the generalized notion of information partition in the line of Bacharach [2] as follows.

Definition 1. The *associated information structure* $(P_t)_{t \in T}$ with awareness structure $\langle \Omega, (A_t), (B_t) \rangle$ is the class of t ’s *associated information functions* $P_t : \Omega \rightarrow 2^\Omega$ defined by $P_t(\omega) = \bigcap_{E \in 2^\Omega} \{E \mid \omega \in B_t E\}$. (If there is no event E for which $\omega \in B_t E$ then we take $P_t(\omega)$ to be undefined.) We denote by $\text{Dom}(P_t)$ the set $\{\omega \in \Omega \mid P_t(\omega) \neq \emptyset\}$, called the *domain* of P_t .

The mapping P_t is called *reflexive* if:

Ref $\omega \in P_t(\omega)$ for every $\omega \in \text{Dom}(P_t)$,

and it is said to be *transitive* if:

Trn $\xi \in P_t(\omega)$ implies $P_t(\xi) \subseteq P_t(\omega)$ for any $\xi, \omega \in \text{Dom}(P_t)$.

Furthermore P_t is called *symmetric* if:

Sym $\xi \in P_t(\omega)$ implies $P_t(\xi) \ni \omega$ for any ω and $\xi \in \text{Dom}(P_t)$.

Remark 1. M. Bacharach [2] introduces the *strong epistemic model* equivalent to the Kripke semantics of the modal logic **S5**. The strong epistemic model is a tuple $\langle \Omega, (K_t)_{t \in T} \rangle$ in which t ’s *knowledge operator* $K_t : 2^\Omega \rightarrow 2^\Omega$ satisfies the five postulates: For every E, F of 2^Ω ,

$$\begin{array}{lll} \mathbf{N} & K_t \Omega = \Omega, & \mathbf{K} \quad K_t(E \cap F) = K_t E \cap K_t F, \quad \mathbf{T} \quad K_t F \subseteq F; \\ \mathbf{4} & K_t F \subseteq K_t K_t F, & \mathbf{5} \quad \Omega \setminus K_t F \subseteq K_t(\Omega \setminus K_t F). \end{array}$$

t ’s associated information function P_t induced by K_t makes a partition of Ω , called t ’s *information partition*, which satisfies the postulates **Ref**, **Trn** and **Sym**. This is just the Kripke semantics corresponding to the logic **S5**; the postulates **Ref**, **Trn** and **Sym** are respectively equivalent to the postulates **T**, **4** and **5**. The strong epistemic model can be interpreted as the awareness structure $\langle \Omega, (A_t), (B_t) \rangle$ such that B_t is the knowledge operator. In this situation it is easily verified that A_t must be the *trivial* operator,³ and that $\text{Dom}(P_t) = \Omega$.

2.2 Economy with Awareness Structure

A pure exchange economy *under uncertainty* is a structure

$$\mathcal{E} = \langle T, \Omega, \mathbf{e}, (U_t)_{t \in T}, (\pi_t)_{t \in T} \rangle$$

³ I.e. $A_t(F) = \Omega$ for every $F \in 2^\Omega$.

consisting of the following structure and interpretations: There are l commodities in each state of the state space Ω ; the consumption set of each trader t is $\mathbf{R}_+^l$; an *initial endowment* is a mapping $\mathbf{e} : T \times \Omega \rightarrow \mathbf{R}_+^l$ with which $\mathbf{e}(t, \cdot) : \Omega \rightarrow \mathbf{R}_+^l$ is called t 's *initial endowment*; $U_t : \mathbf{R}_+^l \times \Omega \rightarrow \mathbf{R}$ is t 's von-Neumann and Morgenstern utility function; π_t is a subjective prior on Ω for $t \in T$. For simplicity, π_t is assumed to be *full support* for all $t \in T$. That is, $\pi_t(\omega) \neq 0$ for every $\omega \in \Omega$.

Definition 2. A pure exchange economy *with awareness structure* is a structure $\mathcal{E}^A = \langle \mathcal{E}, (A_t)_{t \in T}, (B_t)_{t \in T}, (P_t)_{t \in T} \rangle$, in which $\mathcal{E}$ is a pure exchange economy under uncertainty, and $\langle \Omega, (A_t)_{t \in T}, (B_t)_{t \in T}, (P_t)_{t \in T} \rangle$ is an awareness structure with $(P_t)_{t \in T}$ the associated information structure. By the *domain* of the economy $\mathcal{E}^A$ we mean $\text{Dom}(\mathcal{E}^A) = \cap_{t \in T} \text{Dom}(P_t)$. We always assume that $\text{Dom}(\mathcal{E}^A) \neq \emptyset$.

Remark 2. An economy under asymmetric information is an economy $\mathcal{E}^A$ with the awareness structure $\langle \Omega, (A_t)_{t \in T}, (B_t)_{t \in T} \rangle$ given by the strong epistemic model, and that $\text{Dom}(\mathcal{E}^A) = \Omega$.

We denote by $\mathcal{F}_t$ the field of $\text{Dom}(P_t)$ generated by $\{P_t(\omega) \mid \omega \in \Omega\}$ and denote by $\Pi_t(\omega)$ the atom containing $\omega \in \text{Dom}(P_t)$. We denote by $\mathcal{F}$ the join of all $\mathcal{F}_t$ ($t \in T$) on $\text{Dom}(\mathcal{E}^A)$; i.e. $\mathcal{F} = \vee_{t \in T} \mathcal{F}_t$, and denote by $\{\Pi(\omega) \mid \omega \in \text{Dom}(\mathcal{E}^A)\}$ the set of all atoms $\Pi(\omega)$ containing ω of the field $\mathcal{F} = \vee_{t \in T} \mathcal{F}_t$. We shall often refer to the following conditions: For every $t \in T$,

- A-1** $\sum_{t \in T} \mathbf{e}(t, \omega) > 0$ for each $\omega \in \Omega$.
- A-2** $\mathbf{e}(t, \cdot)$ is $\mathcal{F}$ -measurable on $\text{Dom}(P_t)$.
- A-3** For each $x \in \mathbf{R}_+^l$, the function $U_t(x, \cdot)$ is at least $\mathcal{F}$ -measurable on $\text{Dom}(\mathcal{E}^A)$, and the function: $T \times \mathbf{R}_+^l \rightarrow \mathbf{R}, (t, x) \mapsto U_t(x, \omega)$ is $\Sigma \times \mathcal{B}$ -measurable where $\mathcal{B}$ is the σ -field of all Borel subsets of $\mathbf{R}_+^l$.
- A-4** For each $\omega \in \Omega$, the function $U_t(\cdot, \omega)$ is strictly increasing on $\mathbf{R}_+^l$, continuous, strictly quasi-concave and *non-satiated* on $\mathbf{R}_+^l$.⁴

2.3 Expectations Equilibrium in Awareness

An *assignment* is a mapping $\mathbf{x} : T \times \Omega \rightarrow \mathbf{R}_+^l$ such that for each $t \in T$, the function $\mathbf{x}(t, \cdot)$ is at least $\mathcal{F}$ -measurable on $\text{Dom}(\mathcal{E}^A)$. We denote by $\mathcal{Ass}(\mathcal{E}^A)$ the set of all assignments for the economy $\mathcal{E}^A$. By an *allocation* we mean an assignment $\mathbf{a}$ such that $\mathbf{a}(t, \cdot)$ is $\mathcal{F}$ -measurable on $\text{Dom}(\mathcal{E}^A)$ for all $t \in T$ and $\sum_{t \in T} \mathbf{a}(t, \omega) \leq \sum_{t \in T} \mathbf{e}(t, \omega)$ for every $\omega \in \Omega$. We denote by $\mathcal{Alc}(\mathcal{E}^A)$ the set of all allocations.

We introduce the revised notion of trader's expectation of utility in $\mathcal{E}^A$. By t 's *ex-ante* expectation we mean $\mathbf{E}_t[U_t(\mathbf{x}(t, \cdot))] := \sum_{\omega \in \text{Dom}(P_t)} U_t(\mathbf{x}(t, \omega), \omega) \pi_t(\omega)$ for each $\mathbf{x} \in \mathcal{Ass}(\mathcal{E}^A)$. The *interim* expectation $\mathbf{E}_t[U_t(\mathbf{x}(t, \cdot) \mid P_t)]$ is defined by $\mathbf{E}_t[U_t(\mathbf{x}(t, \cdot) \mid P_t)](\omega) := \sum_{\xi \in \text{Dom}(P_t)} U_t(\mathbf{x}(t, \xi), \xi) \pi_t(\{\xi\} \cap A_t(\{\xi\}) \mid P_t(\omega))$ on

⁴ That is, for any $x \in \mathbf{R}_+^l$ there exists an $x' \in \mathbf{R}_+^l$ such that $U_i(x', \omega) > U_i(x, \omega)$.

$\text{Dom}(P_t)$. It should be noted that we use not the usual notion of posterior $\pi_t(\{\xi\}|P_t(\omega))$ but the revised one $\pi_t(\{\xi\} \cap A_t(\{\xi\})|P_t(\omega))$.⁵

A *price system* is a non-zero function $p : \Omega \rightarrow \mathbf{R}_+^l$ which is $\mathcal{F}$ -measurable on $\text{Dom}(\mathcal{E}^A)$. We denote by $\Delta(p)$ the partition on Ω induced by p , and denote by $\sigma(p)$ the field of Ω generated by $\Delta(p)$. The *budget set* of a trader t at a state ω for a price system p is defined by $B_t(\omega, p) := \{x \in \mathbf{R}_+^l \mid p(\omega) \cdot x \leq p(\omega) \cdot \mathbf{e}(t, \omega)\}$. Define the mapping $\Delta(p) \cap P_t : \text{Dom}(P_t) \rightarrow 2^\Omega$ by $(\Delta(p) \cap P_t)(\omega) := \Delta(p)(\omega) \cap P_t(\omega)$. We denote by $\text{Dom}(\Delta(p) \cap P_t)$ the set of all states ω in which $\Delta(p)(\omega) \cap P_t(\omega) \neq \emptyset$. Let $\sigma(p) \vee \mathcal{F}_t$ be the smallest σ -field containing both the fields $\sigma(p)$ and $\mathcal{F}_t$.

Definition 3. An *expectations equilibrium in awareness* for an economy $\mathcal{E}^A$ with awareness structure is a pair $(p, \mathbf{x})$, in which p is a price system and $\mathbf{x}$ is an assignment satisfying the following conditions:

EA1. $\mathbf{x}$ is an allocation;

EA2. For all $t \in T$ and for every $\omega \in \Omega$, $\mathbf{x}(t, \omega) \in B_t(\omega, p)$;

EA3. For all $t \in T$, if $\mathbf{y}(t, \cdot) : \Omega \rightarrow \mathbf{R}_+^l$ is $\mathcal{F}$ -measurable on $\text{Dom}(\mathcal{E}^A)$ with $\mathbf{y}(t, \omega) \in B_t(\omega, p)$ for all $\omega \in \Omega$, then

$$\mathbf{E}_t[U_t(\mathbf{x}(t, \cdot)) | \Delta(p) \cap P_t](\omega) \geq \mathbf{E}_t[U_t(\mathbf{y}(t, \cdot)) | \Delta(p) \cap P_t](\omega)$$

pointwise on $\text{Dom}(\Delta(p) \cap P_t)$;

EA4. For every $\omega \in \text{Dom}(\mathcal{E}^A)$, $\sum_{t \in T} \mathbf{x}(t, \omega) = \sum_{t \in T} \mathbf{e}(t, \omega)$.

The allocation $\mathbf{x}$ in $\mathcal{E}^A$ is called an *expectations equilibrium allocation* in awareness for $\mathcal{E}^A$.

We denote by $EA(\mathcal{E}^A)$ the set of all the expectations equilibria of a pure exchange economy $\mathcal{E}^A$, and denote by $\mathcal{A}(\mathcal{E}^A)$ the set of all the expectations equilibrium allocations in awareness for the economy.

3 The Results

Let $\mathcal{E}^A$ be the economy with awareness structure and $\mathcal{E}^A(\omega)$ the economy with complete information $\langle T, (\mathbf{e}(t, \omega))_{t \in T}, (U_t(\cdot, \omega))_{t \in T} \rangle$ for each $\omega \in \Omega$. We denote by $\mathcal{W}(\mathcal{E}^A(\omega))$ the set of all competitive equilibria for $\mathcal{E}^A(\omega)$.

3.1 Existence of Equilibrium in Awareness

Theorem 1. Let $\mathcal{E}^A$ be a pure exchange economy with awareness structure satisfying the conditions **A-1**, **A-2**, **A-3** and **A-4**. Then there exists an expectations equilibrium in awareness for the economy; i.e., $EA(\mathcal{E}^A) \neq \emptyset$.

⁵ A discussion why this improvement of the notion of posterior is needed is given in T. Matsuhisa and S.-S. Usami, *Awareness, belief and agreeing to disagree*, Far East Journal of Mathematical Sciences 2(6) (2000) 833–844.

Before proceeding with the proof we shall note that:

Lemma 1. *The event $(\Delta(p) \cap P_t)(\omega)$ can be decomposed into the disjoint union $(\Delta(p) \cap P_t)(\omega) = \cup_{k=1}^p \Pi(\xi_k)$. Furthermore, for $\mathbf{x} \in \text{Ass}(\mathcal{E}^A)$, $\mathbf{E}_t[U_t(\mathbf{x}(t, \cdot)) | \Delta(p) \cap P_t](\omega) = \sum_{k=1}^p \frac{\pi_t(\Pi(\xi_k))}{\pi_t((\Delta(p) \cap P_t)(\omega))} U_t(\mathbf{x}(t, \xi_k), \xi_k)$. $\square$*

Proof of Theorem 1. In view of the existence theorem of a competitive equilibrium for an economy with complete information,⁶ it follows that there exists a $(p^*(\omega), \mathbf{x}^*(\cdot, \omega)) \in \mathcal{W}(\mathcal{E}^A(\omega))$ for each $\omega \in \Omega$ by the conditions **A-1**, **A-2**, **A-3** and **A-4**. Define the pair $(p, \mathbf{x})$ as follows: For each $\omega \in \Omega$, denote $\mathbf{x}(t, \xi) := \mathbf{x}^*(t, \omega)$ for all $\xi \in \Pi(\omega)$ and $\omega \in \text{Dom}(\mathcal{E}^A)$, and set $p(\xi) := p^*(\omega)$ for all $\xi \in \Pi(\omega)$ and $\omega \in \text{Dom}(\mathcal{E}^A)$, $p(\xi) := p^*(\omega)$ for $\omega \notin \text{Dom}(\mathcal{E}^A)$. Then we can verify that $(p, \mathbf{x})$ is an expectations equilibrium in awareness for $\mathcal{E}^A$: For **EA3**. On noting that $\mathcal{E}^A(\xi) = \mathcal{E}^A(\omega)$ for any $\xi \in \Pi(\omega)$, it follows that $(p(\xi), \mathbf{x}(t, \xi)) \in \mathcal{W}(\mathcal{E}^A(\omega))$ for every $\omega \in \Omega$, and thus we can observe **EA3** by Lemma 1. The other conditions in Definition 3 are easily verified. $\square$

3.2 Fundamental Theorem for Welfare Economics

An allocation $\mathbf{x}$ in $\mathcal{E}^A$ is said to be *ex-ante Pareto-optimal* if there is no allocation $\mathbf{a}$ such that $\mathbf{E}_t[U_t(\mathbf{a}(t, \cdot))] \geq \mathbf{E}_t[U_t(\mathbf{x}(t, \cdot))]$ for all $t \in T$ with at least one inequality strict. We can now state our main theorem.

Theorem 2. *Let $\mathcal{E}^A$ be an economy with awareness structure satisfying the conditions **A-1**, **A-2**, **A-3** and **A-4**. An allocation is ex-ante Pareto optimal if and only if it is an expectations equilibrium allocation in awareness for some initial endowment $\mathbf{w}$ with respect to some price system such that $\sum_{t \in T} \mathbf{w}(t, \omega) = \sum_{t \in T} \mathbf{e}(t, \omega)$ for each $\omega \in \text{Dom}(\mathcal{E}^A)$.*

Proof. Follows immediately from Propositions 1 and 2 as below. $\square$

Proposition 1. *Let $\mathcal{E}^A$ be an economy with awareness structure satisfying the conditions **A-1**, **A-2**, **A-3** and **A-4**. Then an allocation $\mathbf{x}$ is ex-ante Pareto optimal if it is an expectations equilibrium allocation in awareness with respect to some price system.*

Proposition 2. *Let $\mathcal{E}^A$ be an economy with awareness structure satisfying the conditions **A-1**, **A-2**, **A-3** and **A-4**. If an allocation $\mathbf{x}$ is ex-ante Pareto optimal in $\mathcal{E}^A$ then there are a price system and an initial endowment $\mathbf{e}'$ such that $\mathbf{x}$ is an expectations equilibrium allocation in awareness with $\sum_{t \in T} \mathbf{e}'(t, \omega) = \sum_{t \in T} \mathbf{e}(t, \omega)$ for each $\omega \in \text{Dom}(\mathcal{E}^A)$.*

⁶ C.f.: Theorem 5 in G. Debreu, *Existence of competitive equilibrium*, in: Handbook of Mathematical Economics, Volume 2, K.J.Arrow and M.D.Intriligator (eds), North-Holland Publishing Company, Amsterdam, 1982, 697–744.

3.3 Proof of Propositions 1 and 2

Before proving the propositions we first establish

Proposition 3. *Let $\mathcal{E}^A$ be an economy with awareness structure satisfying the conditions **A-1**, **A-2**, **A-3** and **A-4**. Then $\mathcal{A}(\mathcal{E}^A) = \{\mathbf{x} \in \text{Alc}(\mathcal{E}^A) \mid \text{There is a price system } p \text{ such that } (p(\omega), \mathbf{x}(\cdot, \omega)) \in \mathcal{W}(\mathcal{E}^A(\omega)) \text{ for all } \omega \in \text{Dom}(\mathcal{E}^A)\}$.*

Proof. Let $\mathbf{x} \in \mathcal{A}(\mathcal{E}^A)$ and $(p, \mathbf{x}) \in EA(\mathcal{E}^A)$. We shall show that $(p(\omega), \mathbf{x}(\cdot, \omega)) \in \mathcal{W}(\mathcal{E}^A(\omega))$ for any $\omega \in \text{Dom}(\mathcal{E}^A)$. Suppose to the contrary that there exist a trader $s \in T$ and states $\omega' \in \text{Dom}(\mathcal{E}^A)$, $\omega_0 \in (\Delta(p) \cap P_s)(\omega')$ with the property: There is an $\mathbf{a}(s, \omega_0) \in B_s(\omega_0, p)$ such that $U_s(\mathbf{a}(s, \omega_0), \omega_0) > U_s(\mathbf{x}(s, \omega_0), \omega_0)$. Define the $\mathcal{F}$ -measurable function $\mathbf{y} : T \times \Omega \rightarrow \mathbf{R}_+^I$ by $\mathbf{y}(t, \xi) := \mathbf{a}(t, \omega_0)$ for $\xi \in \Pi(\omega_0)$, and $\mathbf{y}(t, \xi) := \mathbf{x}(t, \xi)$ otherwise. It follows immediately by Lemma 1 that $\mathbf{E}_s[U_s(\mathbf{x}(s, \cdot)) | \Delta(p) \cap P_s](\omega') < \mathbf{E}_s[U_s(\mathbf{y}(s, \cdot)) | \Delta(p) \cap P_s](\omega')$, in contradiction.

The converse will be shown: Let $\mathbf{x} \in \text{Ass}(\mathcal{E}^A)$ with $(p(\omega), \mathbf{x}(\cdot, \omega)) \in \mathcal{W}(\mathcal{E}^A(\omega))$ for any $\omega \in \text{Dom}(\mathcal{E}^A)$. Set the price system $p^* : \Omega \rightarrow \mathbf{R}_+^I$ by $p^*(\xi) := p(\omega)$ for all $\xi \in \Pi(\omega)$ and $\omega \in \text{Dom}(\mathcal{E}^A)$, and $p^*(\xi) := p(\omega)$ for $\omega \notin \text{Dom}(\mathcal{E}^A)$. We shall show that $(p^*, \mathbf{x}) \in EA(\mathcal{E}^A)$: $\mathbf{x}(t, \cdot)$ is $\mathcal{F}$ -measurable and $\mathbf{x}(t, \omega) \in B_t(\omega, p^*)$ on $\text{Dom}(\mathcal{E}^A)$ for all $t \in T$. It can be plainly observed that **EA1**, **EA2** and **EA4** are all valid. For **EA3**: Let $\mathbf{y}(t, \cdot) : \Omega \rightarrow \mathbf{R}_+^I$ be an $\mathcal{F}$ -measurable function with $\mathbf{y}(t, \omega) \in B_t(\omega, p^*)$ for all $\omega \in \text{Dom}(\mathcal{E}^A)$. Since $(p^*(\omega), \mathbf{x}(\cdot, \omega)) \in \mathcal{W}(\mathcal{E}^A(\omega))$ it follows that $U_t(\mathbf{x}(t, \omega), \omega) \geq U_t(\mathbf{y}(t, \omega), \omega)$ for all $t \in T$ and for each $\omega \in \text{Dom}(\mathcal{E}^A)$. By Lemma 1, $\mathbf{E}_t[U_t(\mathbf{x}(t, \cdot)) | \Delta(p^*) \cap P_t](\omega) \geq \mathbf{E}_t[U_t(\mathbf{y}(t, \cdot)) | \Delta(p^*) \cap P_t](\omega)$ for all $\omega \in \text{Dom}(\Delta(p^*) \cap P_t)$, and so $(p^*, \mathbf{x}) \in EA(\mathcal{E}^A)$, in completing the proof. $\square$

Proof of Proposition 1. It follows from Proposition 3 that $(p(\omega), \mathbf{x}(\cdot, \omega)) \in \mathcal{W}(\mathcal{E}^A(\omega))$ at each $\omega \in \text{Dom}(\mathcal{E}^A)$. By the fundamental theorem of welfare in the economy $\mathcal{E}^A(\omega)$, we can plainly observe that for all $\omega \in \text{Dom}(\mathcal{E}^A)$, $\mathbf{x}(\cdot, \omega)$ is Pareto optimal in $\mathcal{E}^A(\omega)$, and thus $\mathbf{x}$ is ex-ante Pareto optimal. $\square$

Proof of Proposition 2. It can be shown that *for each $\omega \in \Omega$ there exists $p^*(\omega) \in \mathbf{R}_+^I$ such that $(p^*(\omega), \mathbf{x}(\cdot, \omega)) \in \mathcal{W}(\mathcal{E}^A(\omega))$ for some initial endowment $\mathbf{e}'(\cdot, \omega)$ with $\sum_{t \in T} \mathbf{e}'(t, \omega) = \sum_{t \in T} \mathbf{e}(t, \omega)$* . *Proof:* First it can be observed that for each $\omega \in \Omega$ there exists $p^*(\omega) \in \mathbf{R}_+^I$ such that $p^*(\omega) \cdot v \leq 0$ for all $v \in G(\omega) = \{ \sum_{t \in T} \mathbf{x}(t, \omega) - \sum_{t \in T} \mathbf{y}(t, \omega) \in \mathbf{R}^I \mid \mathbf{y} \in \text{Ass}(\mathcal{E}^A) \text{ and } U_t(\mathbf{y}(t, \omega), \omega) \geq U_t(\mathbf{x}(t, \omega), \omega) \text{ for all } t \in T \}$ for each $\omega \in \text{Dom}(\mathcal{E}^A)$: In fact, on noting that that $G(\omega)$ is convex and closed in $\mathbf{R}_+^I$ by the conditions **A-1**, **A-2**, **A-3** and **A-4**, the assertion immediately follows from the fact that $v \leq 0$ for all $v \in G(\omega)$ by the separation theorem⁷: Suppose to the contrary. Let $\omega_0 \in \Omega$ and $v_0 \in G(\omega_0)$ with $v_0 > 0$. Take $\mathbf{y}^0 \in \text{Ass}(\mathcal{E}^A)$ such that for all t , $U_t(\mathbf{y}^0(t, \omega), \omega_0) \geq U_t(\mathbf{x}(t, \omega_0), \omega_0)$ and $v_0 = \sum_{t \in T} \mathbf{x}(t, \omega_0) - \sum_{t \in T} \mathbf{y}^0(t, \omega_0)$. Let $\mathbf{z} \in \text{Alc}(\mathcal{E}^A)$ be defined by $\mathbf{z}(t, \xi) := \mathbf{y}^0(t, \omega_0) + \frac{v_0}{|T|}$ if $\xi \in \Pi(\omega_0)$, $\mathbf{z}(t, \xi) := \mathbf{x}(t, \xi)$ if not. By **A-4**

⁷ C.f.: Lemma 8, Chapter 4 in K. J. Arrow and F. H. Hahn, *General competitive analysis*, North-Holland Publishing Company, Amsterdam, 1971. p.92.

it follows that for all $t \in T$, $\mathbf{E}_t[U_t(\mathbf{z})] \geq \mathbf{E}_t[U_t(\mathbf{x})]$, in contradiction to which $\mathbf{x}$ is ex-ante Pareto optimal. By a similar argument in the proof of the second fundamental theorem of welfare economics,⁸ we can verify that $(p^*(\omega), \mathbf{x}(\cdot, \omega)) \in \mathcal{W}(\mathcal{E}^A(\omega))$ for some initial endowment $\mathbf{e}'$ with $\sum_{t \in T} \mathbf{e}'(t, \omega) = \sum_{t \in T} \mathbf{e}(t, \omega)$.

Now, let p be the price system defined by: $p(\xi) := p^*(\omega)$ for all $\xi \in \Pi(\omega)$ and $\omega \in \text{Dom}(\mathcal{E}^A)$, $p(\xi) := p^*(\omega)$ for $\omega \notin \text{Dom}(\mathcal{E}^A)$. Further we extend $\mathbf{e}'$ to the initial endowment $\mathbf{w}$ for $\mathcal{E}^A$ by $\mathbf{w}(t, \xi) := \mathbf{e}'(t, \omega)$ for all $\xi \in \Pi(\omega)$ and $\omega \in \text{Dom}(\mathcal{E}^A)$. It can be observed that $\mathbf{w}(t, \cdot)$ is $\mathcal{F}$ -measurable with $\sum_{t \in T} \mathbf{w}(t, \omega) = \sum_{t \in T} \mathbf{e}'(t, \omega)$.

To conclude the proof we shall show that $(p, \mathbf{x}) \in EA(\mathcal{E}^A)$. *Proof:* For each $\omega \in \text{Dom}(\mathcal{E}^A)$, there exists ξ such that $\xi \in (\Delta(p) \cap P_t)(\omega) = \Delta(p)(\xi) = \Pi(\xi)$, and so we can observe by **A-3** that for each $\mathbf{x} \in \text{Alc}(\mathcal{E}^A)$, $\mathbf{E}_t[U_t(\mathbf{x}(t, \cdot))](\Delta(p) \cap P_t)(\omega) = U_t(\mathbf{x}(t, \xi), \xi)$. We shall verify **EA3** only: Suppose to the contrary that there exists $s \in T$ with the two properties: (i) there is an $\mathcal{F}$ -measurable function $\mathbf{y}(s, \cdot) : \Omega \rightarrow \mathbf{R}_+^l$ such that $\mathbf{y}(s, \omega) \in B_s(\omega, p)$ for all $\omega \in \Omega$; and (ii) $\mathbf{E}_s[U_s(\mathbf{y}(s, \cdot))](\Delta(p) \cap P_s)(\omega_0) > \mathbf{E}_s[U_s(\mathbf{x}(s, \cdot))](\Delta(p) \cap P_s)(\omega_0)$ for some $\omega_0 \in \text{Dom}(\Delta(p) \cap P_s)$. In view of the above equation it follows from (ii) that there exists $\xi \in (\Delta(p) \cap P_t)(\omega_0)$ with $U_s(\mathbf{y}(s, \xi), \xi) > U_s(\mathbf{x}(s, \xi), \xi)$, and thus $\mathbf{y}(s, \xi) > \mathbf{x}(s, \xi)$ by **A-4**. Thus $p(\xi) \cdot \mathbf{y}(s, \xi) > p(\xi) \cdot \mathbf{x}(s, \xi)$, in contradiction. $\square$

4 Concluding Remarks

Our real concern in this article is about relationship between players' beliefs and their decision making, especially when and how the players take corporate actions under their decisions. We focus on extending the fundamental theorem of welfare economics into an economy with traders having 'awareness and belief' model. We have shown that the nature of the theorem is dependent not on common-belief nor on the partition structure of traders' information, but on the structure of awareness and belief when each player receives information.

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⁸ C.f.: Proposition 16.D.1 in A. Mas-Colell, M. Whinston, and J. Green, *Microeconomics Theory*. Oxford University Press, 1995, pp. 552–554.