The Dynamics of General Fuzzy Cellular Automata[1]

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Abstract. We continue the investigation into the dynamics and evolution of fuzzy rules, obtained by the fuzzification of the disjunctive normal form, and initiated for rule 90 in [2], for rule 110 in [10] and for rule 30 in [3]. We present general methods for detecting the evolution and dynamics of any one of the 255 fuzzy rules and apply this theory to fuzzy rules 30, 110, 18, 45, and 184, each of which has a Boolean counterpart with interesting features. Finally, it is deduced that (except for at most nine cases) no fuzzy cellular automaton admits chaotic behavior in the sense that no sensitive dependence on the initial string can occur.

1 Introduction

This work is motivated by a question originally posed by Andy Wuensche [1] regarding the convergence of fuzzy rules induced by fuzzy cellular automata (CA). He asked whether the results in [2] for fuzzy rule 90 presented in the conference cited in [1] could be extended to fuzzy rule 30 in the generality obtained in [2]. Although this was answered in [3] we choose to go beyond this and provide a framework for discovering the global evolution of an arbitrary fuzzy CA, cf., [4]. We develop some methods for obtaining limiting information about any one of the 255 fuzzy rules. Recent work in this new area includes some variations on the game of Life in [5] and applications to pattern recognition [6]. In addition, such CA have been used to investigate the result of perturbations, for example, noisy sources, computation errors, mutations, etc. on the evolution of boolean CA (cf., [7], [4], [2], etc.).

Recall some basic terminology from [2]. A CA is a collection of cells arranged on a graph; all cells share the same local space, the same neighborhood structure and the same local function (i.e., the function defining the effect of neighbors on each cell). Given a linear bi-infinite lattice of cells, the local Boolean space $\{0, 1\}$, the neighborhood structure (left neighbor, itself, right neighbor), and a

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local rule $g : \{0,1\}^3 \rightarrow \{0,1\}$, the global dynamics of an *elementary CA* are defined by (cf., [8]) $f : \{0,1\}^{\mathbb{Z}} \rightarrow \{0,1\}^{\mathbb{Z}}$ and $f(x)_i = g(x_{i-1}, x_i, x_{i+1})$, for all i. The *local rule* is defined by the 8 possible local configurations that a cell detects in its neighborhood: $(000, 001, 010, 011, 100, 101, 110, 111) \rightarrow (r_0, \cdots, r_7)$, where each triplet represents a local configuration of the left neighbor, the cell itself, and the right neighbor. In general, the value $\sum_{i=0}^{7} 2^i r_i$ is used as the *name* of the rule. As usual, the local rule of any Boolean CA is expressed as a *disjunctive normal form*: $g(x_1, x_2, x_3) = \vee_{i|r_i=1} \wedge_{j=1}^{3} x_j^{d_{ij}}$ where d_{ij} is the j-th digit, from left to right, of the binary expression of i, and x^0 (resp. x^1) stands for $\neg x$ (resp. x). A *Fuzzy CA* is obtained by *fuzzification* of the local function of a Boolean CA: in the disjunctive normal form by redefining $(a \vee b)$ as $(a+b)$, $(a \wedge b)$ as (ab), and $(\neg a)$ as $(1-a)$. The usual fuzzification of the expression $a \vee b$ is $max\{1, a+b\}$ so as to ensure that the result is not larger than 1. Note, however, that taking $(a+b)$ for the CA fuzzification does not lead to values greater than 1 since the sum of all the expressions for rule 255 is 1 (*i.e.,* $g_{255}(x, y, z) = 1$), and so every (necessarily non-negative) partial sum must be bounded by 1. Since every fuzzy rule is obtained by adding one or more of these partial sums it follows that every fuzzy rule is bounded below by 0 and above by 1. We will be analyzing the behavior of an *odd* fuzzy rule, rule 45, towards the end of this paper.

As an example, we note that rule $45 = 2^0 + 2^2 + 2^3 + 2^5$ has the local rule $(000, 001, 010, 011, 100, 101, 110, 111) \rightarrow (1, 0, 1, 1, 0, 1, 0, 0)$. Its canonical expression is $g_{45}(x_1, x_2, x_3) = (\neg x_1 \wedge \neg x_2 \wedge \neg x_3) \vee (\neg x_1 \wedge x_2 \wedge x_3) \vee (x_1 \wedge \neg x_2 \wedge x_3) \vee (\neg x_1 \wedge x_2 \wedge \neg x_3)$ and its fuzzification gives $g_{45}(x_1, x_2, x_3) = 1 - x_1 - x_3 + x_2 x_3 + 2x_1 x_3 - 2x_1 x_2 x_3$. In the same way we derive the local rules for rules 45 and 184. One of the exceptional rules (one of nine that defies assumption (I), stated in the next section), is rule 184 (see [4]) which we will analyze at the very end. The dynamics of these nine rules are interesting in that the methods presented herein require some modification, yet even so, it cannot be asserted at this time that we can determine their dynamics in general. See the last subsection for details.

Let $g_n(x_1, x_2, x_3)$, $1 \leq n \leq 255$, denote the canonical expression of fuzzy rule n. We know that the disjunctive normal form for a fuzzy rule is given by $g_n(x_1, x_2, x_3) = \vee_{i|r_i=1} \wedge_{j=1}^{3} x_j^{d_{ij}}$ where $0 \leq d_{ij} \leq 1$ is the integer defined above. Since $x_j^0 = 1 - x_j$, $x_j^1 = x_j$, and the disjunction is an additive operation it follows that g_n is a linear map in each variable separately and so satisfies Laplace's equation (see [9]). Thus, maximum principles (see [9], Chapter 4) can be used to derive properties of such rules under iterations.

2 The Long Term Dynamics of General Rules

We fix the notation and recall definitions from [2]. The *light cone* from a cell x_i^t is the set of $\{x_j^{t+p} \mid p \geq 0 \, \text{and} \, j \in \{i-p, \cdots, i+p\}\}$. In this case, the light cone is the boundary of an infinite triangle whose vertex is at the singleton a and whose boundary consists of all the other a's. Thus, $x_{\pm n}^m$ will denote the cell at $\pm n$ steps to the right/left of the zero state at time m. The single cell x_0^0 will be denoted by a and generally we will take it that $0 < a \leq 1$, since $a = 0$ is clear. The method we present will allow us to determine the long term dynamics of any fuzzy rule $g_n(x, y, z)$ where $1 \leq n \leq 255$ where the various asymptotic estimates are found via successive iterations. In a nutshell, the basic idea here is to distinguish a single diagonal for a starting point, use the rule to derive basic theoretical estimates, use continuity to prove the existence of the various limits, when applicable, and finally use an iterative scheme to compute all subsequent limits.

We will always assume that $g : [0, 1]^3 \to [0, 1]$ is continuous on the unit cube $\mathcal{U}$ and not necessarily the canonical expression of a fuzzy rule. This is for simplicity only since, in reality, any compact set may be used in lieu of $[0, 1]$. The symbol $G^m(a)$ denotes the usual m^{th} iterate of G at a, where $G(a) \equiv g(0, 0, a)$ and $a \in (0, 1)$ is a given fixed number, sometimes called a *seed*. Similarly, we define $H(a) \equiv g(a, 0, 0)$. The value of "a" here measures in some sense the degree of fuzziness in that $a = 0$ gives trivial evolutions while $a = 1$ gives Boolean evolution. We assume, as a further restriction on g, that

- **(I)** The equations $x - g(x, y, z) = 0$, $y - g(x, y, z) = 0$, and $z - g(x, y, z) = 0$ may each be solved uniquely for x, y, z respectively, for given values of (y, z), (x, z), (x, y) respectively in $[0, 1]^2$, and that the resulting functions of (y, z), (x, z), (x, y) are continuous on $[0, 1]^2$.
- **(II)** The limits $G^m(a) \to L_0^-(a)$ and $H^m(a) \to L_0^+(a)$ each exist and are finite as $m \to \infty$.

Remark. Condition (I) is generally satisfied for fuzzy rules as considered here. The only exceptions that require modifications to the technique presented here are the nine fuzzy rules 170, 172, 184, 202, 204, 216, 226, 228, and 240. They are to be distinguished because they violate (I). In general, the implicit function theorem may be used here to guarantee conditions under which (I) holds, for arbitrary local rules. Secondly, we note that the full force of hypothesis (I) is not necessary for many of the fuzzy rules and that weaker assumptions can be made by restricting the class of rules. Nevertheless, we will assume it throughout for expository reasons and introduce modifications as the need arises. For most fuzzy rules (II) clearly holds because of our basic assumptions. For odd rules this assumption (II) may fail, but the techniques herein can then be applied to subsequences (see fuzzy rule 45 below).

2.1 Evolution and Dynamics for a Single Seed in a Zero Background

We assume that all cells but one (denoted by a) are initially set at zero. Writing $g(0, 0, a) \equiv G(a)$, we see that the main left-diagonal $x_{-m}^m = G^m(a)$ for each $m \geq 1$, where the symbol $G^m(a)$ denotes the usual m^{th} iterate of G at a. The cells of the first left-diagonal (given by x_{-m}^{m+1}, $m \geq 0$) then satisfy $x_m^{m+1} = g\left(0, G^m(a), x_{-(m-1)}^m\right)$. Passing to the limit as $m \to \infty$ in the previous display and using (II), we see that $L_1^-(a) = g\left(0, L_0^-(a), L_1^-(a)\right)$, by (I), and so this relation defines this limit $L_1^-(a)$ uniquely. Now that we know both $L_0^-(a)$ and $L_1^-(a)$ we can find $L_2^-(a)$ since $x_{-n}^{n+2} = g\left(x_{-(n+1)}^{n+1}, x_{-n}^{n}, x_{-(n-1)}^{n+1}\right)$, for each $n \geq 1$. Passing to the limit as $n \to \infty$ we find the special relation $L_2^-(a) = g\left(L_0^-(a), L_1^-(a), L_2^-(a)\right)$. By (I) this equation can be solved uniquely for $L_2^-(a)$, since the other two quantities are known. Proceeding inductively we observe that

$$L_{k+1}^-(a) = g\left(L_{k-1}^-(a), L_k^-(a), L_{k+1}^-(a)\right) \tag{1}$$

holds for each $k \geq 1$ and this defines the limit $L_{k+1}^-(a)$ uniquely, by (I), in terms of the preceding limits. If we set $g(a,0,0) \equiv H(a)$, then $x_m^m = H^m(a)$ for each $m \geq 1$. Arguing as in the left-diagonal case and using (II) we get that the existence of $\lim_{m\to\infty} H^m(a) = L_0^+(a)$ implies the existence of $L_1^+(a)$. The limit $L_2^+(a)$ is now found recursively as the unique solution of $L_2^+(a) = g\left(L_2^+(a), L_1^+(a), L_0^+(a)\right)$ whose existence is guaranteed by (I). Finally, an inductive argument gives us that subsequent limits, like $L_{k+1}^+(a)$, are given recursively by solving

$$L_{k+1}^+(a) = g\left(L_{k+1}^+(a), L_k^+(a), L_{k-1}^+(a)\right) \tag{2}$$

uniquely, using (I), for each $k \geq 1$.

As for the limits of the right-vertical sequences of the form $\{x_m^{m+p}\}_{p=0}^\infty$, $m = 1, 2, 3, \ldots$, we use the fact that the union of two sets of points each of which has exactly one (possibly the same) point of accumulation (or limit point) also has exactly two (maybe the same) points of accumulation. This result is easily derived using topology and, in fact, it also holds for any countable family of sets each of which has exactly one point of accumulation (by the Axiom of Choice). In the case under discussion we note that the right-half of the infinite cone, $\mathcal{C}^+$, whose vertex is at x_0^1 (the set that excludes the zero background and the zeroth left- and right-diagonals) can be written as $\mathcal{C}^+ = \bigcup_{i=1}^\infty \mathcal{S}_m^+ = \bigcup_{i=1}^\infty \{\{x_j^p\}_{j=0}^\infty | \ p \geq m\}$, and is therefore the countable union of sets of (right-diagonal) sequences each of which converges to some $L_k^+(a)$ and so the only points of accumulation of $\mathcal{C}^+$ must lie in the union of the set consisting of all the limits $L_k^+(a)$, for $k \geq 1$. Since right-vertical sequences are infinite subsequences of $\mathcal{C}^+$, we get that all such sequences may or may not have a limit but if they do, it must be one of the $L_k^+(a)$'s where the k generally depends on the choice of the column. A similar discussion applies to left-vertical sequences.

Remark. If the limits $L_k^\pm$ themselves converge as $k \to \infty$ to $L^\pm$, say, then $L^\pm$ must be a fixed point of the rule, that is $L^\pm = g\left(L^\pm, L^\pm, L^\pm\right)$. This is clear by (1), (2) and the continuity of g. The same observation applies to limits of vertical sequences. In the case of fuzzy rule 30 all columns (left or right) converge to one and the same value, namely, 1/2 (see [3]). On the other hand, fuzzy rule 18 (see below) has the property that, generally speaking, *none* of its vertical columns converges at all (since each column has exactly two limit points).

2.2 A General Theory in the Finite Support Case

This case is very similar to the single support case and so need only be sketched. We now assume that all cells but a finite number are initially set at zero. We take it that the initial string of cells is given by $x_{-k}^0, \ldots, x_0^0, \ldots, x_q^0$, where $x_{\pm i}^0 \in (0,1)$. Without loss of generality we will assume that $k \geq 1$ and $q \geq 0$ so that our initial string contains at least two elements.

We now distinguish two vertical columns, that is, $\mathcal{V}_{-k}$ and $\mathcal{V}_q$, that is those two infinite columns whose top cell is x_{-k}^0, x_q^0, respectively. We need to describe the evolution of the *half*-light cones emanating to the left and down from x_{-k}^0 (and to the right and down from x_q^0). Suppressing the variables in the expression of a limit for the purpose of clarity we will write $L_{-k}^-(x_{-k}^0, \ldots, x_0^0, \ldots, x_q^0)$ as L_{-k}^-. As before the zeroth left-diagonal, consisting of the value x_{-k}^0 only, necessarily converges to the same value. Hence, $L_0^- = x_{-k}^0$. It is helpful to think of the cell value x_{-k}^0 as playing a role analogous to a in the single support case.

Consider now the first left-diagonal S_1^-, originating at the cell $x_{-(k-1)}^0$ of the initial string. By definition every term of S_1^- on or to the left of $\mathcal{V}_{-k}$ is of the form $x_{-(k+m)}^{m+1} = g\left(0, x_{-k}^0, x_{-(k+m-1)}^m\right)$, where $m \geq 0$. Passing to the limit as $m \to \infty$, using the continuity assumptions of g at the outset, and hypotheses (I) and (II) we see that $L_1^- = g(0, L_0^-, L_1^-)$, from which we get the existence and uniqueness of L_1^-. The remaining limits, $L_2^-, L_3^-, \ldots$ are found recursively as in the single support case. Thus, $L_2^- = g(L_0^-, L_1^-, L_2^-)$, and so this limit exists and is unique, etc. The verification of a recursion similar to (1) is straightforward in this case.

The finite support right-diagonal case is handled precisely as the right-diagonal case of the single support configuration except that we distinguish the column $\mathcal{V}_q$ and, without loss of generality, consider all right-diagonal sequences as originating on $\mathcal{V}_q$. This only leaves out a finite number of terms and so the asymptotic dynamics are not affected. In this case one can show that $L_0^+ = x_q^0$, L_1^+ exists and is unique, and as before there holds a relation similar to (2) for all subsequent limits each one of which can be calculated explicitly on account of (I). Vertical sequences are handled as before so that, generally, one can only guarantee the existence of various limit points, even for sequences in the "dark area", that is that area enclosed by those columns between $\mathcal{V}_{-k}$ and $\mathcal{V}_q$.

Remark. All but nine fuzzy rules (mentioned above) satisfy the conditions of continuity and (I), (II) above, so the analysis captures much of the dynamics of essentially every single rule. The exceptions thus noted are distinguished by the fact that their diagonal function has every point in $[0,1]$ as a fixed point! More refined estimates as to the rate of convergence of a diagonal, questions of convergence in the dark area, etc. may be obtained on a case-by-case basis. It follows that there are no random or chaotic fuzzy rules in this context (except for the 9 undetermined ones) since all existing limits are continuous functions of the initial data. Chaos can occur when the iterates $G^m(a)$ fail to converge thus violating (II) or, if they do converge, they admit sensitive dependence upon a because the original *nonlinear* rule (not necessarily related to a fuzzy CA) admits chaotic sequences under iterations. For example, the "rule" defined by $g(x, y, z) = 4x(1-x^2)$ produces a chaotic sequence in the case of a single support initial configuration with fixed cell value $a \in (0, 1)$. The spatio-temporal evolution of fuzzy rule 30 can be found in [3] and these results follow immediately from our methods. We note that the long term dynamics of fuzzy rule 110 were obtained in [10] using special arguments pertaining to the form of the rule itself along with its representation as a Taylor polynomial. This also follows from our methods.

2.3 The Dynamics of Fuzzy Rule 18

As for new phenomena we exhibit, for example, the long term dynamics of fuzzy rules 18, 45 and 184 (a typical representative of the exceptional list of fuzzy rules that defy (I)) below, in this and the next subsections. The canonical expression for fuzzy rule 18 is given by $g_{18}(x, y, z) = (1 - y)(x + z - 2xz)$. The methods presented above can be applied easily here so the following result is presented without proof.

Theorem 1 *Let $a \in (0, 1)$ be a single seed in a zero background. Then the long term dynamics of fuzzy rule 18 are given as follows:*

- $L_0^-(a) = a$, $L_1^-(a) = 0$, *and* $L_{2n}^-(a) = 1/2$, *for each* $n \geq 1$, *while* $L_{2n+1}^-(a) = 0$, *for each* $n \geq 0$.
- $L_0^+(a) = a$, $L_1^+(a) = 0$, *and* $L_{2n}^+(a) = 1/2$, *for each* $n \geq 1$, *while* $L_{2n+1}^+(a) = 0$, *for each* $n \geq 0$.
- *Vertical columns fail to converge as they always have two limit points, either 0 or* $1/2$.

Some may argue that the vertical columns actually represent asymptotically periodic sequences. Either way, there is no definite convergence. The symmetry about the central column in the evolution of rule 18 is due to the relation $g(x, y, z) = g(z, y, x)$ satisfied by this rule for every $x, y, z \in \mathcal{U}$. The asymptotics of the finite support case are governed by the first and last cells of the initial configuration and rule 18's dynamics are identical, by the theory above, to those of the single cell case.

2.4 The Dynamics of Fuzzy Rule 45

In order not to focus our examples on *even* rules we present an example of the application of our techniques to the evolution of fuzzy rule 45, an *odd* rule. The canonical form of this rule is given by the expression $g_{45}(x, y, z) = 1 - x - z + yz + 2xz - 2xyz$. Its *diagonal function* $d(x) \equiv g_{45}(x, x, x)$ is given by $d(x) = -2x^3 + 3x^2 - 2x + 1$. It has only one real fixed point, $x = 1/2$, which is attracting. For a single seed, a, in a zero background observe that, by induction, $x_{-2n}^{2n} = 0 = \ldots = x_{n+1}^{2n}$ for each $n \geq 1$. In addition, $x_{-(2n+1)}^{2n+1} = 1 = \ldots =$

$x_{-(n+2)}^{2n+1}$ for $n \geq 1$. Thus, no left-diagonal sequence converges in the strict sense although it is a simple matter to see that we have eventually periodic behavior (or an eventual 2-cycle) for the left diagonals. The right-diagonals lead to interesting phenomena. Note that condition (I) is untenable for this rule, that is, the limit L_0^+ does not exist (since the zeroth right-diagonal sequence alternates between 0 and 1, or represents a 2-cycle), and the same can be said of L_1^+ (since the first right-diagonal sequence alternates between a and $1-a$, another 2-cycle). However, the limit L_2^+ *does* exist and, in fact, $L_2^+ = 1/2$. The following proof of the preceding result is typical for odd rules. Let n be an even integer. Then

$$x_{n-2}^{n} = g_{45}(x_{n-3}^{n-1}, x_{n-2}^{n-1}, x_{n-1}^{n-1}). \tag{3}$$

However, for even n, $x_{n-1}^{n-1} \to 1$ while $x_{n-2}^{n-1} \to 1-a$. Taking the limit in (3) we get that $L_2^{even} = g_{45}(L_2^{even}, 1-a, 1)$. Solving for L_2^{even} we get that $L_2^{even} = 1/2$, as stated. If n is an odd integer, then (3) is still in force, but $x_{n-2}^{n-1} \to a$ while $x_{n-1}^{n-1} \to 0$ as $n \to \infty$ through odd numbers. It follows that $L_2^{odd} = g_{45}(L_2^{odd}, a, 0)$. Solving for L_2^{odd} we find that $L_2^{odd} = 1/2$ as well. From this we see that the common limit $L_2^{even} = L_2^{odd}$ is, in fact the limit, $L_2^+ = 1/2$. To find L_3^+ we proceed as usual, noting that the non-existence of L_1^+ is unimportant. For example, if $a < 1$, passing to the limit inferior in (3) we deduce that $L_{3,inf}^+ = g_{45}(L_{3,inf}^+, L_2^+, a)$ where $L_{3,inf}^+$ is the limit inferior of the sequence x_{n-2}^n. Observe that one can solve for $L_{3,inf}^+$ uniquely in the preceding display provided $a \neq 2$, which is necessarily the case (since the range of g_{45} is contained in $[0,1]$). The unique value thus obtained is $L_{3,inf}^+ = 1/2$. A similar argument using the limit superior gives us that $L_{3,sup}^+ = 1/2$. Since these two limits agree, the sequence itself has a limit and this limit must be $L_3^+ = 1/2$. We see that $L_k^+ = 1/2$, for each $k \geq 3$, by induction.

2.5 The Dynamics of Fuzzy Rule 184

We consider the dynamics of a rule that fails to satisfy (I). As we pointed out earlier there are exactly nine such (so-called exceptional) rules, including the present one. The canonical form of rule 184 is given by the expression $g_{184}(x, y, z) = x - xy + yz$. Its *diagonal function* $d(x) \equiv g_{45}(x, x, x)$ is given by $d(x) = x$. Thus, *every* real number in $[0,1]$ is a fixed point (it is *this* fact that characterizes the other exceptional rules). Next, for a single seed a in a zero background observe that, by induction, $x_n^n = a$ for each $n \geq 1$, so that this rule is a right-shift. Clearly, for a single seed its dynamics are trivial. The difficulty occurs when we pass to the case of finite support/random initial data.

Consider the case of two seeds, $a, b \in (0,1)$ in a zero background. We take it that $x_0^0 = a, x_1^0 = b$. Of special interest in this case is the convergence of the right-diagonals and the dynamics of the rule along them. Note that $x_n^{n-1} = b$ for all $n \geq 1$, so that the limit of this sequence (or zeroth diagonal, $\mathcal{V}_0$), is $L_0^+ = b$. Next, the terms of the first right-diagonal, $\mathcal{V}_1$, are given by $x_n^n = a(1-b)^{n-1}$, a result that is easily verified by induction. It follows that its limit, $L_1^+ = 0$, except in the special case where $b = 0$, in which case this reduces to the single seed scenario already discussed above. Difficulties arise in the discussion bearing on the next diagonal, $\mathcal{V}_2$. Applying our technique to this situation we find $L_2^+ = g_{184}(L_2^+, L_1^+, L_0^+) = g_{184}(L_2^+, 0, b) = L_2^+$. Thus, no *a priori* information regarding L_2^+ is obtainable using our methods, as they stand.

In order to circumvent this difficulty we suggest the following approach. We suspect that L_2^+ may be obtained by passage to the limit of a small parameter $\varepsilon > 0$ using the claim that $L_2^+ = g_{184}(L_2^+, \varepsilon, b)$, holds for every $\varepsilon > 0$. This then results in the equality $L_2^+ \varepsilon = b\varepsilon$, from which we conclude that $L_2^+ = b$. This argument is supported by numerical calculations of this specific limit. Since $L_2^+ = b$ we get $L_3^+ = g_{184}(L_3^+, L_2^+, L_1^+) = g_{184}(L_3^+, b, 0) = L_3^+ - bL_3^+$ and so $L_3^+ = 0$. Continuing in this way we find the sequence of limits, $L_k^+ = b$ if k is even, and $L_k^+ = 0$ if k is odd. Once again these limiting values are supported by calculations in this two-seed case.

Remark. We note that rigorous justification for our technique of *passage to the limit of a small parameter* is lacking, except for the fact that it yields the correct dynamics for this rule. We suspect that this yields correct limiting behavior for finite random initial data in rule 184. However, it is an open question whether this technique will produce the correct limiting behavior for the other eight remaining exceptional fuzzy rules.

Concluding Remarks. An iterative method for finding the dynamics of all fuzzy CA as defined in [4] is produced. This gives a road-map for determining the global evolution of a general fuzzy rule. It is likely that all fuzzy CA lead to deterministic behavior and there are no chaotic or random rules (except possibly for rules 170, 172, 184, 202, 204, 216, 226, 228, and 240; see [7]). The dynamics of these nine rules remain undetermined at this time even though the methods used in rule 184 above may be used, it is not clear that this will work in all cases. Minor modifications show that the techniques presented here apply to neighborhood structures with an arbitrary fixed number of cells (*e.g.,* 5 or 7) and finite support (or random) initial configurations.

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