Modeling of Electromagnetic Waves in Media with Dirac Distribution of Electric Properties

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Abstract. We develop a new numerical approach of the electromagnetic wave propagation in a complex media model. We use a new Finite Element Method for space discretization and a finite difference method in time domain. This study is applied to TDR (Time Domain Reflectometry) which appears as a wave guide. In this context, the experimental approach is very uneasy and the numerical study brings about very interesting results, particularly for local values of the electromagnetic field. Moreover we test models of an heterogeneous medium. Dielectric properties of the medium are represented by an homogeneous component associated with Dirac distribution. We present wave front profiles with useable signals simulation.

Keywords: Finite Element Method, Backward Difference Method, Maxwell's equations.

1 Introduction

We present a model of electromagnetic wave propagation in a wave guide containing a complex medium. This work is a contribution to the interpretation of the TDR (Time Domain Reflectometry) probe's signals. The TDR probe [1] is a wave guide constituted by two parallel metallic rods. At initial time, a step of electric field is applied between the electrodes and intensity is measured at the entrance of the wave guide. Measurement and processing of the signal obtained by reflection should theoretically allow the determination of medium properties. An interesting application of this technology concerns soil science. In practice, a soil is composed of a complex mixture with inhomogeneous electric permittivities.

The theoretical approach is formulated using Maxwell's equations. They are solved in the time domain using the Finite Element Method. Moreover we consider an homogeneous medium in which random space distribution of electric properties is added to some finite element nodes with Dirac distribution. Each numerical simulation is performed using a given ratio r of Dirac dedicated nodes. The main objective is then to study the influence of ratio r on the electric signal.

2 General Presentation of the Model

2.1 Maxwell Equations

The 2D wave guide is represented by two parallel plate electrodes whose electric conductivity is infinite. The free space around the electrodes is constituted by a

complex medium. With the usual notations, the dimensionless Maxwell equations are formulated as follows [2]:

$$\begin{aligned}\mu_r \cdot \frac{\partial \vec{H}}{\partial t} &= \overrightarrow{\text{curl}} \vec{E} & \varepsilon_r \cdot \frac{\partial \vec{E}}{\partial t} &= \overrightarrow{\text{curl}} \vec{H} \\ \text{div}(\mu_r \cdot \vec{H}) &= 0 & \text{div}(\varepsilon_r \cdot \vec{E}) &= 0\end{aligned}\quad (1)$$

2.2 Finite Element Formulation

The Finite Element method is applied to Maxwell's equations. In our formulation, we use the weak formulation, presented by Chambarel et al [3], with Galerkin's ponderation [4], in order to obtain integral forms for the electromagnetic equations. Then, with the Finite Element formulation of the integral forms for a first order form of Maxwell's equations, we can get an approximate solution of the weak formulation, with discretization of the electric and the magnetic fields. The weighted residual method can be written:

$$\begin{aligned}\int_{(\Omega)} \overrightarrow{\delta H} \cdot \mu_r \cdot \frac{\partial \vec{H}}{\partial t} \cdot d\Omega &= - \int_{(\Omega)} \overrightarrow{\delta H} \cdot \overrightarrow{\text{curl}} \vec{E} \cdot d\Omega \\ \int_{(\Omega)} \overrightarrow{\delta E} \cdot \varepsilon_r \cdot \frac{\partial \vec{E}}{\partial t} \cdot d\Omega &= \int_{(\Omega)} \overrightarrow{\delta E} \cdot \overrightarrow{\text{curl}} \vec{H} \cdot d\Omega\end{aligned}\quad (2)$$

In formula (2), ε_r incorporates the Dirac distribution.

The medium arises as a medium whose permittivity is homogeneous but in which a distribution of singularities is inserted. For the Dirac formulation of electric permittivity we define the following density θ :

$$\varepsilon_r = \frac{d\theta}{d\Omega} \quad \varepsilon_r = \varepsilon_H + \sum_p \theta_p \cdot \delta(x - x_p) \quad (3)$$

where index H denotes the homogeneous component of the medium. So the weak formulation becomes :

$$\begin{aligned}\int_{(\Omega)} \overrightarrow{\delta H} \cdot \mu_r \cdot \frac{\partial \vec{H}}{\partial t} \cdot d\Omega &= - \int_{(\Omega)} \overrightarrow{\delta H} \cdot \overrightarrow{\text{curl}} \vec{E} \cdot d\Omega \\ \int_{(\Omega)} \overrightarrow{\delta E} \cdot \varepsilon_H \cdot \frac{\partial \vec{E}}{\partial t} \cdot d\Omega &+ \sum_p \left(\overrightarrow{\delta E} \cdot \theta_p \cdot \frac{\partial \vec{E}}{\partial t} \right)_{x=x_p} = \int_{(\Omega)} \overrightarrow{\delta E} \cdot \overrightarrow{\text{curl}} \vec{H} \cdot d\Omega\end{aligned}\quad (4)$$

In that way, we can define the discretization. Let $n_i(x)$ be a base of work space and index n denote the elementary nodal values. For an isoparametrical element the polynomial interpolation can be written:

$$x_i^h = n_j \cdot x_{ij}^n, \quad H_i^h = n_j \cdot H_{ij}^n \quad \text{and} \quad E_i^h = n_j \cdot E_{ij}^n$$

The discretization of the electric fields is as follows :

$$\{E_h\} = \begin{bmatrix} \langle n_i \rangle & \langle 0 \rangle & \langle 0 \rangle \\ \langle 0 \rangle & \langle n_i \rangle & \langle 0 \rangle \\ \langle 0 \rangle & \langle 0 \rangle & \langle n_i \rangle \end{bmatrix} \begin{Bmatrix} E_x^n \\ E_y^n \\ E_z^n \end{Bmatrix} = [N] \{E^n\} \quad (5)$$

So the discretization of the curl operator is easy and we have :

$$\overrightarrow{\text{curl}} E_h = \begin{bmatrix} \langle 0 \rangle & -\langle \frac{\partial n_i}{\partial z} \rangle & \langle \frac{\partial n_i}{\partial y} \rangle \\ \langle \frac{\partial n_i}{\partial z} \rangle & \langle 0 \rangle & -\langle \frac{\partial n_i}{\partial x} \rangle \\ -\langle \frac{\partial n_i}{\partial y} \rangle & \langle \frac{\partial n_i}{\partial x} \rangle & \langle 0 \rangle \end{bmatrix} \begin{Bmatrix} E_x^n \\ E_y^n \\ E_z^n \end{Bmatrix} = [R] \{E^n\} \quad (6)$$

We can write similar relations for the magnetic field. The weak formulation can be written in matricial form:

$$\int_{(\Omega)} \overrightarrow{\delta H} \cdot \mu_r \cdot \frac{\partial \vec{H}}{\partial t} d\Omega = \langle \delta H^n \rangle \cdot \int_{(\Omega)} N^T \cdot \mu_r \cdot N \cdot d\Omega \cdot \left\{ \frac{\partial H^n}{\partial t} \right\} \quad (7)$$

$$\int_{(\Omega)} \overrightarrow{\delta H} \cdot \text{rot} \vec{E} d\Omega = \langle \delta H^n \rangle \cdot \int_{(\Omega)} N^T \cdot R \cdot d\Omega \cdot \{E^n\} \quad (8)$$

$$\int_{(\Omega)} \overrightarrow{\delta E} \cdot \varepsilon_r \cdot \frac{\partial \vec{E}}{\partial t} d\Omega = \langle \delta E^n \rangle \cdot \left[\int_{(\Omega)} N^T \cdot \varepsilon_H \cdot N \cdot d\Omega + \sum_{k \in (\Omega_e)} \{n(x_k)\} \theta_k \cdot \langle n(x_k) \rangle \right] \left\{ \frac{\partial E^n}{\partial t} \right\} \quad (9)$$

$$\int_{(\Omega)} \overrightarrow{\delta E} \cdot \text{rot} \vec{H} d\Omega = \langle \delta E^n \rangle \cdot \int_{(\Omega)} N^T \cdot R \cdot d\Omega \cdot \{H^n\} \quad (10)$$

2.3 Matricial Formulation

$$\sum_{i=1}^n \langle \delta H^n, \delta E^n \rangle \cdot \left(\begin{bmatrix} m_\mu & 0 \\ 0 & m_\varepsilon \end{bmatrix} \cdot \begin{Bmatrix} \frac{\partial H^n}{\partial t} \\ \frac{\partial E^n}{\partial t} \end{Bmatrix} + \begin{bmatrix} 0 & k_i \\ -k_i & 0 \end{bmatrix} \cdot \begin{Bmatrix} H^n \\ E^n \end{Bmatrix} \right) = 0 \quad (11)$$

$$\text{or } \sum_{N_e} \langle \delta u \rangle \cdot \left([me_H + me_{Dirac}] \cdot \left\{ \frac{due}{dt} \right\} + [ke] \cdot \{ue\} - \{fe\} \right) = 0$$

After an assembling process, we obtain :

$$[M_H + M_{Dirac}] \cdot \frac{d}{dt} \{U\} = \{F\} - [K] \{U\} \quad \text{with} \quad \{U\} = \{H, E\} \quad (12)$$

3 Numerical Resolution

We obtain a large size time-differential system. By inversion of the mass matrix $[M]$, the general formulation of this differential system is :

$$\frac{d}{dt}\{U\}=[M]^{-1}.\{\Psi(U,t)\} \quad \text{where} \quad \{\Psi(U,t)\}=\{F\}-[K].\{U\} \quad (13)$$

For the numerical quadrature of formula (11) we choose the nodes of the element as integration points [3]. Consequently, the mass matrix $[M]$ is diagonal and this inversion is an easy procedure. It is a necessary condition for the efficiency of the methods described below. For numerical time-resolution, we modify the Backward Difference Method [5]. With the propose formulation one can choose the time order of discretization and the upward scheme. Under these conditions we can test a semi-implicit method. With the matrix-free technique, the mass matrix and the stiffness matrix are never built: only the elementary matrices in (11) are calculated. So we note a high performance level both for the CPU and the storage costs.

3.1 Semi-implicit Method

The corresponding k order algorithm is as follows :

$$\begin{aligned} & t_n = 0 \\ & \text{while} \quad (t_n \leq t_{\max}) \\ & \quad \left\{ \begin{aligned} \left\{ \Delta U_n^i \right\} &= \Delta t_n \cdot \sum_{j=0}^{k-1} \lambda_j \cdot [M_{n-j}^i]^{-1} \cdot \left\{ \Psi_{n-j} \left(U_{n-j} + \alpha_j \cdot \Delta U_{n-j}^{i-1}, t_n + \alpha_j \cdot \Delta t_n \right) \right\} \\ & \quad i = 1, 2, \dots \quad \text{until} \quad \left\| \Delta U_n^i - \Delta U_n^{i-1} \right\| \leq \text{tolerance} \end{aligned} \right\} \\ & \quad \{U_{n+1}\} = \{U_n\} + \{\Delta U_n\} \\ & \quad t_{n+1} = t_n + \Delta t_n \\ & \text{end while} \end{aligned} \quad (14)$$

where α is the upward time-parameter.

This method requires inner iterations for each time-step with index i , for ΔU determination, and the convergence criteria can be written for example at the first order [5]:

$$\alpha \cdot \Delta t \cdot \left\| \frac{\partial}{\partial U} \{[M]^{-1} \cdot \Psi(U,t)\} \right\| < 1 \quad \text{let} \quad \Delta t < \Delta t_0 \quad (15)$$

But we always find a time-step value for the convergence of the process. We note the good stability of the scheme in the implicit case. If $\alpha < 0.5$ a CFL condition is also required [6]. In this way we choose a time step as follows [7]:

$$\Delta t \leq \text{Min}(\Delta t_0, \Delta t_{\text{CFL}})$$

The advantage of this method is the matrix free technique associated with an iterative method. The initial solution of each time step is close to the next solution. So the

number of inner iterations is very low, two or three in practice. Initially the electromagnetic field is zero in full domain. For time t_k , the time step Δt_k can be changed at each step with a CFL condition.

3.2 The Software

For this method to work with efficacy, we must adapt each formulation in accordance with the boundary conditions. For easy adaptation of the code to the different formulations described above we must have a dedicated software. We use efficient C++ Objects-Oriented Programming for the Finite Element code called FAFEMO (Fast Adaptive Finite Element Modular Object). All details are given in reference [3].

Figure 1 shows the general structure of the compact code. It is organized in three classes corresponding to the functional blocks of the FEM's different stages. With these classes we built three objects that are connected by a single heritage. So the transmission of the parameters between the objects is defined by a list technique. We adapt the "class elementary matrices" for Dirac distribution.

Fig. 1. Structure of the software

This constitutes a general Finite Element solver for the Maxwell equations.

4 Numerical Results

We study the transverse magnetic mode -T.M.- in the case of a 2D model. Figure 2 shows the meshing of the domain. We use here a high density of elements because the wave front propagates in the whole domain, but does not return significantly in the wave guide. In this context we develop stationary waves and we notice at each period

that a part of the electromagnetic energy is scattered in the cavity. So the amplitude of the signal decreases (Fig. 3). After the electromagnetic field computation, we can deduce the value of the signal by Ampere's theorem. For TDR technology only intensity at the entrance of the wave guide is measurable. So we simulate the useable signal.

Fig. 2. The meshing

Fig. 3. The useable signal

Another test studies the effect of a single Dirac distribution. Each distribution concentrates the electromagnetic energy. Figure 4 gives the details around a distribution and Figure 5 shows the electromagnetic energy in the wave guide with several distributions.

Fig. 4. Electromagnetic energy around a single Dirac distribution

Fig. 5. Electromagnetic energy in the wave guide

The curve in Figure 6a shows the differential signal obtained by a single Dirac distribution visible in the wave guide (Fig.6b). The progressive wave reflects on the Dirac distribution and a signal returns towards the entrance. In practice our numerical

model is constituted by many distributions with random localizations (Fig. 7a) (Fig. 7b). The superposition of the multiple signals associated to each distribution is associated to a mean electric property of the complex medium. So it is possible to choose a random profile of Dirac distributions according to a complex medium model as a soil.

Fig. 6a. Signal at the entrance of the wave guide obtained with a single Dirac distribution

Fig. 6b. Electromagnetic energy in the guide around a single Dirac distribution

Fig. 7a. Electromagnetic energy in the wave guide in the case of some Dirac distributions with random localizations

Fig. 7b. Electromagnetic energy out of the wave guide

5 Conclusion and Further Works

We have presented a numerical study of the propagation of electromagnetic waves in a complex medium. First we have developed a set of numerical techniques applied to Maxwell's equations. A special algorithm has been developed for wave front study. Secondly we have tested a new model of complex medium by introduction of Dirac

distributions in the physical properties. Also we hope to modelize the electric properties of a soil. This numerical simulation is performed by Object-Oriented Programming. In this context we obtain efficient and low cost calculus. This set of techniques constitutes a very interesting tool for soil science.

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