EFFICIENT COUNTERMEASURES AGAINST POWER ANALYSIS FOR ELLIPTIC CURVE CRYPTOSYSTEMS

Kouichi Itoh, Tetsuya Izu, and Masahiko Takenaka

FUJITSU LABORATORIES Ltd.

{kito,izu,takenaka}@labs.fujitsu.com

Abstract The power analysis on smart cards is a real threat for cryptographic applications. In spite of continuous efforts of previous countermeasures, recent improved and sophisticated attacks against Elliptic Curve Cryptosystems are not protected. This paper proposes two new countermeasures, the Randomized Linearly-transformed Coordinates (RLC) and the Randomized Initial Point (RIP) against the attacks including the Refined Power Analysis (RPC) by Goubin and the Zero-value Point Analysis (ZPA) by Akishita-Takagi. Proposed countermeasures achieve notable speed-up without reducing the security level.

Keywords: Smart cards, power analysis, Elliptic Curve Cryptosystems, countermeasure

1. Introduction

Smart cards are becoming a new infrastructure in the coming IT society for their applications such as the SIM cards for mobile phones, identification cards for entrance systems and electronic tickets for movies. However, the power analysis attacks against these devices are real threats for these applications. In these attacks, an adversary observes traces of the power consumption of the device, and then, he detects a correlation between this information and some secret information hidden in the device. The simple power analysis (SPA) and the differential power analysis (DPA) are classical but typical examples [17, 18, 22]. Fortunately, various countermeasures which is not only secure but also efficient, have been proposed before 2002 [5, 15].

Recently, improved and sophisticated power analysis on Elliptic Curve Cryptosystems (ECC) have been proposed. In 2003, Goubin presented a new analysis, the Refined Power Analysis (RPA), which detects special points with 0-coordinate on the curve by chosen messages [9]. Then, Akishita-Takagi extended RPA to the Zero-value Point Analysis (ZPA), which detects 0 value in additions and doublings [2]. Some previous countermeasures resist RPA

and ZPA [1, 5, 6], however, they require larger amount of processing time. A practical countermeasure was proposed by Ciet-Joye [7]. Note that Smart's countermeasure resists only RPA [32]; it does not always resist ZPA [3].

In this paper, we propose two practical countermeasures for ECC, the Randomized Linearly-transformed Coordinates countermeasure (RLC) and the Randomized Initial Point countermeasure (RIP), which resist power analysis including above newer attacks, and provide efficient processing speed for scalar multiplications. Proposed countermeasures achieve notable speed-up without reducing the security level.

The rest of this paper is organized as follows: we briefly review Elliptic Curve Cryptosystems (ECC) in section 2. Side channel attacks and countermeasures are in section 3. Then section 4 describes our proposed countermeasures. A comparison of countermeasures are described in section 5. Concrete algorithms of proposed countermeasures are in the appendix.

2. Elliptic Curve

This section describes a brief review of elliptic curves defined over finite fields with prime elements. This is just for a simplicity and basic ideas of most attacks and countermeasures can be applied to other finite fields.

Elliptic Curve Cryptosystems (ECC)

Let K be a finite field with prime elements $p > 3$. An elliptic curve over K is represented by an equation $E(K) := \{(x, y) \in K \times K \mid y^2 = x^3 + ax + b, a, b \in K, 4a^3 + 27b^2 \neq 0\} \cup \{\mathcal{O}\}$. The special point $\mathcal{O}$ is called the point at infinity. In this representation, a point on the curve (different from $\mathcal{O}$) is represented as a pair of K -elements (affine coordinates). An elliptic curve $E(K)$ forms an additive group by the following addition law: the point $\mathcal{O}$ is a neutral element, and an inverse point of $P = (x, y)$ is given by $-P = (x, -y)$. We call $P_1 + P_2$ ($P_1 \neq P_2$) the elliptic curve addition (ECADD) and $P_1 + P_2$ ($P_1 = P_2$), that is $2P_1$, the elliptic curve doubling (ECDBL). Let $P_1 = (x_1, y_1)$, $P_2 = (x_2, y_2)$ be two elements of $E(K)$ different from $\mathcal{O}$ and satisfying $P_1 \neq -P_2$. Then the addition $P_3 = P_1 + P_2 = (x_3, y_3)$ is defined by $x_3 = \lambda^2 - x_1 - x_2$, $y_3 = \lambda(x_1 - x_3) - y_1$, where $\lambda = (y_2 - y_1)/(x_2 - x_1)$ for $P_1 \neq P_2$, and $\lambda = (3x_1^2 + a)/(2y_1)$ for $P_1 = P_2$.

Elliptic Curve Cryptosystems (ECC) are one of the standard technologies in the area of cryptography [24]. The biggest advantage of ECC is the key length; it is currently chosen much shorter than that of existing other cryptosystems such as RSA and ElGamal. This feature is quite suitable for smart card implementations. Let d be an integer (scalar) and P be a point on the elliptic curve $E(K)$. The point P is called the base point. A *scalar multiplication* is an operation to compute a point $dP = P + \dots + P$ ($d - 1$ additions). Computing d

Table 1. Add-and-double method

INPUT: $d, P,$	OUTPUT: dP
1: $T[0] = 0, T[1] = P$	
2: for $i=0$ upto $n-1$	
3: if ($d[i] == 1$) {	
4: $T[0] = \text{ECADD}(T[0], T[1])$	
5: }	
6: $T[1] = \text{ECDBL}(T[1])$	
7: }	
8: return $T[0]$	

Table 2. Add-and-double-always method

INPUT: $d, P,$	OUTPUT: dP
1: $T[0] = 0, T[2] = P$	
2: for $i=0$ upto $n-1$ {	
3: $T[1] = \text{ECADD}(T[0], T[2])$	
4: $T[2] = \text{ECDBL}(T[2])$	
5: $T[0] = T[d[i]]$	
6: }	
7: return $T[0]$	

from dP and P is called the elliptic curve discrete logarithm problem, and the hardness of this problem assures the security of ECC. In most ECC schemes, d is used as a secret key. A dominant computation of all encryption/decryption and signature generation/verification algorithms of ECC is the scalar multiplication. Efficiency of a scalar multiplication strongly depends on a choice of addition chains and coordinate systems for the elliptic curve representation.

Addition Chain and Coordinate System

Let $d = d_{n-1}2^{n-1} + \dots + d_12^1 + d_0$ be a binary expression of a scalar d with $d_{n-1} = 1$. Then the add-and-double (AD) method from the least significant bit (LSB) for a scalar multiplication is shown in Table 1, which requires $n - 1$ ECDBLs and $n/2$ ECADDs in average. The number of ECADDs can be reduced by using the signed-binary expression of d , however, the technique is vulnerable to side channel attacks and we are not interested in it in this paper. A similar algorithm can be constructed from the most significant bit (MSB), however, we omit it because of the space limitation.

Coordinate systems for the elliptic curve representation effect on the processing speed of ECADD/ECDBL and hence a scalar multiplication [8]. In the affine coordinates $\mathcal{A}$, the addition formula always requires an inverse in the definition field K . In some environments, especially in prime fields (our situation), computing inverses are expensive. Since the projective coordinates and the Jacobian coordinates provide division-free formulas, these coordinates are widely used in practice. In the *projective coordinates* $\mathcal{P}$, a point is represented as a tuple (X, Y, Z) where two points (X, Y, Z) and $(\lambda X, \lambda Y, \lambda Z)$ ($\lambda \in K^*$) are identical. A curve equation is obtained by setting $x = X/Z$, $y = Y/Z$ in the curve equation. In the *Jacobian coordinates* $\mathcal{J}$, a point is also represented as a tuple (X, Y, Z) where two points (X, Y, Z) and $(\lambda^2 X, \lambda^3 Y, \lambda Z)$ ($\lambda \in K^*$) are identical. A curve equation is obtained by setting $x = X/Z^2$, $y = Y/Z^3$ in the curve equation. The addition formulas for general cases (namely, $P_1, P_2 \neq \mathcal{O}$ and $P_1 \neq -P_2$ for ECADD) in the Jacobian coordinates are given in Table 3. Note that in these coordinates, the point at infinity is represented as a tuple, which is the only point whose Z -coordinate value equals 0.

Table 3. Addition formulas in the Jacobian coordinates (general cases)

ECADD (in the Jacobian coordinates)	ECDBL (in the Jacobian coordinates)
Input: $P_1 = (X_1, Y_1, Z_1), P_2 = (X_2, Y_2, Z_2)$	Input: $P_1 = (X_1, Y_1, Z_1), a$
Output: $P_3 = P_1 + P_2 = (X_3, Y_3, Z_3)$	Output: $P_4 = 2P_1 = (X_4, Y_4, Z_4)$
$U_1 \leftarrow X_1 Z_2^2, U_2 \leftarrow X_2 Z_1^2, S_1 \leftarrow Y_1 Z_2^3$	$W \leftarrow a Z_1^3, M \leftarrow 3X_1^2 + W$
$S_2 \leftarrow Y_2 Z_1^3, W \leftarrow U_2 - U_1, R \leftarrow S_2 - S_1$	$S \leftarrow 4X_1 Y_1^2, T \leftarrow 8Y_1^4$
$T \leftarrow U_2 + U_1, M \leftarrow S_2 + S_1$	$X_4 \leftarrow M^2 - 2S$
$X_3 \leftarrow R^2 - TW^2, V \leftarrow 3TW^2 - 2R^2$	$Y_4 \leftarrow M(S - X_4) - T$
$Y_3 \leftarrow (VR - MW^3)/2, Z_3 \leftarrow Z_1 Z_2$	$Z_4 \leftarrow 2Y_1 Z_1$

Table 4. Processing times of ECADD and ECDBL

Coordinates	ECADD		ECDBL	
	$Z \neq 1$	$Z = 1$	$a \neq -3$	$a = -3$
$\mathcal{A}$	$2M + 1S + 1I$	—	$2M + 2S + 1I$	
$\mathcal{P}$	$12M + 2S$	$9M + 2S$	$7M + 5S$	$7M + 3S$
$\mathcal{J}$	$12M + 4S$	$8M + 3S$	$4M + 6S$	$4M + 4S$

Required processing time of ECADD/ECDBL is summarized in Table 4, where M , S , I denotes the processing time of a multiplication, a squaring, and an inverse in the definition field K , respectively. These processing time can be improved when one of Z -coordinate values in $\mathcal{P}, \mathcal{J}$ equals to 1 or the coefficient of the definition equation satisfies $a = -3$.

3. Power Analysis

The power analysis is a powerful attack against cryptographic schemes on some devices such as smart cards. An adversary observes leaked power consumption, and detects a correlation between this information and some secret information hidden in the device. The simple power analysis (SPA) [17] and the differential power analysis (DPA) [18,22] are typical examples.

Simple Power Analysis

The binary methods compute ECADD only when $d_i = 1$. Hence an adversary easily detects this irregular procedure by observing power consumption and obtains the value of d_i . The attack is called the *simple power analysis* (SPA) [17]. A standard approach to resist SPA is to use so called the *add-and-double-always* method (Table 2) [5], in which both ECADD and ECDBL are computed repeatedly (in step 3 and step 4). Since a pattern of the power trace becomes fixed, the adversary cannot detect the bit information d_i by SPA. As a variant of this countermeasure, the Montgomery ladder (Table 5) [21] is sometimes used [4,11,12,16,26,27], because it provides good processing speed by using the specialized addition formula, the **x -coordinate-only** addition formula [4,12,11] which will be discussed later.

Table 5. Montgomery ladder

INPUT: $d, P,$	OUTPUT: dP
1: $T[0] = P$	
2: $T[1] = \text{ECDBL}(T[0])$	
3: for $i=n-2$ downto 0 {	
4: $T[2] = \text{ECDBL}(T[d[i]])$	
5: $T[1] = \text{ECADD}(T[0], T[1])$	
6: $T[0] = T[2-d[i]]$	
7: $T[1] = T[1+d[i]]$	
8: }	
9: return $T[0]$	

Table 6. ADA and RPC

INPUT: $d, P,$	OUTPUT: dP
1: $T[0] = 0, T[2] = \text{RPC}(P)$	
2: for $i=0$ upto $n-1$ {	
3: $T[1] = \text{ECADD}(T[0], T[2])$	
4: $T[2] = \text{ECDBL}(T[2])$	
5: $T[0] = T[d[i]]$	
6: }	
7: return $\text{invRPC}(T[0])$	

Note that some SPA-countermeasures use window-based methods for efficiency [14, 19, 20, 30, 31]. However, the memory requirement is severe in most cases on smart cards and we do not follow these approaches in this paper.

Differential Power Analysis

In the *differential power analysis* (DPA) [18, 22], an adversary guesses $d_i = 0$, for example, and simulates the computation repeatedly. Then he/she divides the power consumption into two sets depending on the assumption, in order to make a bias of hamming weights of the internal information between these sets. If the assumption is correct, a difference of the power consumption of two sets can be observed (as a spike) in the trace. Note that even if a scheme is SPA-resistant, it is not always DPA-resistant.

In order to resist DPA, a randomization of parameters is a well-known technique to make the simulation impossible [5]. One approach is to randomize a scalar d or an addition chain for d . Coron proposed the scalar blinding countermeasure, in which d is replaced by $d + r\phi$ for a random integer r and the order of the point ϕ [5]. Original r (20-bit) does not provide sufficient security [27]. Larger r will provide higher security and less speed. Coron also proposed the base point blinding countermeasure, in which dP is computed by $dP = d(P + R) - dR$ for a random point R . However biases are pointed out [27]. Oswald-Aigner used the randomized addition-subtraction chain [25], but effect of the randomization is not enough [28, 29, 33].

Clavier-Joye splits the scalar randomly, in which dP is computed by $dP = rP + (d - r)P$ for a random integer r (the exponent splitting, ES) [6]. As no security problems have been pointed out, ES requires at least twice of the processing time than without it. Ciet-Joye proposed another splitting method, in which dP is computed by $[d/r](rP) + (d \bmod r)P$ for a random integer r [7]. Here r is as long as half of d and, with Shamir's trick, it provides practical processing speed without reducing the security.

Another approach for resisting DPA is to randomize a point representation. Coron proposed the randomized projective coordinates (RPC) countermeasure

[5]. Let $P = (X, Y, Z)$ be a base point in the projective coordinates. Then (X, Y, Z) equals to (rX, rY, rZ) for all $r \in K^*$ mathematically; but they all are different data as bit sequences. The power consumption of a scalar multiplication is randomized if (X, Y, Z) is randomized to (rX, rY, rZ) for a randomly chosen integer $r \in K^*$. An example of RPC for the add-and-double-always method is shown in Table 6, where a function `RPC` outputs a randomized point by RPC and a function `invRPC` denotes its inverse function. Joye-Tymen proposed another DPA-countermeasure based on the randomization of point representation, the randomized curve (RC) countermeasure [15]. RC uses an isomorphism of elliptic curves with which a curve equation and a base point are transformed with holding the group structure. Again, let $P = (X, Y, Z)$ be a base point in the projective coordinates. Then (X, Y, Z) and (r^2X, r^3Y, Z) correspond to each other under the isomorphism. Two points are the same (under the isomorphism) mathematically; but different as bit sequences. Thus the power consumption will be randomized if the curve and the base point is randomized by a randomly chosen integer $r \in K^*$. A sample algorithm is easily obtained similarly to Table 6 by changing functions `RPC`, `invRPC` to `RC`, `invRC`. `RPC` and `RC` provide efficient scalar multiplications because the conversions are quite cheap compared to the main multiplications.

Note that all countermeasures of ES, `RPC`, `RC` does not resist SPA. These countermeasures should be used combined with SPA-countermeasures.

Refined Power Analysis and Zero-value Point Attack

In 2003, Goubin presented a new power analysis against ECC, the *refined power analysis* (RPA), which reveals the secret scalar by detecting special points with 0-coordinates on the curve by chosen messages [9]. Let $E(K)$ be an elliptic curve which has the special point $Q_y = (0, y)$, for example. Suppose scalar multiplications are computed by add-and-double-always method from LSB (Table 2). When an adversary knows the bit information $d_0, \dots, d_{j-1}$ of the secret key d , he/she tries to reveal the next bit d_j . At the end of the $(j-1)$ -th loop in Table 2, we have $T[2] = 2^j P$, $T[0] = \sum_{i=0}^{j-1} d_i 2^i P$. Then, in the j -th loop, we will compute

$$T[1] \leftarrow \left(\sum_{i=0}^{j-1} d_i 2^i + 2^j \right) P, \quad T[0] \leftarrow \begin{cases} T[0] = \sum_{i=0}^{j-1} d_i 2^i P & \text{if } d_j = 0, \\ T[1] = (\sum_{i=0}^{j-1} d_i 2^i + 2^j) P & \text{if } d_j = 1. \end{cases}$$

Then the adversary pre-computes $\bar{P} = \left(\sum_{i=0}^{j-1} d_i 2^i + 2^j \right)^{-1} Q_y$, where the inverse is computed modulo the order of the curve, and computes a scalar multiplication on input d and $\bar{P}$. If $d_i = 1$, at the end of the j -th loop, we have $T[0] = \left(\sum_{i=0}^{j-1} d_i 2^i + 2^j \right) \bar{P} = Q_y = (0, y)$ and the adversary easily detects d_i , because the power consumption of “0” is quite distinctive. This is a basic

idea of RPA. The attack can be applied to the add-and-double-always method from MSB and the Montgomery ladder [9]. Note that the pre-computation is very easy for the adversary because elliptic curve parameters are public.

In the same year, Akishita-Takagi extended Goubin's RPA to the *zero-value point attack* (ZPA) [2]. A basic idea of ZPA is to observe 0 in the functions ECADD and ECDBL rather than in the addition chain where RPA paid attention. Akishita-Takagi discussed conditions when such 0 can be observed. In order to resist ZPA, implementers should avoid operations which possibly output 0 in ECADD and ECDBL.

Countermeasures against RPA/ZPA

A key of RPA is to observe 0 in the process. The most noteworthy property of RPA is that RPC and RC cannot resist RPA, because in RPC and RC, the special point $Q_x = (x, 0)$, $Q_y = (0, y)$ are converted to point $(X, 0, Z)$, $(0, Y, Z)$ by the conversion with keeping the specialty "0". Because of the same reasons as RPA, RPC and RC cannot resist ZPA.

Smart proposed an RPA-countermeasures using isogeny of the curve [32]. However, the countermeasure does not resist ZPA [3]. Avanzi analyzed the resistance of Coron's scalar blinding and base point blinding countermeasure against RPA, and concluded that they are resistant to RPA [1]. Similarly, they also resist ZPA. However, these Countermeasures require larger amount of computing time than without them. The exponent splitting countermeasure [6] and its improved version [7] also resist both RPA and ZPA. Especially, Ciet-Joye's countermeasure has almost no penalty from a viewpoint of efficiency.

4. Proposed Countermeasures

This section proposes two practical Countermeasures for ECC against power analysis including DPA, RPA and ZPA: the *randomized linearly-transformed coordinates* (RLC) and the *randomized initial point* (RIP) Countermeasures.

Randomized Linearly-transformed Coordinates (RLC)

Before proposing the countermeasure, we introduce the linearly-transformed coordinates (LC) in the following:

DEFINITION 1 *In the linearly-transformed coordinates (LC) with a parameter set $(\lambda_X(\lambda), \lambda_Y(\lambda), \lambda_Z(\lambda), \mu_X, \mu_Y, \mu_Z)$, a point (X, Y, Z) and (X', Y', Z') are identified if there exists $\lambda \in K^*$ such that*

$$\begin{aligned} X - \mu_X &= \lambda_X(\lambda)(X' - \mu_X), & Y - \mu_Y &= \lambda_Y(\lambda)(Y' - \mu_Y), \\ Z - \mu_Z &= \lambda_Z(\lambda)(Z' - \mu_Z), \end{aligned}$$

where $\lambda_X(\lambda)$, $\lambda_Y(\lambda)$, $\lambda_Z(\lambda)$ are functions of λ and $\mu_X, \mu_Y, \mu_Z \in K$.

Table 7. Addition formulas in LC for a parameter set $S_1 = (\lambda^2, \lambda^3, \lambda, \mu, 0, 0)$ (general case)

LCECADD for $S_1 = (\lambda^2, \lambda^3, \lambda, \mu, 0, 0)$	LCECDBL for $S_1 = (\lambda^2, \lambda^3, \lambda, \mu, 0, 0)$
Input: $P_1 = (X_1, Y_1, Z_1),$ $P_2 = (X_2, Y_2, Z_2), \mu$	Input: $P_1 = (X_1, Y_1, Z_1), a, \mu$
Output: $P_3 = P_1 + P_2 = (X_3, Y_3, Z_3)$	Output: $P_4 = 2P_1 = (X_4, Y_4, Z_4)$
$U_1 \leftarrow (X_1 - \mu)Z_2^2, U_2 \leftarrow (X_2 - \mu)Z_1^2$	$W \leftarrow aZ_1^4$
$S_1 \leftarrow Y_1Z_2^3, S_2 \leftarrow Y_2Z_1^3$	$M \leftarrow 3(X_1 - \mu)^2 + W$
$W \leftarrow U_2 - U_1, R \leftarrow S_2 - S_1$	$S \leftarrow 4(X_1 - \mu)Y_1^2$
$T \leftarrow U_2 + U_1, M \leftarrow S_2 + S_1$	$T \leftarrow 8Y_1^4$
$X_3 \leftarrow R^2 - TW^2 + \mu$	$X_4 \leftarrow M^2 - 2S + \mu$
$V \leftarrow 3TW^2 - 2R^2$	$Y_4 \leftarrow M(S - X_4 + \mu) - T$
$Y_3 \leftarrow (VR - MW^3)/2, Z_3 \leftarrow Z_1Z_2W$	$Z_4 \leftarrow 2Y_1Z_1$

The projective coordinates $\mathcal{P}$ and the Jacobian coordinates $\mathcal{J}$ are examples of the linearly-transformed coordinates; the parameter set for $\mathcal{P}$ is $(\lambda, \lambda, \lambda, 0, 0, 0)$ and for $\mathcal{J}$ is $(\lambda^2, \lambda^3, \lambda, 0, 0, 0)$. One of the advantage of LC is a variation of representations. Especially, if we set $\mu_X, \mu_Y \neq 0$, the special point $Q_y = (0, y)$ and $Q_x = (x, 0)$ are converted to some points whose coordinate values are different from 0. However, the processing speed of ECADD/ECDBL become slower in general than previous coordinates.

As an example, we give explicit addition formulas LCECADD, LCECDBL which compute ECADD and ECDBL in the linearly-transformed coordinates for a parameter set $S_1 = (\lambda^2, \lambda^3, \lambda, \mu, 0, 0)$ in Table 7. Here the coordinates are very similar to the formulas for the Jacobian case (Table 3).

Proposed Countermeasure. An idea of our countermeasure is quite simple. For given functions $\lambda_X(\lambda), \lambda_Y(\lambda), \lambda_Z(\lambda)$, we randomly choose μ_X, μ_Y, μ_Z and convert the base point into the transformed coordinates with the parameter set $(\lambda_X(\lambda), \lambda_Y(\lambda), \lambda_Z(\lambda), \mu_X, \mu_Y, \mu_Z)$ in the beginning of a scalar multiplication. After processing the scalar multiplication in LC, we re-convert the result to the original coordinates (*randomized linearly-transformed coordinates*, RLC). A generic algorithm of RLC is obtained by changing RPC, invRPC to RLC, invRLC, where a function RLC converts a point P into a point with a chosen parameter set, and invRLC re-converts to the affine coordinates. Note that RLC can be applied to all addition chains mentioned in section 2.

Let us consider the security of RLC. Basically, we have to use the add-and-double-always methods or the Montgomery ladder for the addition chain, for RLC does not resist SPA itself. On the other hand, RLC resists DPA if a parameter set is properly chosen, because RLC is a natural extension of RPC. But RPA will be successful when we choose $\mu_X = 0$ or $\mu_Y = 0$ (for speeding up), because points with 0 coordinate values can appear. In RLC, by choosing $\mu_X, \mu_Y \neq 0$, such special points will not appear in the process of scalar multiplications. This is the same against ZPA. Consequently, RLC resists SPA, DPA, RPA, and ZPA if the parameter set is chosen suitably (namely $\mu_X, \mu_Y \neq 0$).

Table 8. ADA and RLC

INPUT: d, P .	OUTPUT: dP
1: $T[0] = 0, T[2] = \text{RLC}(P)$	
2: for $i=0$ upto $n-1$ {	
3: $T[1] = \text{LCECADD}(T[0], T[2])$	
4: $T[2] = \text{LCECDBL}(T[2])$	
5: $T[0] = T[d[i]]$	
6: }	
7: return $\text{invRLC}(T[0])$	

Table 9. ADA and RIP

INPUT: d, P .	OUTPUT: dP
1: $R = \text{randompoint}()$	
2: $T[0] = R, T[2] = P$	
3: for $i=0$ upto $n-1$ {	
4: $T[1] = \text{ECADD}(T[0], T[2])$	
5: $T[2] = \text{ECDBL}(T[2])$	
6: $T[0] = T[d[i]]$	
7: }	
8: return $T[0]-R$	

However, a straight-forward implementation of RLC may be vulnerable to RPA. For example, when we implement LCECADD in Table 7, direct computations of $X_1 - \mu$ and $X_2 - \mu$ should be avoided, because these values equal to 0 for a chosen base point $\bar{P}$ by RPA. Thus we have to compute W, T without computing $X_i - \mu$ and U_i . Realizations of concrete secure algorithms are shown in the following sections.

In the followings, we combine RLC and three addition chains concretely, and then discuss the security against possible attacks. As we will consider over a finite field with prime elements, the condition $\mu_X \neq 0$ should be satisfied.

Combination with Add-and-double-always Method from MSB. Applying RLC to the add-and-double-always method from MSB is easy (so we omit a concrete algorithm because of the space limitation). Here both LCECADD and LCECDBL should be implemented not to compute $X_i - \mu_X$ as mentioned before. Remember that if μ_X is chosen randomly (different from 0), the probability that $X_i - \mu_X = 0$ is negligible. Concrete secure algorithms LCECADD1 and LCECDBL1 for a parameter set $S_1 = (\lambda^2, \lambda^3, \lambda, \mu, 0, 0)$, $\mu \in K^*$ are shown in Table A1 and Table A2 in the appendix. Note that these algorithms are constructed not to output 0 values after each operation in order to resist ZPA. The processing speed of LCECADD1 is $14M + 4S$ (Table A1) and LCECDBL1 is $8M + 7S$ (Table A2). An average processing speed for each bit is $T^{\text{MSB}} = (14M + 4S) + (8M + 7S) = 22M + 11S$.

Combination with Add-and-double-always Method from LSB. Applying RLC to the add-and-double-always method from LSB is also easy. A concrete algorithm is shown in Table 8. In this case, a variable $T[2]$ has no relation to the secret key d . Hence there is no need for $T[2]$ to be protected. In other words, we are allowed to compute $X_2 - \mu_X$ in LCECADD, and allowed not to use secure algorithm for LCECDBL. Thus we only take care not to compute $X_1 - \mu_X$ in LCECADD. A concrete algorithm LCECADD1' for a parameter set $S_1 = (\lambda^2, \lambda^3, \lambda, \mu, 0, 0)$, $\mu \in K^*$ is shown in Table A3 in the appendix. Note that the algorithm is constructed not to output 0 values after each operation in order to resist ZPA. The processing speed of LCECADD1' is $13M + 4S$ (Table

A1) and LCECDBL1 is $4M + 6S$ (Table 3). An average processing speed for each bit is $T^{\text{LSB}} = (13M + 4S) + (4M + 6S) = 17M + 10S$.

When the secret key d has successive 0 in lowest bits, the number of 0 can be detected, because variables $T[0]$, $T[2]$ are not randomized by insecure LCECDBL. Once 1 appears in d , $T[0]$ is randomized in LCECADD and no information will be leaked moreover. One solution for the leakage is to use secure LCECDBL and compute $X - \mu_X$ directly. Since the effect of λ 's are remained, the security is improved without reducing the processing speed.

In the Jacobian coordinates, Itoh et al. proposed an efficient algorithm called the *iterated ECDBL* [13], which computes $2^k P$ for a given integer $k \geq 1$ more efficiently than computing ECDBL k times by recycling intermediate values; the iterated ECDBL requires $4kM + (4k + 2)S$, while each ECDBL requires $4M + 6S$ (Table 4). In our countermeasure for the parameter set S_1 , a variable $T[2]$ only concerns a sequence $P, 2P, 2^2P, \dots$ and the iterated ECDBL can be applied efficiently. In this case, an averaged processing speed for each bit is optimized to $T_0^{\text{LSB}} = (13M + 4S) + (4M + 4S) = 17M + 8S$.

Combination with Montgomery Ladder. Applying RLC to the Montgomery ladder is also easy. In this case, all variables $T[0]$, $T[1]$, $T[2]$ depend on the secret key and both LCECADD and LCECDBL should be implemented not to compute $X_i - \mu_X$.

When we use secure algorithms LCECADD1, LCECDBL1 for the parameter set S_1 (Table A1 and Table A2 in the appendix), an average processing speed is $T^{\text{Mon}} = (14M + 4S) + (6M + 6S) = 20M + 10S$.

The Montgomery ladder provides good processing speed by using the specialized addition formula, the ***x-coordinate-only*** addition formula [4, 12, 11], which computes ECADD and ECDBL with only ***x-coordinate*** value in the affine coordinates, or X - and Z -coordinate values in the projective or the Jacobian coordinates. We can easily establish similar formulas $\times\text{LCECADD}$, $\times\text{LCECDBL}$, which compute ECADD and ECDBL in our linearly-transformed coordinates by the ***x-coordinate-only*** formula.

Moreover, Izu-Takagi proposed an encapsulated formula ECADDDBL, which takes P_1, P_2 on input and outputs $P_1 + P_2$ and $2P_1$, and $\times\text{ECADDDBL}$ computed by the ***x-coordinate-only*** formula. In this case, an improved version of the Montgomery ladder is required [11]. A concrete algorithm $\times\text{LCECADDDBL2}$, which computes $\times\text{ECADDDBL}$ in LC by the ***x-coordinate-only*** method for a parameter set $S_2 = (\lambda, \lambda, \lambda, \mu, 0, 0)$, is shown in Table A4 (where the additive ECADD is used [12]). Here the algorithm is constructed not to output 0 values after each operation in order to resist ZPA, and the processing speed for each bit is $T_0^{\text{Mon}} = 20M + 5S$.

Randomized Initial Point Countermeasure (RIP)

Next, we propose another countermeasure against power analysis.

Proposed Countermeasure. In the add-and-double always method from LSB (Table 2), an initial value of $T[0]$ is set to $\mathcal{O}$. If this initial value is changed to a randomized point R , intermediate information will be randomized and thus it resists power analysis. (Of course, R should be subtracted from a result of a scalar multiplication.) This countermeasure is called the *randomized initial point* countermeasure (RIP). A generic algorithm of RIP combined with Table 2 is shown in Table 9. Here a function `randompoint()` generates a random point on the curve. Note that RIP can be applied only to the add-and-double always method from LSB.

Selection of Initial Point. The security of RIP depends on how to select an initial point R . The simplest way is to generate the **x -coordinate** value of R and compute corresponding **y -coordinate** value. One can obtain such y with probability $1/2$, namely it might require large amount of time to obtain R . However, R can be obtained in short time practically.

The other way is to convert a fixed and stored point Q into a randomized point R by using RPC. As the conversion is very cheap, it provides an efficient generation of R .

Security Consideration. Firstly, as RIP does not resist SPA (similarly to RLC), we have to use the add-and-double always method for the addition chain (Table 9). When an initial point R is chosen properly, as a representation of R would be randomized, RIP resists DPA (contrary, RIP is vulnerable to DPA if R is fixed). Here there is no need to apply RPC or RC. On the other hand, RIP also resists RPA and ZPA, because the all intermediate values are randomized by the initial point R .

Processing Speed. When R is set from Q by RPC, the processing time of generating R is negligible. Thus, the processing time of RIP per bit equals to a sum of that of ECADD and ECDBL. It is $(12M+2S)+(7M+5S) = 19M+7S$ for the projective coordinates, and $(12M+4S)+(4M+6S) = 16M+10S$ for the Jacobian coordinates.

When the Jacobian coordinates are used, the iterated ECDBL can be also applied similarly to RLC. In this case, the processing time of RIP per a bit equals $(12M + 4S) + (4M + 6S) - 2S = 16M + 8S$.

Table 10. Comparison of secure countermeasures

Method	Coordinates	Speed in each bit	
iES [7]	$\mathcal{J}$	$16M + 10S$	(24.0M)
iES with Montgomery's trick	$\mathcal{J}$	$12M + 9S$	(19.2M)
RLC with MSB	$\mathcal{J}$	$22M + 11S$	(30.2M)
RLC with LSB	$\mathcal{J}$	$17M + 10S$	(25.0M)
RLC with LSB and iECDBL	$\mathcal{J}$	$17M + 8S$	(23.4M)
RLC with Montgomery ladder	$\mathcal{J}$	$20M + 5S$	(24.0M)
RIP with LSB	$\mathcal{P}$	$19M + 7S$	(24.6M)
RIP with LSB	$\mathcal{J}$	$16M + 10S$	(24.0M)
RIP with LSB and iECDBL	$\mathcal{J}$	$16M + 8S$	(22.4M)

5. Comparison

Let us compare countermeasures against power analysis that resist all of SPA, DPA, RPA and ZPA. We provide practical processing time for the improved exponent splitting (iES) [7] and our proposed two countermeasures (RLC and RIP). Processing speed are given averaged for each bit of the scalar d and is estimated with the assumption that $1S = 0.8M$ [27, 12].

As in the table, iES with Shamir's trick and Montgomery's trick provides the most efficient processing speed. However, our proposed countermeasures provide comparable speed. A difference between iES and RIP is the number of registers for points on the curve; iES requires 5 registers while our RIP requires only 4 registers for a scalar multiplication. Note that, in RLC, the seed of an initial point R is required. That is, RLC requires extra memory compared to other methods. On the other hand, RLC requires no extra memory.

6. Concluding Remarks

In this paper, two new practical countermeasures against power analysis are proposed, the randomized linearly-transformed coordinates countermeasure and the randomized initial point countermeasure. These methods improves the processing speed without reducing the security.

Since RPA and ZPA are newer attacks, the property of the attacks and of the countermeasures are not well studied yet. Theoretically, iES provides a better processing speed, however, because it uses a small table of pre-computed points, the resistance against the address-bit DPA [10] and the 2nd order DPA [28] should be analyzed. On the other hand, applying RIP to other addition chains will be for future work. Very recently, Morimoto-Mamiya-Miyaji attempted to apply RIP to the binary method from MSB [23], for which further analysis will be required.

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Appendix: Algorithms for Proposed Countermeasures

The appendix shows detailed concrete algorithms for `LCECADD1`, `LCECDBL1`, `LCECADD1'`, and `xLCECADDDBL2` for RLC introduced in section 4.1. The parameter set for `LCECADD1`,

Table A1. ECADD in LC with a parameter set $S_1 = (\lambda^2, \lambda^3, \lambda, \mu, 0, 0)$

LCECADD1 (in 14M+4S)	
Input: $P_1 = (X_1, Y_1, Z_1)$, $P_2 = (X_2, Y_2, Z_2)$, μ	
Output: $P_3 = P_1 + P_2 = (X_3, Y_3, Z_3)$	
$T_1 \leftarrow Z_1^2$	$(= Z_1^2)$
$T_2 \leftarrow Z_1 \times T_1$	$(= Z_1^3)$
$T_3 \leftarrow Z_2^2$	$(= Z_2^2)$
$T_4 \leftarrow Z_2 \times T_3$	$(= Z_2^3)$
$T_5 \leftarrow X_1 \times T_3$	$(= X_1 Z_2^2)$
$T_6 \leftarrow \mu \times T_3$	$(= \mu Z_2^2)$
$T_7 \leftarrow X_2 \times T_1$	$(= X_2 Z_1^2)$
$T_8 \leftarrow \mu \times T_1$	$(= \mu Z_1^2)$
$T_9 \leftarrow Y_1 \times T_4$	$(= Y_1 Z_2^3 = S_1)$
$T_{10} \leftarrow Y_2 \times T_2$	$(= Y_2 Z_1^3 = S_2)$
$T_{11} \leftarrow T_7 - T_5$	$(= X_2 Z_1^2 - X_1 Z_2^2)$
$T_{12} \leftarrow T_{11} - T_8$	$(= U_2 - X_1 Z_2^2)$
$T_{13} \leftarrow T_{12} + T_6$	$(= U_2 - U_1 = W)$
$T_{14} \leftarrow T_7 + T_5$	$(= X_2 Z_1^2 + X_1 Z_2^2)$
$T_{15} \leftarrow T_{14} - T_8$	$(= U_2 + X_1 Z_2^2)$
$T_{16} \leftarrow T_{15} - T_6$	$(= U_2 + U_1 = T)$
$T_{17} \leftarrow T_{10} - T_9$	$(= S_2 - S_1 = R)$
$T_{18} \leftarrow T_{10} + T_9$	$(= S_2 + S_1 = M)$
$T_{19} \leftarrow T_{17}^2$	$(= R^2)$
$T_{20} \leftarrow T_{13}^2$	$(= W^2)$
$T_{21} \leftarrow T_{16} \times T_{20}$	$(= TW^2)$
$T_{22} \leftarrow T_{19} + \mu$	$(= R^2 + \mu)$
$X_3 \leftarrow T_{22} - T_{21}$	$(= R^2 - TW^2 + \mu)$
$T_{23} \leftarrow 2 * X_3$	
$T_{24} \leftarrow T_{23} - T_{21}$	
$T_{25} \leftarrow 2 * \mu$	$(= 2\mu)$
$T_{26} \leftarrow T_{25} - T_{24}$	$(= V)$
$T_{27} \leftarrow T_{25} \times T_{17}$	$(= VR)$
$T_{28} \leftarrow T_{18} \times T_{13}$	$(= MW)$
$T_{29} \leftarrow T_{28} \times T_{20}$	$(= MW^3)$
$2Y_3 \leftarrow T_{27} - T_{29}$	
$T_{30} \leftarrow Z_1 \times Z_2$	$(= Z_1 Z_2)$
$Z_3 \leftarrow T_{30} \times T_{13}$	$(= Z_1 Z_2 W)$

Table A2. ECDBL in LC with a parameter set $S_1 = (\lambda^2, \lambda^3, \lambda, \mu, 0, 0)$

LCECDBL1 (in 8M+7S)	
Input: $P_1 = (X_1, Y_1, Z_1)$, a , μ	
Output: $P_4 = 2P_1 = (X_4, Y_4, Z_4)$	
$T_1 \leftarrow Z_1^2$	$(= Z_1^2)$
$T_2 \leftarrow T_1^2$	$(= Z_1^4)$
$T_3 \leftarrow a \times T_2$	$(= aZ_1^4 = W)$
$T_4 \leftarrow X_1^2$	$(= X_1^2)$
$T_5 \leftarrow 2 * X_1$	$(= 2X_1)$
$T_6 \leftarrow \mu - T_5$	$(= \mu - 2X_1)$
$T_7 \leftarrow \mu \times T_6$	$(= \mu(\mu - 2X_1))$
$T_8 \leftarrow 3 * T_4$	$(= 3X_1^2)$
$T_9 \leftarrow 3 * T_7$	$(= 3\mu(\mu - 2X_1))$
$T_{10} \leftarrow T_3 + T_8$	$(= W + 3X_1^2)$
$T_{11} \leftarrow Y_1^2$	$(= Y_1^2)$
$T_{12} \leftarrow X_1 \times T_{11}$	$(= X_1 Y_1^2)$
$T_{13} \leftarrow \mu \times T_{11}$	$(= \mu Y_1^2)$
$T_{14} \leftarrow T_{11}^2$	$(= Y_1^4)$
$T_{15} \leftarrow 8 * T_{14}$	$(= 8Y_1^4 = T)$
$T_{16} \leftarrow T_9^2$	$(= (3\mu(\mu - 2X_1))^2)$
$T_{17} \leftarrow T_9 \times T_{10}$	
$T_{18} \leftarrow 2 * T_{17}$	
$T_{19} \leftarrow T_{10}^2$	$(= (W + 3X_1^2)^2)$
$T_{20} \leftarrow \mu + T_{16}$	
$T_{21} \leftarrow T_{20} + T_{18}$	
$T_{22} \leftarrow T_{21} + T_{19}$	$(= \mu + M^2)$
$T_{23} \leftarrow 8 * T_{12}$	$(= 8X_1 Y_1^2)$
$T_{24} \leftarrow 8 * T_{13}$	$(= 8\mu Y_1^2)$
$T_{25} \leftarrow T_{22} - T_{23}$	$(= M^2 - 8X_1 Y_1^2 + \mu)$
$X_4 \leftarrow T_{22} + T_{24}$	$(= M^2 - 2S + \mu)$
$T_{26} \leftarrow 4 * T_{12}$	$(= 4X_1 Y_1^2)$
$T_{27} \leftarrow 4 * T_{13}$	$(= 4\mu Y_1^2)$
$T_{28} \leftarrow T_{26} + \mu$	$(= 4X_1 Y_1^2 + \mu)$
$T_{29} \leftarrow X_4 - T_{28}$	$(= 4X_1 Y_1^2 - X_4 + \mu)$
$T_{30} \leftarrow T_{29} - T_{27}$	$(= S - X_4 + \mu)$
$T_{31} \leftarrow T_9 \times T_{30}$	
$T_{32} \leftarrow T_{10} \times T_{30}$	
$T_{33} \leftarrow T_{31} - T_{15}$	
$Y_4 \leftarrow T_{33} + T_{32}$	$(= M(S - X_4 + \mu) - T)$
$T_{34} \leftarrow Y_1 \times Z_1$	$(= Y_1 Z_1)$
$Z_4 \leftarrow 2 * T_{34}$	$(= 2Y_1 Z_1)$

LCECDBL1, LCECADD1' is $S_1 = (\lambda^2, \lambda^3, \lambda, \mu, 0, 0)$ and that for $\times$ LCECADDDBL2 is $S_2 = (\lambda, \lambda, \lambda, \mu, 0, 0)$. In the algorithms, $\times$ denotes a general multiplication, while $*$ denotes a multiplication by a constant. We ignore the processing time for $*$ in the evaluations.