RESOURCE BOUNDED IMMUNITY AND SIMPLICITY *

Extended Abstract

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Abstract Revisiting the thirty years-old notions of resource-bounded immunity and simplicity, we investigate the structural characteristics of various immunity notions: strong immunity, almost immunity, and hyperimmunity as well as their corresponding simplicity notions. We also study ***k*-immunity** and ***k*-simplicity** and their extensions: feasible ***k*-immunity** and feasible ***k*-simplicity**. Finally, we propose the ***k*-immune** hypothesis as a working hypothesis that ensures the existence of simple sets in NP.

Keywords: immune set, simple set, complete set, forcing, generic oracle, random oracle

1. Foundations of Immunity and Simplicity

The original notions of immunity and simplicity date back to mid 1940s. Post [17] first constructed a simple set for the class of recursively enumerable sets. The new breed of resource-bounded immunity and simplicity waited to be introduced until mid 1970s by an early work of Flajolet and Steyaert [5]. In their seminal paper, Flajolet and Steyaert constructed various recursive sets that, for instance, have no infinite ***DTIME(t(n))-subsets*** under the term “***DTIME(t(n))-immune*** sets.” Later, Ko and Moore [11] studied the polynomial-time bounded immunity, which is now preferably called P-immune sets. Subsequently, Balcázar and Schöning [2] considered P-bi-immune sets, which are P-immune sets whose complements are also P-immune. Homer and Maass [8] extensively discussed the cousin of P-immune sets, known as NP-simple sets. The importance of these notions was widely recognized in 1980s. Since these notions can be easily expanded to any complexity class ***C***, we begin with

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an introduction of the general notions of $\mathcal{C}$ -immune sets, $\mathcal{C}$ -bi-immune sets, and $\mathcal{C}$ -simple sets. These notions are further expanded in various manners in later sections.

DEFINITION 1 Let $\mathcal{C}$ be any complexity class of languages over alphabet Σ .

- 1 A set S is **$\mathcal{C}$ -immune** if S is infinite and there is no infinite subset of S in $\mathcal{C}$.
- 2 A set S is **$\mathcal{C}$ -bi-immune** if S and $\bar{S}$ are both $\mathcal{C}$ -immune.
- 3 A set S is **$\mathcal{C}$ -simple** if S belongs to $\mathcal{C}$ and $\bar{S}$ is $\mathcal{C}$ -immune.

Note that the existence of a $\mathcal{C}$ -simple set immediately implies $\mathcal{C} \neq \text{co-}\mathcal{C}$; however, the separation $\mathcal{C} \neq \text{co-}\mathcal{C}$ does not necessarily guarantee the existence of $\mathcal{C}$ -simple sets.

Throughout this paper, we set our alphabet Σ to be $\{0,1\}$. Let $\mathbb{N}$ (or ω) denote the set of all nonnegative integers and set $\mathbb{N}^+ = \mathbb{N} - \{0\}$. All *logarithms* are taken to base 2 and a *polynomial* means a multivariate polynomial with integer coefficients. We assume a standard bijection from $\Sigma^{<\omega}$ to Σ^* that is polynomial-time computable and polynomial-time invertible, where $\Sigma^{<\omega}$ is the set of all finite sequences of strings over Σ . This bijection enables us to identify $\Sigma^{<\omega}$ with Σ^* . We use *multi-tape off-line Turing machines* (TMs, in short) as a model of computation. Assumed is the reader's familiarity with basic complexity classes, such as P, NP, E (linear exponential time), and EXP (polynomial exponential time). This paper focuses mostly on the complexity classes lying in the *polynomial-time hierarchy*¹ $\{\Delta_k^P, \Sigma_k^P, \Pi_k^P \mid k \in \mathbb{N}\}$ [12].

We mainly use “partial” functions and all functions are presumed to be single-valued. Since total functions are also partial functions, we will define function classes as collections of *partial* functions and, whenever we need total functions, we will explicitly indicate the *totality* of functions. Now, fix $k \in \mathbb{N}^+$. The notation $\text{F}\Delta_k^P$ denotes the collection of all single-valued partial functions f such that there exist a set $B \in \Sigma_{k-1}^P$ and a polynomial-time deterministic oracle TM M satisfying the following condition: for every x , if $x \in \text{dom}(f)$ then $M^B(x)$ halts in an accepting state and outputs $f(x)$ and otherwise, $M^B(x)$ halts in a rejecting state (in this case, $f(x)$ is *undefined*). In particular, write FP for $\text{F}\Delta_1^P$.

A set A is called Δ_k^P -*m-reducible* to another set B via a reduction f if f is a total $\text{F}\Delta_k^P$ -function from Σ^* to Σ^* and $A = \{x \mid f(x) \in B\}$. If in addition f is honest², then we say that A is $h\text{-}\Delta_k^P$ -*m-reducible* to B . Moreover, a set A is Δ_k^P -*tt-reducible* to B via a reduction triplet (ν, f, α) if (i) ν is a total $\text{F}\Delta_k^P$ -function from Σ^* to $\{0\}^*$, (ii) f is a total $\text{F}\Delta_k^P$ -function from Σ^* to Σ^* such that, for every x , $f(x) = \langle y_1, y_2, \dots, y_k \rangle$ for certain strings $y_1, y_2, \dots, y_k$, where $k = |\nu(x)|$, (iii) α is a total $\text{F}\Delta_k^P$ -function from $\Sigma^* \times \Sigma^*$ to $\{0,1\}$ such that $A = \{x \mid \alpha(x, B(f(x))) = 1\}$, where $B(f(x))$ is an abbreviation of the k -bit string $B(y_1)B(y_2) \cdots B(y_k)$ when $f(x) = \langle y_1, \dots, y_k \rangle$. If in addition f is componentwise honest³, then A is $h\text{-}\Delta_k^P$ -*tt-reducible* to B . The notion of *completeness* can be induced from its corresponding reducibility.

¹The polynomial-time hierarchy consists of the classes defined in the following fashion: $\Delta_0^P = \Sigma_0^P = \Pi_0^P = \text{P}$, $\Sigma_{k+1}^P = \text{NP}^{\Sigma_k^P}$, and $\Pi_{k+1}^P = \text{co-}\Sigma_{k+1}^P$ for any number $k \in \mathbb{N}$.

²A partial function f from Σ^* to Σ^* is *polynomially honest* (*honest*, for short) if there exists a polynomial p such that $|x| \leq p(|f(x)|)$ for all strings $x \in \text{dom}(f)$.

³A partial function from Σ^* to $\Sigma^{<\omega}$ (which is identified with Σ^*) is *componentwise honest* if there exists a polynomial p such that, for every $x \in \text{dom}(f)$, $|x| \leq p(|y_i|)$ for all $i \in \{1, 2, \dots, k\}$, provided that $f(x) = \langle y_1, y_2, \dots, y_k \rangle$.

It is well-known that P-immune sets exist even in the class E. In particular, Ko and Moore [11] constructed a P-immune set that is also P-tt-complete for E. Note that no h-P-m-complete set for NP can be P-immune since the image of a P-immune set by a polynomial-time computable reduction is either finite or P-immune. Using a relativization technique, Bennett and Gill [3] showed that a P-immune set exists in NP relative to a random oracle with probability 1. A recursive oracle relative to which NP contains P-immune sets was later constructed by Homer and Maass [8]. Torenvliet and van Emde Boas [22] strengthened their results by demonstrating a relativized world where NP has a P-immune set which is also NP-simple.

The notion of **C-immunity** is closely related to various other notions, which include complexity cores [13] and instance complexity [15]. We can naturally expand these characterizations to more general Σ_k^P -immune and Δ_k^P -immune sets. Balcázar and Schöning [2] also built a bridge between P-bi-immune sets and finite-to-one reductions. Expanding their argument, we give in Lemma 2 a characterization of **C-bi-immunity** as well as **C-immunity**.

For any partial function f from Σ^* to Σ^* , the set $\text{Graph}(f) = \{\langle x, f(x) \rangle \mid x \in \text{dom}(f)\}$ is called the *graph* of f . Let $\Sigma_0^P \text{SV} = \text{FP}$ and let $\Sigma_k^P \text{SV}$ denote the class of all single-valued partial functions f such that f is polynomially bounded⁴ and $\text{Graph}(f)$ is in Σ_k^P . For brevity, we write NPSV for $\Sigma_1^P \text{SV}$. For any $k \in \mathbb{N}$ and any $A, B \subseteq \Sigma^*$, a single-valued partial function f from Σ^* to Σ^* is called a Σ_k^P -*m-quasireduction* (Δ_k^P -*m-quasireduction*, resp.) from A to B if (i) f is in $\Sigma_k^P \text{SV}$ (Δ_k^P , resp.), (ii) $\text{dom}(f)$ is infinite, and (iii) for any string $x \in \text{dom}(f)$, $x \in A$ iff $f(x) \in B$. For any string $u \in \Sigma^*$, the *inverse image* $f^{-1}(u)$ of f at u is the set $\{x \in \text{dom}(f) \mid f(x) = u\}$. Notice that $f^{-1}(u) = \emptyset$ if $u \notin \text{ran}(f)$.

LEMMA 2 *Let $\mathcal{C} \in \{\Delta_k^P, \Sigma_k^P \mid k \in \mathbb{N}\}$ and $S \subseteq \Sigma^*$.*

- 1 *S is **C-immune** if and only if (i) S is infinite and (ii) for every set B , every $\mathcal{C}$ -*m-quasireduction* f from S to B , and every u in B , $f^{-1}(u)$ is finite.*
- 2 *S is **C-bi-immune** if and only if (i) S is infinite and (ii) for every set B , every $\mathcal{C}$ -*m-quasireduction* f from S to B , and every u in Σ^* , $f^{-1}(u)$ is finite.*

The characterization given in Lemma 2(2) led Balcázar and Schöning [2] to introduce a stronger notion of P-bi-immunity: strong P-bi-immunity. A more general notion, called strong **C-immunity**, will be introduced in Section 2.

Whether an NP-simple set exists is one of the long-standing open problems because such a set separates NP from co-NP. Nonetheless, NP-simple sets are known to exist in various relativized worlds. In early 1980s, Homer and Maass [8] and Balcázar [1] constructed relativized worlds where an NP-simple set exists. Later, Vereshchagin [24] proved that, relative to a random oracle, an NP-simple set exists with probability 1. From Theorem 9 in Section 2, for instance, it immediately follows that an NP-simple set exists relative to a generic oracle. Torenvliet [21] built an oracle relative to which a Σ_2^P -simple set exists. For a much higher level k of the polynomial-time

⁴A partial function f from Σ^* to Σ^* is *polynomially bounded* if there exists a polynomial p such that $|f(x)| \leq p(|x|)$ for any string $x \in \text{dom}(f)$.

hierarchy, Bruschi [4] constructed an oracle relative to which Σ_k^P -simple sets exist using the size lower bounds of certain nonuniform constant-depth circuits.

In the rest of this section, we focus on closure properties of the class of all Σ_k^P -immune sets because no such closure property has been systematically studied in the literature. A complexity class $\mathcal{C}$ is said to be *closed downward under a reduction $\leq_r$ on infinite sets* if, for any pair of infinite sets A and B , $A \leq_r B$ and $B \in \mathcal{C}$ imply $A \in \mathcal{C}$. Here, we study three reducibilities. Let $k \in \mathbb{N}$. A set A is Δ_k^P -d-reducible to B via f if f is a total $\mathbf{F}\Delta_k^P$ -function from Σ^* to $\Sigma^{<\omega}$ (which is identified with Σ^*) and $A = \{x \mid B \cap \text{set}(f(x)) \neq \emptyset\}$, where $\text{set}(\langle y_1, y_2, \dots, y_m \rangle) = \{y_1, y_2, \dots, y_m\}$. By contrast, A is Δ_k^P -c-reducible to B via f if A is Δ_k^P -d-reducible to $\bar{B}$ via f . For any fixed $i \in \mathbb{N}^+$, A is Δ_k^P -itt-reducible to B via (f, α) if A is Δ_k^P -tt-reducible to B via (ν, f, α) , where $\nu(x) = 0^i$ for any x . For any reducibility r using computation $\mathcal{C}$, we say that A is $\mathbf{h}\mathcal{C}$ - r -reducible to B if A is $\mathcal{C}$ - r -reducible to B via f (or (f, α)) such that f is componentwise honest.

Now, we claim that the class of all Σ_k^P -immune sets is closed downward under $\mathbf{h}\Delta_k^P$ -c-reductions on infinite sets; however, we cannot replace this conjunctive reducibility by disjunctive reducibility.

THEOREM 3 *Let $k \in \mathbb{N}^+$.*

- 1 *The class of all Σ_k^P -immune sets is closed downward under $\mathbf{h}\Delta_k^P$ -c-reductions on infinite sets.*
- 2 *The class of all NP-immune sets is not closed under h-P-d-reductions or h-P-tt-reductions on infinite sets.*

The first claim of Theorem 3 is easy and is shown as follows. Assume that an infinite set A is $\mathbf{h}\Delta_k^P$ -c-reducible to a Σ_k^P -immune set B via a componentwise-honest reduction f . If A contains an infinite Σ_k^P -subset C , then consider the set $D = \{y \mid \exists x \in C[|x| \leq p(|y|) \wedge y \in \text{set}(f(x))]\}$, where p is a polynomial such that $|x| \leq p(|y|)$ for all x and all $y \in \text{set}(f(x))$. Clearly, D is an infinite Σ_k^P -subset of B , a contradiction. Therefore, A is Σ_k^P -immune.

How complex are Σ_k^P -simple sets? Intuitively, $\mathcal{C}$ -simple sets are “thin” and thus cannot be “complete” for the class $\mathcal{C}$. As an immediate consequence of Theorem 3(1), we obtain the following corollary.

COROLLARY 4 *Let $k \in \mathbb{N}^+$. No Σ_k^P -simple set is $\mathbf{h}\Delta_k^P$ -d-complete for Σ_k^P .*

Recently, Agrawal (cited in [19]) showed, using the NP-levelability of $\overline{\text{SAT}}$ (assuming $\text{SAT} \notin \mathbf{P}$), that no NP-simple set is h-P-btt-complete for NP, where SAT is the set of all satisfiable quantifier-free Boolean formulas. His argument will be generalized in Section 4 in connection to $\mathcal{C}$ -hyperimmune sets.

2. Strong Immunity and Strong Simplicity

Following the introduction of P-bi-immunity, Balcázar and Schöning [2] stepped forward to introduce the notion of strongly P-bi-immunity, which comes from the quasireducibility-characterization of P-bi-immunity given in Lemma 2(2). While P-bi-immunity requires its quasireductions to be finite-to-one, strong P-bi-immunity requires the quasireductions to be almost one-to-one, where a quasireduction f is called

almost one-to-one on a set S if the collision set $\{(x, y) \in (\text{dom}(f) \cap S)^2 \mid x < y \wedge f(x) = f(y)\}$ is finite. Such strongly P-bi-immune sets are known to exist even in the class E [2].

Generalizing the notion of P-bi-immunity, we can introduce strong **C-bi-immunity** for any complexity class $\mathcal{C}$ lying in the polynomial-time hierarchy. Moreover, we newly introduce the notions of strong **C-immunity** and strong **C-simplicity**. Recall that Σ_k^P -**m-quasireductions** are all single-valued functions in $\Sigma_k^P \text{SV}$ for each $k \in \mathbb{N}^+$.

DEFINITION 5 Let $\mathcal{C} \in \{\Delta_k^P, \Sigma_k^P \mid k \in \mathbb{N}\}$.

- 1 A set S is strongly **C-immune** if (i) S is infinite and (ii) for every set B and every **C-m-quasireduction** f from S to B , f is almost one-to-one on S .
- 2 A set S is strongly **C-bi-immune** if S and $\bar{S}$ are both strongly **C-immune**.
- 3 A set S is strongly **C-simple** if S is in $\mathcal{C}$ and $\bar{S}$ is strongly **C-immune**.

In particular, when $\mathcal{C} = \text{P}$, Definition 5(2) coincides with the notion of P-bi-immunity originally given in [2].

LEMMA 6 For any complexity class $\mathcal{C} \in \{\Delta_k^P, \Sigma_k^P \mid k \in \mathbb{N}\}$, every strongly **C-immune** set is **C-immune** and every strongly **C-simple** set is **C-simple**.

A major difference between **C-immunity** and strong **C-immunity** is shown in the following example. For any NP-immune set A , the disjoint union⁵ $A \oplus A$ is also NP-immune; on the contrary, it is not strongly NP-immune because $A \oplus A$ can be reduced to A by the almost two-to-one function f defined as $f(\lambda) = \lambda$ and $f(xb) = x$ for $b \in \{0, 1\}$, where λ is the empty string. Therefore, the class of all strongly NP-immune sets is not closed under the disjoint-union operator. Historically, using the structural difference between these two notions, Balcázar and Schöning [2] constructed a set in E which is P-bi-immune but not strongly P-bi-immune.

We show a closure property of the class of strongly Σ_k^P -immune sets. If A is Δ_k^P -**m-reducible** to B via a one-to-one honest reduction f , we say that A is $h\text{-}\Delta_k^P$ -**1-reducible** to B via f .

PROPOSITION 7 Let $k \in \mathbb{N}^+$.

- 1 The class of all strongly Σ_k^P -immune sets is closed downward under $h\text{-}\Delta_k^P$ -1-reductions on infinite sets.
- 2 The class of all strongly NP-immune sets is not closed downward under $h\text{-P-m-reductions}$ on infinite sets.

COROLLARY 8 For each level $k \in \mathbb{N}^+$, there is no strongly Σ_k^P -simple set that is $h\text{-}\Delta_k^P$ -1-complete for Σ_k^P .

Finally, we turn our interest to relativization. For each $k \in \mathbb{N}^+$, it is easy to show that a strongly Σ_k^P -simple set exists relative to a recursive oracle (similar to Proposition

⁵ The disjoint union of A and B is the set $\{0x \mid x \in A\} \cup \{1x \mid x \in B\}$.

26). Even relative to a random oracle, there exists a strongly NP-simple set with probability 1 (similar to Proposition 27). Employing weak forcing, we now prove the following relativization result.

THEOREM 9 *A strongly NP^G -simple set exists relative to a generic oracle G .*

3. Almost Immunity and Almost Simplicity

We have shown in the previous section that strong $\mathcal{C}$ -immunity and its simplicity strengthen the ordinary notion of $\mathcal{C}$ -immunity and $\mathcal{C}$ -simplicity. In contrast to these notions, Orponen [14] and Orponen, Russo, and Schöning [16] expanded P-immunity to the new notion of almost P-immunity. The complementary notion of almost P-immunity under the term P-levelability (a more general term “levelable sets” was first used by Ko [10] in a resource-bounded setting) was extensively discussed by Orponen et al. [16]. Naturally, we can generalize these notions to almost $\mathcal{C}$ -immunity and $\mathcal{C}$ -levelability for any complexity class $\mathcal{C}$. Furthermore, we newly introduce the notion of almost $\mathcal{C}$ -bi-immunity and almost $\mathcal{C}$ -simplicity.

DEFINITION 10 *Let $\mathcal{C}$ be any complexity class.*

- 1 *A set S is almost $\mathcal{C}$ -immune if S is the union of a $\mathcal{C}$ -immune set and a set in $\mathcal{C}$.*
- 2 *An infinite set is $\mathcal{C}$ -levelable if it is not almost $\mathcal{C}$ -immune.*
- 3 *A set S is almost $\mathcal{C}$ -bi-immune if S and $\bar{S}$ are both almost $\mathcal{C}$ -immune.*
- 4 *A set S is almost $\mathcal{C}$ -simple if S is an infinite set in $\mathcal{C}$ and $\bar{S}$ is the union of a set A in $\mathcal{C}$ and a $\mathcal{C}$ -immune set B , where the difference $B \setminus A$ is infinite.*

It follows from Definition 10(1) that every almost $\mathcal{C}$ -immune set is infinite since so is every $\mathcal{C}$ -immune set. The definition of almost $\mathcal{C}$ -simplicity in Definition 10(4) is slightly different from other simplicity definitions because the infinity condition of the difference $B \setminus A$ is necessary to guarantee $\mathcal{C} \neq \text{co-}\mathcal{C}$, provided that an almost $\mathcal{C}$ -simple set exists.

LEMMA 11 *Let $\mathcal{C}$ be any complexity class closed under finite variations, finite union and finite intersection. If an almost $\mathcal{C}$ -simple set exists, then $\mathcal{C} \neq \text{co-}\mathcal{C}$.*

Lemma 11 is shown as follows. Take any set S such that $\bar{S} = A \cup B$ for a set $A \in \mathcal{C}$ and a $\mathcal{C}$ -immune set B . Suppose $\mathcal{C} = \text{co-}\mathcal{C}$. Note that $B \setminus A \subseteq B$ and $B \setminus A = \bar{S} \setminus A \in \mathcal{C}$. Since B is $\mathcal{C}$ -immune, $B \setminus A$ must be finite.

The following lemma is immediate from Definition 10.

LEMMA 12 *For any complexity class $\mathcal{C}$, every $\mathcal{C}$ -immune set is almost $\mathcal{C}$ -immune and every $\mathcal{C}$ -simple set is almost $\mathcal{C}$ -simple.*

Several characterizations of almost P-immunity and P-levelability are shown in [16] in terms of maximal P-subsets and P-to-finite reductions. We can naturally expand these characterizations to almost Δ_k^P -immunity and Δ_k^P -levelability (but not to the Σ -level classes of the polynomial-time hierarchy).

To understand the characteristics of almost **C-immunity**, we begin with a simple observation. It is known in [16] that any honestly paddable⁶ set not in P is P-levelable. As observed in [18], the essence of this assertion is that if $A \notin P$ and A is length-increasing P-m-selfreducible then A is P-levelable, where A is *length-increasing C-m-selfreducible* if A is **C-m-reducible** to A via a certain length-increasing reduction. This observation can be generalized to Δ_k^P -levelable sets in the following lemma.

LEMMA 13 *Let $k \in \mathbb{N}^+$ and $A \subseteq \Sigma^*$. Assuming that $A \notin \Delta_k^P$, if A is length-increasing Δ_k^P -m-selfreducible, then A and $\bar{A}$ are both Δ_k^P -levelable. Thus, if $\Delta_k^P \neq \Sigma_k^P$ then Σ_k^P as well as Π_k^P has a Δ_k^P -levelable set.*

Most known NP-m-complete sets are known to be honestly paddable and thus, by Lemma 13, the complements of these sets are P-levelable sets, which are also NP-levelable. Therefore, most known NP-m-complete sets cannot be almost NP-simple. This result can be compared to Proposition 16.

Now, we assume a standard effective enumeration $\{\varphi_s\}_{s \in \Sigma^*}$ of all nondeterministic TMs φ_s . For each index s , define the set $W_s = \{x \mid \varphi_s(x) \downarrow = 1\}$, where “ $\varphi_s(x) \downarrow$ ” means that φ_s eventually halts on input x . Fix $k \in \mathbb{N}$. Let $\text{NP}_{(k)}$ denote the collection of all sets W_s such that, for any string $x \in W_s$, the running time of φ_s on input x is at most $|s| \cdot |x|^k + |s|$. Moreover, we set $\text{INDEX}_{(k)} = \{s \mid W_s \in \text{NP}_{(k)}\}$. Note that $\text{NP} = \bigcup_{k \in \mathbb{N}} \text{NP}_{(k)}$.

Earlier, Ko and Moore [11] considered the resource-bounded notion of “productive sets.” Another formulation based on $\text{NP}_{(k)}$ was later given by Joseph and Young [9], who used the terminology of **k-creative**⁷ sets, where k is any number in $\mathbb{N}^+$. They showed that every **k-creative** set is P-m-complete for NP. Orponen et al. [16] showed that, unless $P = \text{NP}$, every honestly **k-creative** set is P-levelable by demonstrating that any honestly **k-creative** set is length-increasing P-m-selfreducible. From Lemma 13, it follows that any honestly **k-creative** set and its complement are both P-levelable. Consequently, we obtain the following result.

COROLLARY 14 *For any $k \in \mathbb{N}^+$, no honestly k-creative set is almost NP-simple.*

Our notion of almost **C-simplicity** is similar to what Uspenskii [23] discussed under the term “pseudosimplicity.” Here, we give a resource-bounded version of his pseudosimplicity. A set S is called **C-pseudosimple** if there is an infinite **C-subset** A of $\bar{S}$ such that $S \cup A$ is **C-simple**. Although **C-simple** sets cannot be **C-pseudosimple** by our definition, any infinite **C-pseudosimple** set is almost **C-simple**. The latter claim is shown as follows. Suppose that S is an infinite **C-pseudosimple** set and A is a **C-subset** of $\bar{S}$ for which $S \cup A$ is **C-simple**. This means that $\bar{S} \setminus A$ is **C-immune**. Therefore, S is almost **C-simple**.

⁶A set S is *polynomially paddable* (paddable, in short) if there is an one-to-one total FP-function *pad* (called the *padding function*) from Σ^* to Σ^* such that, for all pairs $(x, y) \in \Sigma^* \times \Sigma^*$, $x \in S$ iff $\text{pad}((x, y)) \in S$. A set S is *honestly paddable* if it is paddable with a padding function that is component-wise honest.

⁷A set S is called **k-creative** if there exists a function $f \in \text{FP}$ such that, for any index $i \in \text{INDEX}_{(k)}$, $f(i) \in S$ iff $f(i) \in W_i$. This function f is called the *productive function* for S . If in addition f is honest, S is called *honestly k-creative*.

The following theorem shows a close connection among simplicity, almost simplicity, and pseudosimplicity.

THEOREM 15 *For each $k \in \mathbb{N}^+$, the following three statements are equivalent.*

- 1 *There exists a Σ_k^P -simple set.*
- 2 *There exists an infinite Σ_k^P -pseudosimple set in P .*
- 3 *There exists an almost Σ_k^P -simple set in P .*

The most essential part of Theorem 15 is the implication from 1 to 2. Assume that S is an almost Σ_k^P -simple set. Let $A_1 = 1S$ and $B_1 = 1\bar{S}$. Clearly, B_1 is infinite. Note that A_1 is a Σ_k^P -subset of $1\Sigma^*$. Since $\bar{S}$ is Σ_k^P -immune and $B_1 \subseteq 1\bar{S}$, B_1 is Σ_k^P -immune. Since $B_1 = 1\Sigma^* \cap \overline{A_1}$, the set $0\Sigma^* \cup A_1$ is Σ_k^P -simple. Hence, $0\Sigma^*$ is Σ_k^P -pseudosimple, as required. Similarly, $1\Sigma^*$ is Σ_k^P -pseudosimple.

Theorem 15 indicates the importance of the structure of P in the course of the study of Σ_k^P -simplicity. In a relativized world where a Σ_k^P -simple set exists [4], since Theorem 15 relativizes, there exists an almost Σ_k^P -simple set within P .

Finally, we briefly discuss a closure property of the class of all almost Σ_k^P -immune sets under polynomial-time reductions. For each number $k \in \mathbb{N}^+$, the class of all almost Σ_k^P -immune sets is closed under h - Δ_k^P - m -reductions on infinite sets. This immediately implies the following consequence.

PROPOSITION 16 *For each number $k \in \mathbb{N}^+$, there is no almost Σ_k^P -simple set that is h - Δ_k^P - m -complete for Σ_k^P .*

4. Hyperimmunity and Hypersimplicity

Since Post [17] constructed a so-called *hypersimple* set, the notions of hyperimmunity and hypersimplicity have played a significant role in the progress of classical recursion theory. A resource-bounded version of these notions was first considered by Yamakami [26] and studied extensively by Schaefer and Fenner [19]. The definition of Schaefer and Fenner is based on the notion of “honest NP-arrays”, which differs from the notion of “strong arrays” in recursion theory, where a strong array is a series of pairwise disjoint finite sets. For our formalization, we demand only “eventually disjointness” for sets in an array rather than “pairwise disjointness.”

A binary string x is said to *represent* a finite set $\{a_1, a_2, \dots, a_k\}$ if and only if $x = \langle a_1, a_2, \dots, a_k \rangle$ and $a_1 < a_2 < \dots < a_k$ in the lexicographic order on Σ^* . For convenience, we say that a set A *surpasses* another set B if there exists a string $z \in A$ satisfying $z > x$ (lexicographically) for all strings x in B .

DEFINITION 17 *Let $k \in \mathbb{N}^+$, $A \subseteq \Sigma^*$, and $C \in \{\Sigma_k^P, \Delta_k^P\}$.*

- 1 *An infinite sequence $\mathcal{D} = \{D_s\}_{s \in \Sigma^*}$ of finite sets is called a Σ_k^P -array (Δ_k^P -array, resp.) if there exists a single-valued partial function f in $\Sigma_k^P\text{SV}$ (Δ_k^P , resp.) such that (i) $\text{dom}(f)$ is infinite, (ii) $D_s \neq \emptyset$ and $f(s)$ represents D_s for any string $s \in \text{dom}(f)$, and (iii) $D_s = \emptyset$ for any string $s \notin \text{dom}(f)$. This f is called the supporting function of $\mathcal{D}$ and the set $\bigcup_{s \in \text{dom}(f)} D_s$ is called the support of $\mathcal{D}$. The width of $\mathcal{D}$ is the supremum of the cardinality $|D_s|$ over all $s \in \text{dom}(f)$.*

- 2 A **C-array** $\mathcal{D}$ has an infinite support if the support of $\mathcal{D}$ is infinite.
- 3 A **C-array** $\{D_s\}_{s \in \Sigma^*}$ via f is polynomially honest (honest, in short) if f is componentwise honest; namely, there exists a polynomial p satisfying that $|s| \leq p(|x|)$ for any $s \in \text{dom}(f)$ and any $x \in D_s$.
- 4 A **C-array** $\{D_s\}_{s \in \Sigma^*}$ via f is eventually disjoint if, for every string x in $\text{dom}(f)$, there exists a string y in $\text{dom}(f)$ such that $y \geq x$ (lexicographically), D_y surpasses D_x , and $D_x \cap D_y = \emptyset$.
- 5 A **C-array** $\{D_s\}_{s \in \Sigma^*}$ via f intersects A if $D_s \cap A \neq \emptyset$ for all $s \in \text{dom}(f)$.

The honesty condition of an **C-array** guarantees that the array is eventually disjoint. In addition, any eventually-disjoint **C-array** has an infinite support because, for any element D in the array, we can always find another disjoint element D' .

A simple relationship between Σ_k^P -simplicity and a honest Σ_k^P -array is given in the following lemma, which was implicitly proven by Yamakami [26] and later explicitly stated in [19] for the case where $k = 1$.

LEMMA 18 *Let $k \in \mathbb{N}^+$ and let A be any Σ_k^P -simple set. For every number $\ell \in \mathbb{N}^+$, there is no honest Σ_k^P -array $\mathcal{D}$ such that (i) the width of $\mathcal{D}$ is at most ℓ and (ii) $\mathcal{D}$ intersects A .*

We introduce below the notions of **C-hyperimmunity** and honest **C-hyperimmunity**.

DEFINITION 19 *Let $\mathcal{C} \in \{\Delta_k^P, \Sigma_k^P \mid k \in \mathbb{N}\}$.*

- 1 A set S is (honestly) **C-hyperimmune** if S is infinite and there is no (honest) **C-array** $\mathcal{D}$ such that $\mathcal{D}$ is eventually disjoint and $\mathcal{D}$ intersects A .
- 2 A set S is (honestly) **C-bi-hyperimmune** if S and $\bar{S}$ are both (honestly) **C-hyperimmune**.
- 3 A set S is (honestly) **C-hypersimple** if $S \in \mathcal{C}$ and $\bar{S}$ is (honestly) **C-hyperimmune**.

Note that “NP-hyperimmunity” defined by Schaefer and Fenner [19] coincides with our honest NP-hyperimmunity. The following relationship between immunity and hyperimmunity can be obtained immediately from Definition 19.

LEMMA 20 *For any complexity class $\mathcal{C} \in \{\Sigma_k^P, \Delta_k^P \mid k \in \mathbb{N}\}$, every honestly **C-hyperimmune** set is **C-immune** and every honestly **C-hypersimple** set is **C-simple**.*

In late 1970s, Selman [20] introduced the notion of P-selective sets, which are analogues of semi-recursive sets in recursion theory. These sets connects P-immunity to P-hyperimmunity. In general, for any class $\mathcal{F}$ of total functions, we say that a set S is **$\mathcal{F}$ -selective** if there exists a function (called the *selector*) f in $\mathcal{F}$ such that, for all pairs $(x, y) \in \Sigma^* \times \Sigma^*$, (i) $f(x, y) \in \{x, y\}$ and (ii) $\{x, y\} \cap S \neq \emptyset$ implies $f(x, y) \in S$. Recall the partial function class $\Sigma_k^P \text{SV}$. We use the notation $\Sigma_k^P \text{SV}_t$ to denote the collection of all total functions in $\Sigma_k^P \text{SV}$.

LEMMA 21 *Let $k \in \mathbb{N}^+$. Every Σ_k^P -immune $\Sigma_k^P \text{SV}_t$ -selective set is honestly Σ_k^P -hyperimmune.*

We give the proof of Lemma 21. Assume that S is $\Sigma_k^P \text{SV}_t$ -selective via a selector f and has a honest Σ_k^P -array $\mathcal{D} = \{D_s\}_{s \in \Sigma^*}$ intersecting S via a supporting function g . For each $y \in \text{dom}(g)$, assuming $D_y = \{x_1, x_2, \dots, x_m\}$ with $x_1 < x_2 < \dots < x_m$, let $y_1 = x_1$ and $y_{i+1} = f(y_i, x_{i+1})$ for every $i \in \{1, 2, \dots, m-1\}$ and then define $h(y) = y_m$. This h is honest and in $\Sigma_k^P \text{SV}$. The set $B = \{x \mid \exists y[y \in \text{dom}(h) \wedge h(y) = x]\}$ is therefore infinite and in Σ_k^P since $\text{dom}(f)$ is infinite and h is honest. Because $B \subseteq S$, B cannot be Σ_k^P -immune. Note that our proof relativizes.

Observe that the complement of a $\Sigma_k^P \text{SV}_t$ -selective set S is also $\Sigma_k^P \text{SV}_t$ -selective because the exchange of the output string of any selector for S gives rise to a selector for $\bar{S}$. It also follows from Lemma 21 that every NP-simple P-selective set is honestly NP-hypersimple since the complement of any P-selective set is also P-selective.

Next, we show that strong P-immunity does not imply honest P-hyperimmunity within the class E. Earlier, Balcázar and Schöning [2] created a strongly P-bi-immune set S in E with the density $|S \cap \Sigma^{\leq n}| = 2^{n+1} - n - 1$ for all $n \in \mathbb{N}$. For each x , let D_x consist of the first $|x| + 1$ elements of $\Sigma^{|x|}$. Clearly, D_x intersects S . This implies that S is not honestly P-hyperimmune. Therefore, we obtain the following proposition.

PROPOSITION 22 *There exists a strongly P-bi-immune set in E that is not honestly P-hyperimmune.*

As a main theorem, we show the P-T-incompleteness of Σ_k^P -hypersimple sets. Generally, we say that A is Δ_k^P -T-reducible to B if there exists an oracle Δ_k^P -machine M which recognizes A with access to B as an oracle. If in addition M on input x makes only queries y to B that satisfy $|x| \leq p(|y|)$, where p is a fixed polynomial, then we say that A is $h\text{-}\Delta_k^P$ -T-reducible to B via M . This reduction machine M is simply called *honest*.

THEOREM 23 *Let $k \in \mathbb{N}^+$.*

- 1 *No Σ_k^P -hypersimple set is P-T-complete for Σ_k^P .*
- 2 *No honestly Σ_k^P -hypersimple set is h-P-T-complete for Σ_k^P .*

Note that it is not clear if we can replace the P-T-completeness in Theorem 23 by the Δ_k^P -T-completeness. Theorem 23 follows from Lemma 24 in the following fashion. We prove only the first claim. Assume that B is a Σ_k^P -hypersimple set that is P-T-complete for Σ_k^P . Thus, $\Delta_k^P \neq \Sigma_k^P$. Clearly, B is in EXP and every Σ_k^P -set is P-T-reducible to B . By Lemma 24, every Σ_k^P -set is almost Δ_k^P -immune. This contradicts Lemma 13. Therefore, B cannot be Σ_k^P -hypersimple.

LEMMA 24 *Let $k \in \mathbb{N}^+$ and let A be any infinite set in Σ_k^P .*

- 1 *If A is P-T-reducible to a Σ_k^P -hyperimmune set in EXP, then A is almost Δ_k^P -immune.*
- 2 *If A is h-P-T-reducible to a honestly Σ_k^P -hyperimmune set, then A is almost Δ_k^P -immune.*

Lemma 24 needs a key idea of Agrawal (mentioned earlier), who showed that no NP-simple set is h-P-btt-complete for NP. We extend his core argument to Lemma

25. For convenience, a complexity class $\mathcal{C}$ is said to be *closed under intersection with Δ_k^P -sets* if, for any set A in $\mathcal{C}$ and any set B in Δ_k^P , the intersection $A \cap B$ is in $\mathcal{C}$.

LEMMA 25 *Let $\mathcal{C}$ be any complexity class containing Δ_k^P such that $\mathcal{C}$ is closed under intersection with Δ_k^P -sets. Let A be any Δ_k^P -levelable set in $\mathcal{C}$. If A is Δ_k^P -T-reducible to B via a reduction machine M , then there exists an infinite set C in $\mathcal{C}$ such that $Q(M, B, x) \cap B \neq \emptyset$ for all $x \in C$.*

Bruschi [4] demonstrated how to construct a recursive oracle relative to which a Σ_k^P -simple set exists. We can easily modify his proof to obtain a P-selective set that is Σ_k^P -simple in a relativized world. Since Lemma 21 relativizes, we obtain the following proposition.

PROPOSITION 26 *For each $k \in \mathbb{N}^+$, there exists a recursive oracle A such that a $\Sigma_k^P(A)$ -hypersimple set exists.*

As Schaefer and Fenner [19] demonstrated, it is relatively easy to prove the existence of an honest NP^G -hypersimple set relative to a generic oracle G . By contrast, Vereshchagin [24] proved the existence of an NP-simple set relative to a random oracle with probability 1. Again, we modify his proof to construct a relativized P-selective NP-simple set. From Lemma 21, the next proposition follows.

PROPOSITION 27 *With probability 1, an honestly NP^X -hypersimple set exists relative to a random oracle X .*

An important open problem is to prove that, at each level k of the polynomial-time hierarchy, honest Σ_k^P -hypersimple sets exist relative to a random oracle with probability 1.

5. Completeness Under Non-Honest Reductions

Immunity has a deep connection to various completeness notions. For example, there is a simple, tt-complete set; however, no simple set is btt-complete. In the previous sections, we have shown that various types of resource-bounded simple sets cannot be complete under certain polynomial-time honest reductions. This section instead focuses on the incompleteness of simple sets under non-honest reductions.

To remove the honesty condition from reductions, we often need to make extra assumptions for similar incompleteness results. In mid 1980s, Hartmanis, Li, and Yesha [6] proved that (i) no NP-immune set in EXP is P-m-hard for NP if $\text{NP} \not\subseteq \text{SUBEXP}$ and (ii) no NP-simple set is P-m-complete if $\text{NP} \cap \text{co-NP} \not\subseteq \text{SUBEXP}$. These results can be expanded to any Δ -level of the polynomial-time hierarchy and of the subexponential-time hierarchy⁸. We also improve the latter claim.

⁸The Δ -level of the subexponential-time hierarchy is defined as: $\text{SUB}\Delta_0^{\text{EXP}} = \text{SUBEXP}$ and $\text{SUB}\Delta_{k+1}^{\text{EXP}} = \text{SUBEXP}^{\Sigma_k^P}$ for every $k \in \mathbb{N}$, where SUBEXP^A denotes $\bigcap_{e>0} \text{DTIME}^A(2^{n^e})$ for any oracle A . When $A = \emptyset$, we simply write SUBEXP for SUBEXP^A .

To describe our expansion, we need the unambiguous complexity class $U(\Sigma_k^P \cap \Pi_k^P)$ introduced by Yamakami [25]. For any complexity class $\mathcal{C}$, a set A is in $U(\mathcal{C})$ (or UC) if there exists a single-valued partial function f such that (i) f is polynomially bounded, (ii) $\text{Graph}(f) \in \mathcal{C}$, and (iii) $A = \text{dom}(f)$ [25].

PROPOSITION 28 *Let j and k be any nonnegative integers.*

- 1 *No Σ_k^P -immune set in Δ_j^{EXP} is Δ_k^P -m-hard for Σ_k^P if $\Sigma_k^P \not\subseteq \text{SUB}\Delta_{\max\{j,k\}}^{\text{EXP}}$.*
- 2 *No Σ_k^P -simple set is Δ_k^P -m-complete for Σ_k^P if $U(\Sigma_k^P \cap \Pi_k^P) \not\subseteq \text{SUB}\Delta_k^{\text{EXP}}$.*

Note that Proposition 28(2) directly follows from Theorem 29(2).

The original result of Hartmanis et al. refers to the P-m-incompleteness of NP-simple sets. Recently, Schaefer and Fenner [19] showed a similar result for the P-1tt-completeness. They proved that no NP-simple set is P-1tt-complete for NP if $UP \not\subseteq \text{SUBEXP}$. A key to their proof is the fact⁹ that $\text{Sep}(\text{SUBEXP}, \text{NP})$ implies $UP \subseteq \text{SUBEXP}$, where $\text{Sep}(\mathcal{C}, \mathcal{D})$ means the separation property in [25] that, for any two disjoint sets $A, B \in \mathcal{D}$, there exists a set $S \in \mathcal{C} \cap \text{co-}\mathcal{C}$ satisfying that $A \subseteq S \subseteq \overline{B}$.

The following theorem shows that the assumption $UP \not\subseteq \text{SUBEXP}$ in [19] can be replaced by $U(\text{NP} \cap \text{co-NP}) \not\subseteq \text{SUBEXP}$.

THEOREM 29 *Let $j, k \in \mathbb{N}^+$.*

- 1 *No Σ_k^P -immune set in Δ_j^{EXP} is Δ_k^P -1tt-hard for $U(\Sigma_k^P \cap \Pi_k^P)$ if $U(\Sigma_k^P \cap \Pi_k^P) \not\subseteq \text{SUB}\Delta_{\max\{j,k\}}^{\text{EXP}}$.*
- 2 *No Σ_k^P -simple set is Δ_k^P -1tt-complete for Σ_k^P if $U(\Sigma_k^P \cap \Pi_k^P) \not\subseteq \text{SUB}\Delta_k^{\text{EXP}}$.*

Theorem 29 follows from the technical lemmas: Lemmas 30 and 31. The proof for its second claim proceeds as follows. Assume that B is Δ_k^P -1tt-complete for Σ_k^P . Choose an infinite set $A \in U(\Sigma_k^P \cap \Pi_k^P) - \text{SUB}\Delta_k^{\text{EXP}}$, which is of the form $\{x \mid \exists y[\langle x, y \rangle \in \text{Graph}(f)]\}$ for a certain polynomially-bounded partial function f whose graph is in $\Sigma_k^P \cap \Pi_k^P$. Similar to [25], set $A_1 = \{\langle x, z \rangle \mid \exists y[z \leq y \wedge \langle x, y \rangle \in \text{Graph}(f)]\}$ and $A_2 = \{\langle x, z \rangle \mid \exists y[z < y \wedge \langle x, y \rangle \in \text{Graph}(f)]\}$. Clearly, A_1 and A_2 are in $U(\Sigma_k^P \cap \Pi_k^P) - \text{SUB}\Delta_k^{\text{EXP}}$. Since A_1 is Δ_k^P -1tt-reducible to B , by Lemma 30, there exists a set $C \in \Delta_k^P$ such that $A_1 \cap C$ is infinite and coinfinite and $A_1 \cap C$ is $\text{h-}\Delta_k^P$ -1tt-reducible to B . Applying Lemma 31(1), we obtain a set $D \in \Delta_k^P$ and a total $\text{F}\Delta_k^P$ -function f such that $\overline{A_1 \cap C} \cap \overline{D}$ is finite. f Δ_k^P -m-reduces $A_1 \cap C$ to $\overline{B}$, and f is honest on the domain D . Since $A_2 \subseteq \overline{A_1 \cap C}$, $f(A_2) \subseteq \overline{B}$. Moreover, $A_2 \cap D$ is infinite. The honesty of f on D implies that $\overline{B}$ has an infinite Σ_k^P -subset $f(A_2 \cap D)$.

The key idea of Hartmanis et al. [6] is to find a set that can be honestly reducible. Lemma 30 is a “1tt” version of a technical part of [6].

LEMMA 30 *Let $j, k \in \mathbb{N}^+$. Assume that $A \notin \text{SUB}\Delta_{\max\{j,k\}}^{\text{EXP}}$ and $B \in \Delta_j^{\text{EXP}}$. If A is Δ_k^P -1tt-reducible to B , then there exists a set C in Δ_k^P such that (i) $A \cap C$ is $\text{h-}\Delta_k^P$ -1tt-reducible to B and (ii) $A \cap C$ and $\overline{A} \cap C$ are infinite and coinfinite.*

⁹Actually, the result of Schaefer and Fenner can be strengthened in the following way: $\text{Sep}(\text{SUBEXP}, U(\text{NP} \cap \text{co-NP}))$ if and only if $U(\text{NP} \cap \text{co-NP}) \subseteq \text{SUBEXP}$. This is obtained by analyzing a similar result in [25].

LEMMA 31 Let $k \in \mathbb{N}^+$ and $A, B \subseteq \Sigma^*$. Assume that B is Σ_k^P -immune and A is $h\text{-}\Delta_k^P$ -1tt-reducible to B .

- 1 If $A \in \Sigma_k^P$, then there exist a set $D \in \Delta_k^P$ and a total function $f \in \text{F}\Delta_k^P$ such that $\overline{A} \cap \overline{D}$ is finite, $f \Delta_k^P$ -m-reduces A to $\overline{B}$, and f is honest on the domain D .
- 2 A belongs to $\Sigma_k^P \cap \Pi_k^P$ if and only if A belongs to Δ_k^P .

6. Limited Immunity and Simplicity

Within our current knowledge, we cannot prove or disprove the existence of an NP-simple set. The difficulty comes from the fact that an NP-immune set requires *every* NP-subset to be finite. If we restrict our attention to certain types of NP-subsets, then we may overcome the difficulty. Under the name of **k -immune** sets, Homer [7] required only **NP $_{(k)}$ -subsets**, for a fixed number k , to be finite. He then demonstrated how to construct a **k -simple** set within NP using Ladner's delayed diagonalization technique.

In this section, we investigate the notions obtained by restricting the requirements for immunity and simplicity. We first review Homer's notions of **k -immunity** and **k -simplicity**.

DEFINITION 32 Let k be any number in $\mathbb{N}^+$.

- 1 A set S is **k -immune** if S is infinite and there is no index i in $\text{INDEX}_{(k)}$ such that W_i is infinite and $W_i \subseteq S$.
- 2 A set S is **k -simple** if S belongs to NP and $\overline{S}$ is **k -immune**.

An "effective" version of immune and simple sets, called effectively immune and effectively simple sets, has been studied in recursion theory. Effectively simple sets are known to be T-complete and there also exists an effectively simple tt-complete set. If A is strongly effectively immune, then $\overline{A}$ cannot be immune. Analogously, we consider a resource-bounded version of such effectively immune and simple sets. Here, we freely identify binary strings with natural numbers using the lexicographic order on Σ^* .

DEFINITION 33 Let $k \in \mathbb{N}^+$.

- 1 A set S is feebly **k -immune** if (i) S is infinite and (ii) there exists a polynomial p such that, for every index i in $\text{INDEX}_{(k)}$, $W_i \subseteq S$ implies $|W_i| \leq 2^{p(i)}$.
- 2 A set S is feebly **k -simple** if S is in NP and $\overline{S}$ is feebly **k -immune**.

We can easily prove the existence of a feebly **k -immune** set in Δ_2^P for each $k \in \mathbb{N}^+$. From Definition 33, every feebly **k -simple** set is **k -simple**. The converse, however, does not hold since there exists a **k -simple** set which is not feebly **k -simple** for each number k in $\mathbb{N}^+$. The theorem below is slightly stronger than this claim since any feebly **k -simple** set is also feebly 1-simple.

THEOREM 34 For each $k \in \mathbb{N}^+$, there exists a **k -simple** set which is not feebly 1-simple.

We return to the old question of whether NP-simple sets exist. There seems no strong evidence that suggests the existence of such a set. Only relativization provides a world where NP-simple sets exist. At the same time, we can also construct another world where these sets do not exist. These relativization results clearly indicate that the question of whether NP-simple sets exist needs unrelativizable proof techniques.

In the past few decades, the Berman-Hartmanis isomorphism conjecture has served as a working hypothesis in connection to NP-complete problems. By contrast, there has been no “natural” working hypothesis that yields the existence of NP-simple sets. For example, the hypothesis $P \neq NP$ does not suffice since Homer and Maass [8] showed a relativized world where the assumption $P \neq NP$ does not imply the existence of an NP-simple set. Motivated by Homer’s ***k*-simplicity** result, we propose the following working hypothesis:

- The ***k*-immune** hypothesis: There exists a positive integer k such that every infinite NP set has an infinite $NP_{(k)}$ -subset.

Under this hypothesis, we can derive the desired consequence: the existence of NP-simple sets.

LEMMA 35 *If the ***k*-immune** hypothesis holds, then there exists an NP-simple set.*

Assume that the ***k*-immune** hypothesis is true; that is, there exists a positive integer k such that every infinite NP-set has an infinite $NP_{(k)}$ -subset. Consider any ***k*-simple** set A . We claim that A is NP-simple. If A is not NP-simple, then $\bar{A}$ has an infinite NP-subset B . By our assumption, B contains an infinite $NP_{(k)}$ -subset. Hence, A cannot be ***k*-simple**, a contradiction. Therefore, A is NP-simple.

To close this section, we claim the following result concerning the ***k*-immune** hypothesis. The proof uses weak forcing.

PROPOSITION 36 *The ***k*-immune** hypothesis fails relative to a generic oracle.*

Final Note. All the proofs that are omitted from this extended abstract will appear in its forthcoming complete version.

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